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# Random dynamical models from time series

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In this work we formulate a consistent Bayesian approach to modeling stochastic (*random*) dynamical systems by time series and implement it by means of artificial neural networks. The feasibility of this approach for both creating models adequately reproducing the observed stationary regime of system evolution, and predicting changes in qualitative behavior of a weakly nonautonomous stochastic system, is demonstrated on model examples. In particular, a successful prognosis of stochastic system behavior as compared to the observed one is illustrated on model examples, including discrete maps disturbed by non-Gaussian and nonuniform noise and a flow system with Langevin force.

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## I. INTRODUCTION

Construction of parametrized models (*global reconstruction*) of deterministic dynamical systems from time series has been broadly discussed in the literature in the past 20 years [1–10]. The mathematical apparatus substantiating such a possibility has been developed. Different methods of constructing models of evolution operators have been proposed; basic limitations have been understood and formulated recently [11]. In particular, some authors demonstrated that these approaches can be used for prediction of changes in the qualitative behavior of a weakly nonautonomous system for times longer than the duration of the observed time series [11–13].

This paper is concerned with the Bayesian approach to reconstructing random (or, in other words, stochastic) dynamical systems (RDS) from time series (TS). Mathematically RDS is an object consisting of a model of noise and a model of the system perturbed by noise [14]. Physically RDS is a dynamical system subject to random external action in the course of evolution. This action is frequently referred to as dynamical or interactive noise [15]. The majority of natural systems are known to be open, that is, subject to numerous external actions. Therefore, it is physically justified to represent natural systems in the form of RDS. We can say that the problem of RDS reconstruction from TS is the necessary and important step toward reconstructing real (natural) systems when their adequate first-principle mathematical models (based on equations of gas- and hydrodynamics, chemical kinetics, balance relations for the quantity of substance, pulse, energy, and so on) are unknown.

Note that, even when it is justified to regard the observed system to be *deterministic*, that is, finite embedding dimension of the attractor can be found, construction of a *deterministic* model of this dimension from the TS generated by such a system and use of this model for a prognosis of qualitative behavior of the system has quite a number of principal restrictions. The first of them is restriction on system complexity. The point is that reconstruction of a phase trajectory following Takens [16] is possible in a phase space of not too small dimension $d_E$: $d_E \geqslant 2d_S + 1$, where $d_S$ is the box-counting dimension of the chaotic attractor of the system that has generated the initial TS [17]. Generally speaking, this means that a model in the form of a deterministic dynamical system (DDS) describes correctly behavior of the reconstructed system on the manifold of dimension $d_S$ that can be much smaller than the dimension of the phase space of the model $d_E$. But it is not guaranteed and generally not true that this manifold is going to be stable in the $(2d_S + 1)$-dimensional phase space of the model. The second restriction is prior information. The available methods [18] for determining such dimensions are inapplicable for analysis of the TS generated by real (e.g., atmospheric) systems. The point is that methods for determining system dimension work poorly when the studied TS contains a random component ("measurement noise"). An exponential dependence between the dimension of the system and the duration of the TS is required for correct determination of system dimension [19], so that the duration of measurements needed for reconstruction of not too simple systems becomes almost unattainable. We believe that the above restrictions explain why only a few works demonstrated the efficiency of global reconstruction by means of DDS models from TS generated by *natural systems*, that is, unavailable in well-controlled experimental conditions (see, for example, [6–8]).

Reconstruction in the form of RDS mitigates or lifts the restrictions mentioned above, thus making the proposed approach more universal. This study is dedicated to the development of a consistent Bayesian approach to such a reconstruction with the ultimate goal of predicting any qualitative changes that may occur in system behavior. We assume that the classification of qualitatively different regimes is application driven, so that an investigator has a classifier function which maps a point in the parameter space of the model to a finite set of possible behaviors of interest.

The current paper consists of seven parts. A general problem of global reconstruction of a random dynamical

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system using the Bayesian approach is formulated in Sec. II. In Sec. III we propose a stochastic model of an evolution operator efficient for applications. In Sec. IV the evolution operator is approximated by an artificial neural network (ANN) [20] and efficiency of reconstructing a system with inhomogeneous, non-Gaussian, and nonwhite *dynamical* noise is demonstrated on a model example. At the end of the section we describe the technique of representing (classifying) regimes of behavior of stochastic systems based on their invariant measures. The proposed approach is generalized to the case of nonautonomous stochastic systems in Sec. V. A possible application of the approach under consideration, namely, prognosis of changes in qualitative behavior of a stochastic system, is formulated in Sec. VI and is illustrated on an example of a system with inhomogeneous and non-Gaussian dynamical noise.

Finally, in Sec. VII we generalize the proposed approach for the case of a continuous time dynamical system with noise (a stochastic flow) and illustrate it using a Lorenz system with Langevin noise source as an example. To conclude, we formulate problem solutions which will enable us to determine applicability boundaries of the new approach and discuss some possible applications.

## II. FORMULATION OF THE PROBLEM

Let us take normalized and centered vector time series $\{\mathbf{U}(t_n)=\mathbf{U}_n\}_{n=1}^N$, $\mathbf{U}(t)\in\mathbb{R}^d$, $\mathbb{E}(\mathbf{U})=0$, $\mathrm{var}(\mathbf{U})=1$ ($\mathbb{E}$ and var mean mathematical expectation and variance, respectively) obtained as a result of successive measurements of the states of a dynamical system having dimension $d$ by a fixed lag at time instants $t_n$. Making use of the definition of a RDS given in Ref. [14] we suppose that these states are coupled by a random evolution operator $\boldsymbol{\varphi}$:

$$\mathbf{U}_{n+1}=\boldsymbol{\varphi}(\boldsymbol{\omega}_n,\mathbf{U}_n),\ \varphi:\Omega\times\mathbb{R}^d\to\mathbb{R}^d,\quad \boldsymbol{\omega}_{n+1}=\theta(\boldsymbol{\omega}_n),\\ \theta:\Omega\to\Omega, \tag{1}$$

where $\Omega$ is the measurable set (sample space) for which $\sigma$-algebra $\Sigma$ and probability measure $P$ are specified, and $\theta$ is the endomorphism in the probability space $(\Omega,\Sigma,P)$.

The Bayesian approach to reconstructing such a random operator consists of determining the probability $P(\mathbf{U}|\boldsymbol{\varphi})$ (also referred to as likelihood) of observation of the measured time series for $\boldsymbol{\varphi}$ of a definite class. We assume that each operator of this class correlates one-to-one with a point from $\mathbb{R}^L$. Such a map will be called parametrization, and the corresponding point will be referred to as the operator parameter; $L$ is the dimension of the space of parameters. Depending on the context, $\boldsymbol{\varphi}$ will be understood either as an operator or its parameters.

According to the Bayes theorem, posterior distribution of parameters $\boldsymbol{\varphi}$ is specified to an accuracy of normalization by the following expression:

$$P_{\text{posterior}}(\boldsymbol{\varphi}|\mathbf{U})\propto P(\mathbf{U}|\boldsymbol{\varphi})P_{\text{prior}}(\boldsymbol{\varphi}), \tag{2}$$

where prior distribution $P_{\text{prior}}(\boldsymbol{\varphi})$ is determined by prior restrictions on operator parameters. Construction and analysis of (2) thereby solves the problem of modeling.

## III. SOLUTION

For the sake of convenience we transform (1) to

$$\mathbf{U}_{n+1}=\mathbf{f}(\mathbf{U}_n)+\boldsymbol{\eta}(\boldsymbol{\omega}_n,\mathbf{U}_n), \tag{3}$$

where $\mathbf{f}(\mathbf{U})=\mathbb{E}(\boldsymbol{\varphi}(\boldsymbol{\omega},\mathbf{U}))$, $\boldsymbol{\eta}(\boldsymbol{\omega},\mathbf{U})=\boldsymbol{\varphi}(\boldsymbol{\omega},\mathbf{U})-\mathbf{f}(\mathbf{U})$. The form (3) allows one to separate explicitly the deterministic and random components ($\mathbf{f}$ and $\boldsymbol{\eta}$, respectively) in the model evolution operator. Physically, reconstruction of the RDS using model (3) means time-scale separation of the processes available in the TS: the deterministic component will be determined primarily by "long correlated" processes, and the random component by processes with a relatively short correlation time. With allowance for the above mentioned, we will represent the random component in the form

$$\boldsymbol{\eta}(\boldsymbol{\omega},\mathbf{U})=\hat{g}(\mathbf{U})\cdot\boldsymbol{\zeta}(\boldsymbol{\omega}),\quad \hat{g}:\mathbb{R}^d\to\mathbb{R}^{d\times M},\quad \boldsymbol{\zeta}:\Omega\to\mathbb{R}^M. \tag{4}$$

Records of vector random process $\boldsymbol{\zeta}_n=\boldsymbol{\zeta}(\boldsymbol{\omega}_n)$ having dimension $M$ will be assumed to be independent or, in other words, white noise described by the probability density $w(\boldsymbol{\zeta})=\mathbb{E}(\delta(\boldsymbol{\zeta}-\boldsymbol{\zeta}(\boldsymbol{\omega})))$. With allowance for (4), Eq. (3) will take on the form

$$\mathbf{U}_{n+1}=\mathbf{f}(\mathbf{U}_n)+\hat{g}(\mathbf{U}_n)\cdot\boldsymbol{\zeta}_n. \tag{5}$$

The matrix function $\hat{g}$ in Eq. (5) describes the distribution of a random component in the phase space of the model [clearly Eq. (5) is the expression close to that used in the classical least-squares method, the only difference is that the error dispersion is supposed to depend on the point in phase space]. The resulting likelihood will have the form

$$P(\mathbf{U}|\boldsymbol{\varphi})=\prod_n P(\mathbf{U}_{n+1}|\mathbf{U}_n)=\prod_n\int\delta[\mathbf{U}_{n+1}-\mathbf{f}(\mathbf{U}_n)\\ -\hat{g}(\mathbf{U}_n)\boldsymbol{\zeta}_n]w(\boldsymbol{\zeta})d\boldsymbol{\zeta}. \tag{6}$$

Hereinafter we will assume that vector $\boldsymbol{\zeta}$ has normally distributed independent components: $\zeta_l\propto N(0,1)$, $l=\overline{1,M}$. We will show by way of example that such simplification of the model enables successful solution of the reconstruction problem in the case of obviously non-Gaussian statistics of the reconstructed system. The substitution of normal distribution into Eq. (6) yields

$$P(\mathbf{U}|\mathbf{f},\hat{G})\propto\prod_n\frac{1}{\sqrt{|\hat{G}(\mathbf{U}_n)|}}\exp\left\{-\frac{1}{2}[\mathbf{U}_{n+1}-\mathbf{f}(\mathbf{U}_n)]^T\right.\\ \left.\times\hat{G}^{-1}(\mathbf{U}_n)[\mathbf{U}_{n+1}-\mathbf{f}(\mathbf{U}_n)]\right\}. \tag{7}$$

An important consequence of (7) is that all the models Eqs. (3) and (4) having equal deterministic components and equal covariance matrices of the stochastic component $\hat{G}=\hat{g}^T\hat{g}$ are equiprobable. This means that we can limit the dimension of random process $\zeta$ to the phase space dimension $d$ without restricting generality. Besides, as $\hat{G}$ is a symmetric matrix, it may be described by $d(d+1)/2$ independent functions of the phase coordinates.

Thus, the likelihood (7) specifies the probability density for the class of functions $\mathbf{f}(\mathbf{U})$ and $\hat{G}(\mathbf{U})$ defined *a priori*, which solves the formulated problem completely.

## IV. RECONSTRUCTION OF AN AUTONOMOUS STOCHASTIC SYSTEM. MODELS OF DETERMINISTIC AND STOCHASTIC COMPONENTS IN THE FORM OF ARTIFICIAL NEURAL NETWORKS

Useful tools for parametrizing the evolution operator in reconstruction of deterministic systems from TS are artificial neural networks (ANN) [20] with the corresponding prior distributions of network parameters [21]. In the current paper we demonstrate potentialities of the approach using the same approximation:

$$\mathbf{A}_{d_{\text{in}}}^{d_{\text{out}}}(\mathbf{U}) = \left[\sum_{i=1}^{m} \alpha_{ki} \tanh\left(\sum_{j=1}^{d_{in}} w_{ij} U_j + \gamma_i\right)\right]_{k=1}^{d_{\text{out}}},$$
$$\mathbf{f}(\mathbf{U}) = \mathbf{A}_d^d(\mathbf{U}), \quad \mathbf{G}(\mathbf{U}) = \mathbf{A}_d^{d(d+1)/2}(\mathbf{U}), \tag{8}$$

where $d_{\text{in}}$ is the number of ANN inputs, $d_{\text{out}}$ is the number of outputs, and $m$ is the number of neurons in the hidden layer. In line with the considerations put forth in Ref. [21] we set prior distributions of network parameters in the form

$$P_{\text{prior}}(\boldsymbol{\alpha},\mathbf{w},\boldsymbol{\gamma}) \propto \exp\left[-\sum_{i=1}^{m}\left(\sum_{k=1}^{d}\frac{\alpha_{ki}^2}{2\sigma_\alpha^2} + \sum_{j=1}^{d}\frac{w_{ij}^2}{2\sigma_w^2} + \frac{\gamma_i^2}{2\sigma_\gamma^2}\right)\right], \tag{9}$$

where $\sigma_\alpha^2 = 1/m, \sigma_w^2 = 1$, and $\sigma_\gamma^2 = d$ are the dispersions of the corresponding parameters. The considerations standing behind these dispersions are quite simple. They are supposed to reflect our prior knowledge (or expectations) about the underlying system. What was implicitly restricted in the problem setup is the system's spatial and temporal scales. With regard to the spatial scale, it is easy to show that the range of the function in the form (8) can be estimated by the expression $\sqrt{m\sigma_\alpha^2}$. $\sigma_w$ defines how steep the model is allowed to be as a function of phase coordinates. Ultimately, it is connected to the shortest time scale of the system. Finally, $\sigma_\gamma$ is responsible for the area of sensitivity of the model. It is chosen so that this area is appropriate to the observed data range.

Thus, being substituted into Eq. (2), the expressions (7)–(9) determine posterior probability density of the neural network parameters. Here we will restrict consideration to analysis of the most probable models, that is, the models corresponding to maximum posterior distribution. In other words, in such a formulation the problem of model construction will consist of finding the maximum posterior probability density by network parameters approximating deterministic and stochastic components.

By way of example, consider first a stochastic dynamical system in the form of a logistic map perturbed by noise:

$$x_{n+1} = f(x_n) + \sigma\eta_n, \quad f(x) = 1 - \lambda x^2. \tag{10}$$

The system (10) becomes unstable if unbounded noise $\eta$ is used. To avoid this instability we derived the noise $\eta$ from white Gaussian process $\{\zeta_n\}, \zeta_n \overset{iid}{\sim} N(0,1)$ by rejecting values $\zeta_i$ leading to transitions of the system beyond the domain $[x_1, -x_1]$, that is, the noise $\eta$ becomes constrained by the condition $|f(x_i) + \sigma\eta_i| \leqslant -x_1$, where $x_1 = -(1+\sqrt{1+4\lambda})/2\lambda$ is the left unstable equilibrium point of the logistic map. However, we have to note that the probabilities of such events are quite small at durations of the time series and noise levels $\sigma$ used below, so that the process $\eta$ can be considered almost Gaussian.

Second, we consider the same system but with records made next but one:

$$x_{n+2} = 1 - \lambda\left(1 - \lambda x_n^2 + \sigma\eta_n\right)^2 + \sigma\eta_{n+1}. \tag{11}$$

System (11) is interesting in that its random component is inhomogeneous and essentially non-Gaussian. The results of the reconstruction of system (11) are presented in Fig. 1. The TS 1000 records generated by system (11) for $\lambda = 1.85$ were used as initial data. It is clear from Fig. 1 that in spite of the non-Gaussian stochastic component of the system, its distribution in phase space has been correctly reconstructed by the models (2), (5), (7), and (9).

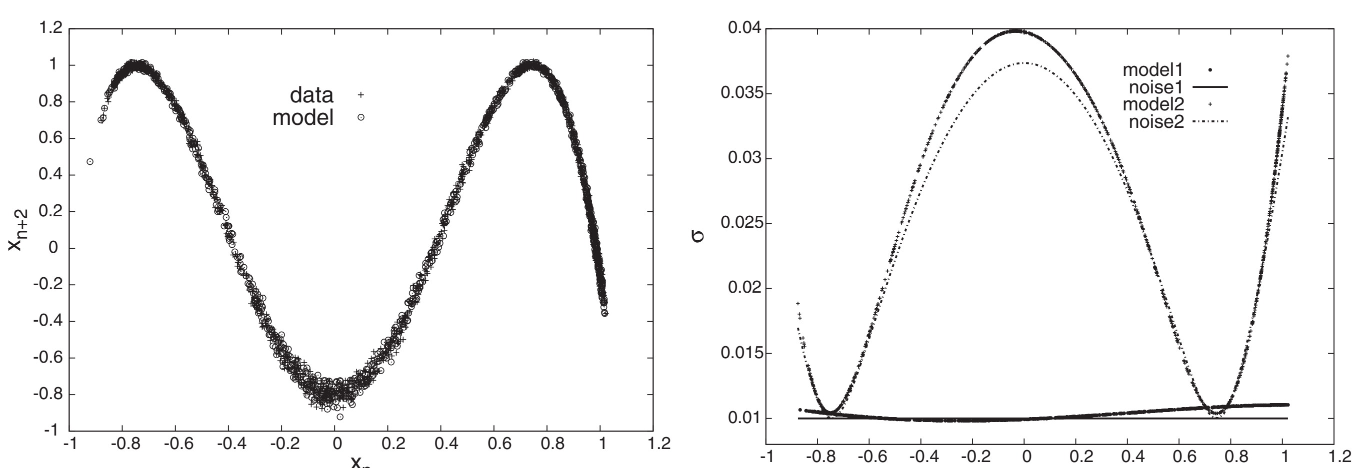


FIG. 1. On the left: Reconstruction of autonomous stochastic system (11) using the ANN model. The ensemble of points labeled "data" is the state of the system, and labeled "model" is the state of the model. Noise level $\sigma = 0.01$. On the right: Comparison of random components of the system and the model: standard deviation of the distribution $\hat{g}(\mathbf{U}) \cdot \boldsymbol{\zeta}$ obtained analytically for system (11) ("noise2"), and calculated by the model in the form of the ANN ("model2"). Points "noise1" and "model1" are the same but for system (10). Noise level $\sigma = 0.01$.

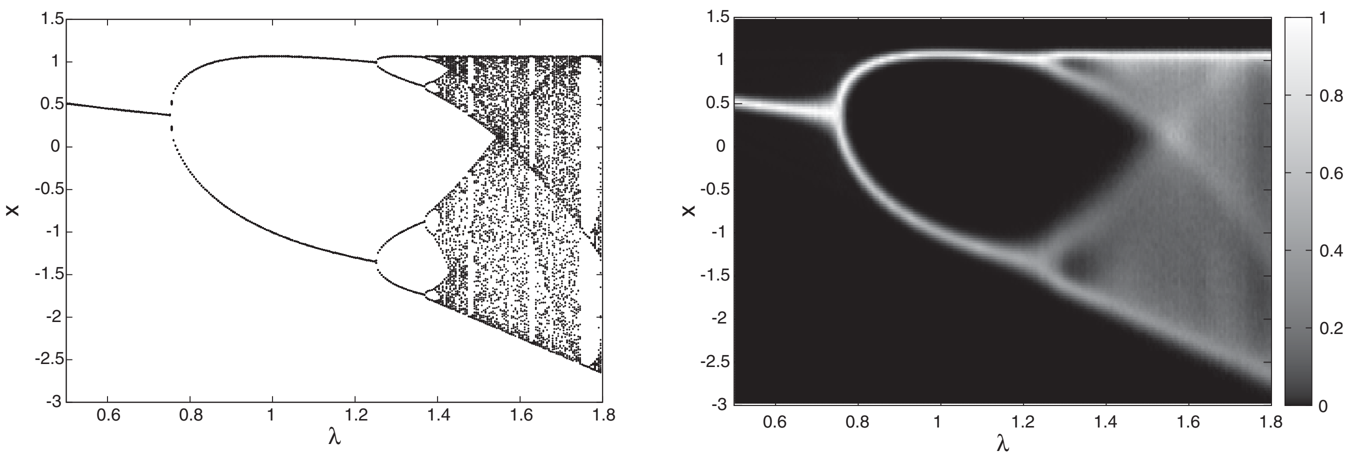


FIG. 2. Invariant measure plot as an analog of bifurcation diagram for stochastic systems on an example of logistic map $x_n = 1 - \lambda x_{n-1}^2 + \sigma\eta_{n-1}$. On the left: bifurcation diagram of deterministic system $\sigma = 0$. On the right: invariant measure plot of stochastic map $\sigma = 0.01$, shades of gray show function $p_x(x,\lambda)$. Hereinafter we use normalized units for $x$ values, that is, the entire time series is reduced to have zero mean and unit variance.

A traditional method of qualitative representation of the dependence of the regime of behavior of a deterministic DS on control parameter is construction of a bifurcation diagram visualizing asymptotic (limiting) regimes of behavior corresponding to different values of parameters (see Fig. 2, left panel as an example). In the case of a stochastic system, limiting the regime of behavior is characterized by invariant measure $p_x(x,\lambda)$ [14] that is probability density of states $x$ in phase space; $\lambda$ is a control parameter of the system. In the current work we represent qualitative behavior of the stochastic system using the invariant measure depicted by shades of gray on the plane of one of the phase coordinates and control parameter (Fig. 2, right panel).

## V. PROGNOSIS OF QUALITATIVE BEHAVIOR OF A WEAKLY NONAUTONOMOUS STOCHASTIC SYSTEM

In this section we will demonstrate capabilities of the proposed approach on an example of the prognosis of qualitative behavior of RDS when its evolution operator depends slowly on time. This means that functions $\mathbf{f}(\mathbf{U})$ and $\hat{G}(\mathbf{U})$ describing the stochastic model must depend explicitly on "slow" time. We assume that the characteristic time scale of this dependence is much longer than the length of the observed TS. Note that this situation is essentially different from one considered in Ref. [22] where the changes of parameters of weakly nonautonomous system were also reconstructed. The difference with our study is that the duration of the time series in Ref. [22] must be (much) greater than nonstationarity time scale, which makes it possible to reconstruct the feature space of the system as an embedding for the system parameters. Thus, if parameters of the system are changing in a cyclic manner, it is possible to reconstruct this cycle in the feature space. In essence it means that it is possible to predict future behavior of the system if it has already been observed in the past. But it is not the case in our situation since when making a prediction we have no information about other possible regimes but those currently observed.

The authors of [13] showed that in the case of interest the functions $\mathbf{f}(\mathbf{U},t)$ and $\hat{G}(\mathbf{U},t)$ may be approximated by the ANN in which output layer parameters depend linearly on time:

$$\mathbf{A}_{d_{\text{in}}}^{d_{\text{out}}}(\mathbf{U},t) = \left[\sum_{i=1}^{m}(\alpha_{ki}+t\beta_{ki})\tanh\left(\sum_{j=1}^{d_{\text{in}}} w_{ij}U_j + \gamma_i\right)\right]_{k=1}^{d_{\text{out}}},$$
$$\mathbf{f}(\mathbf{U},t) = \mathbf{A}_d^d(\mathbf{U},t),\quad G(\mathbf{U},t)=\mathbf{A}_d^{d(d+1)/2}(\mathbf{U},t). \tag{12}$$

Like in the previous section, prior distributions of network parameters are supposed to be Gaussian and, analogously to (9), the dispersion $\sigma_\alpha^2 = \sigma_\beta^2 = 1/m, \sigma_w^2 = 1$, and $\sigma_\gamma^2 = d$. With allowance for the explicit time dependence, the likelihood (7) transforms to

$$P(\mathbf{U}|\mathbf{f},\hat{G}) \propto \prod_n \frac{1}{\sqrt{|\hat{G}(\mathbf{U}_n,t_n)|}} \exp\left\{-\frac{1}{2}[\mathbf{U}_{n+1} - \mathbf{f}(\mathbf{U}_n,t_n)]^T \times \hat{G}^{-1}(\mathbf{U}_n,t_n)[\mathbf{U}_{n+1} - \mathbf{f}(\mathbf{U}_n,t_n)]\right\}. \tag{13}$$

By substituting (12) and (13) into Eq. (2) and finding the maximum of posterior probability density by network parameters, we obtain the most probable model of the system in which slow time $t$ is a *unique* control parameter. Similar to the work in Ref. [13], this model may be used for prognosis of changes in system behavior, which corresponds to time extrapolation of model parameters outside the observation interval. Model examples of such a prognosis will be given below.

## VI. PROGNOSIS IN THE CASE OF INHOMOGENEOUS AND NON-GAUSSIAN NOISE

In the examples presented in this section, we used as the initial data nonstationary TS 1000 records generated by a nonautonomous RDS in the form of the stochastic map (11) in which the control parameter $\lambda$ was varied linearly in the interval [1.7,1.4]. As was mentioned above, this RDS is an example of a stochastic system with non-Gaussian and

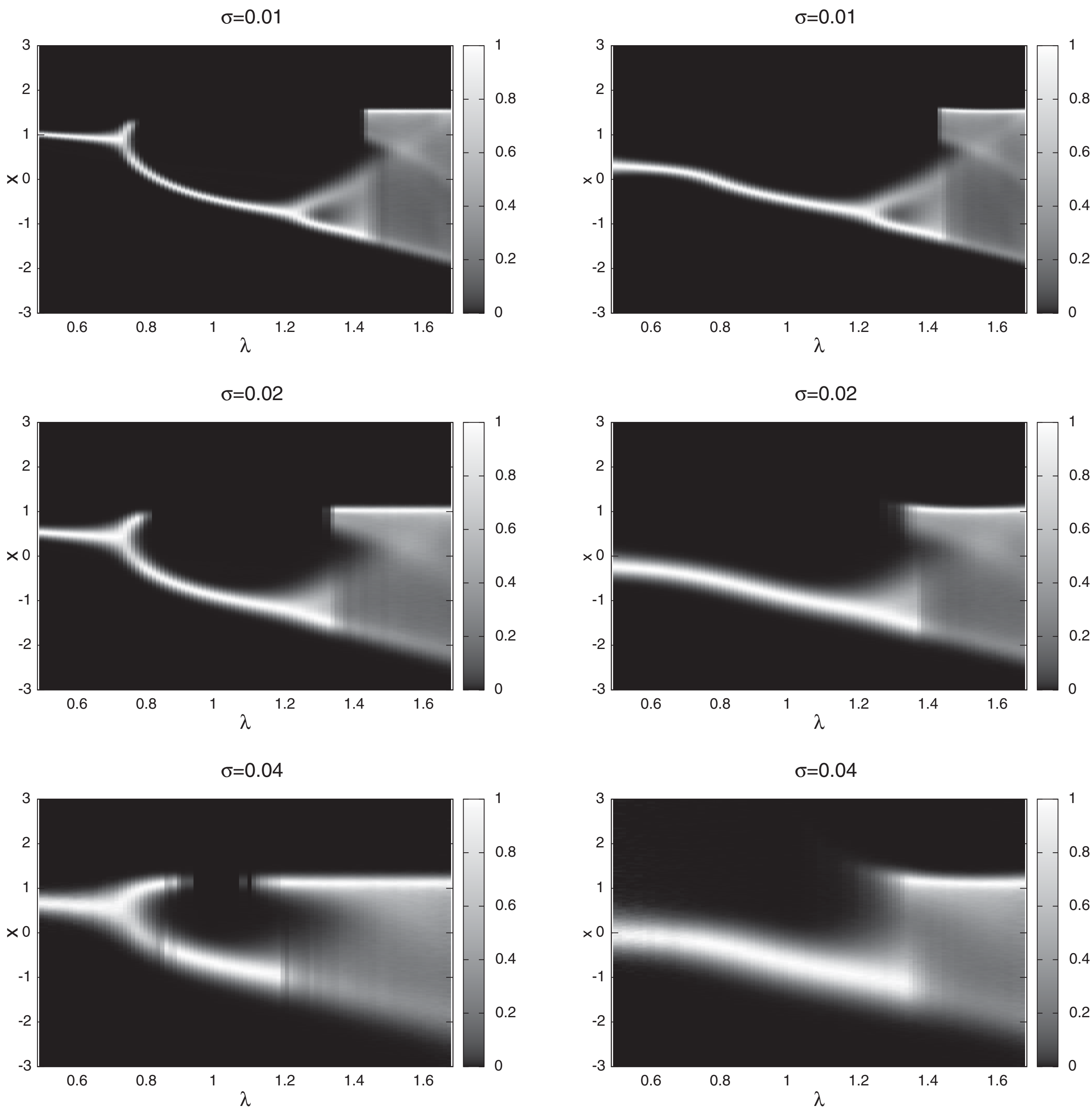


FIG. 3. On the left: invariant measure plot of system (11) corresponding to the true behavior of the system at slow variation of parameter $\lambda$. On the right: the same for the model constructed by the TS corresponding to the range $\lambda \in [1.7; 1.4]$. The noise level is shown above each figure. The number of neurons in the hidden layer of the model $[\mathbf{f}(\mathbf{U})$ and $G(\hat{\mathbf{U}})]$ is $m = 6$.

nonwhite noise distributed nonuniformly over the attractor (see Sec. III). From the initial time series we constructed the nonautonomous models (1), (5), (12), and (13) and its parameters were extrapolated to the "future," to the times equivalent to changes of parameter $\lambda$ in the interval [1.4,0.5]. Figure 3 shows the invariant measure plot of the original system (11) (on the left) and model (on the right) for different noise levels. It is well seen in these figures that the model adequately describes behavior of the nonautonomous RDS throughout the range of variation of the control parameter.

Note that there exist principal restrictions on a prognosis of deterministic system behavior: it is impossible to make a prognosis "from simple to complex" in a deterministic system since only a limiting regime and no transients are available. By from simple to complex we mean that the bifurcation we are going to predict results in an increase in minimal embedding dimension. For instance, we could not predict a period-doubling transition in the unperturbed ($\sigma = 0$) system (10) at $\lambda \approx 0.75$ analyzing the time series corresponding to the stable equilibrium point at $\lambda < 0.75$. The situation is

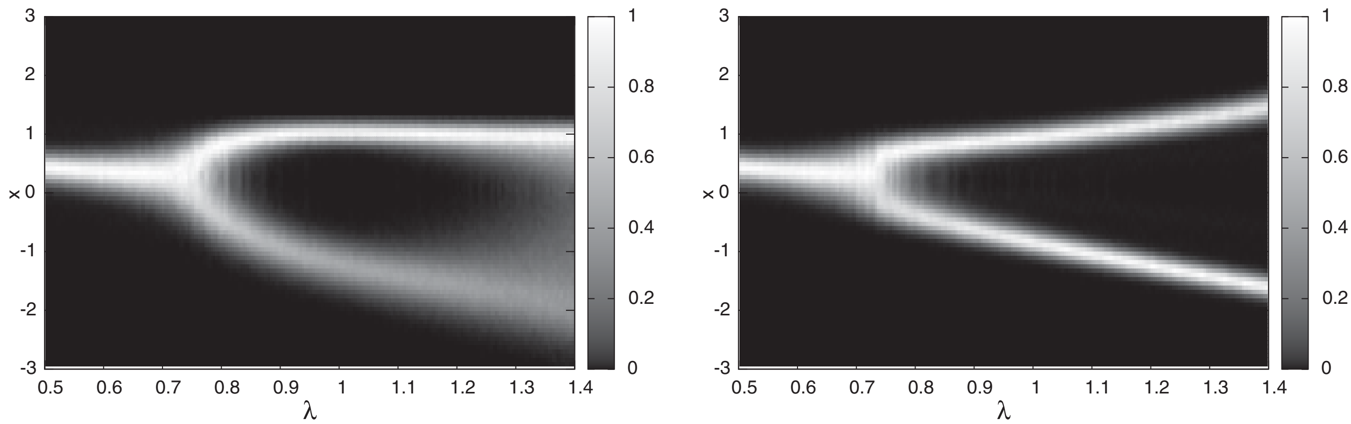


FIG. 4. On the left: invariant measure plot of system (10) corresponding to the true behavior of the system at slow variation of parameter λ. On the right: the same for the *model* constructed by the TS corresponding to the range $\lambda \in [0.5; 0.74]$. Noise level $\sigma = 0.04$. Number of neurons in the hidden layer of the model $[\mathbf{f}(\mathbf{U})$ and $\hat{G}(\mathbf{U})]$ is $m = 6$.

quite different if $\sigma > 0$. Technically, the dimension of the trajectory reconstructed from stochastic time series is always infinitely large. In case of small enough noise we still can see structure similar to what the corresponding deterministic system produces, but it is not really a complication in case of a stochastic system (noise resets the initial conditions so we have sort of transient dynamics). In this section we will demonstrate that it is possible to make a prognosis of qualitative behavior for RDS in both directions. We took as the initial data nonstationary TS 1000 records generated by a nonautonomous RDS in the form of the stochastic logistic map (10) in which the control parameter λ changes linearly in the interval [0.5; 0.74]. From the initial TS we constructed the nonautonomous model (13) and the model parameters were extrapolated to the future for the times equivalent to changes of parameter λ in the interval [0.74; 1.4]. The invariant measure plots of the initial system (on the left) and the model (on the right) demonstrating results of the prognosis of behavior of a nonautonomous RDS system are given in Fig. 4. Clearly the model provides a correct prediction of the change of the type of behavior of the system closest in time, including prediction of the bifurcation point. Eventually the model ceases to adequately describe system behavior and does not reproduce the transition of the system to more complicated regimes.

## VII. PROGNOSIS OF THE DYNAMICS OF A STOCHASTIC FLOW

We now demonstrate how the described approach can be used for modeling the stochastic systems with continuous time using as an illustration the classical Lorenz system [23] with a Langevin noise source in the third equation:

$$\begin{aligned} \dot{x} &= 10(y - x), \\ \dot{y} &= x(r - z) - y, \\ \dot{z} &= xy - \tfrac{8}{3}z + s\xi. \end{aligned} \tag{14}$$

Here $r$ is control parameter and $\xi$ is white Gaussian noise. The bifurcation scenario of this system at $s = 0$ (hereinafter referred as an "unperturbed system") with changing $r$ is well investigated [24]: for $r \in (1,13.9)$ in the phase space of the system (14) there exist three fixed points—two stable focuses and a saddle at (0,0,0). At $r \approx 13.93$ two saddle loops appear, and then after further increase of $r$ two unstable limit cycles are born from these loops, and a strange repeller appears simultaneously. Finally, at $r \approx 24.06$ the transition to chaos occurs, and soon (at $r \approx 24.74$) the focuses lose their stability. Besides, at high $r$ the system exhibits self-sustained oscillations. Thus, an invariant measure of the system (14) at $s = 0$ is a superposition of two $\delta$ functions corresponding to the stable focuses up to $r \approx 24.06$. But if $s > 0$, this is no longer the case since transitions (or switches) occur between basins of attraction of two stable focuses due to noise. In this case a narrow bridge between two maxima appears in the invariant measure. The time the phase trajectory spends in the vicinity of each maximum depends, first, on the noise level: the more the noise the more often the transitions occur; and second, on the value of control parameter $r$ which determines the basins of attraction in the phase space of the unperturbed system.

We generated two scalar time series $y_i$ of the variable $y$ of the system (14) each 2000 time units long with the parameter $r$ linearly changing in time from 28 to 22 for the first time series and from 7 to 13 for the second one. The sampling time step was $\Delta t = 0.3$ in both cases. Then we used the described approach to construct a nonautonomous stochastic model with the ultimate goal to predict the changes in the system behavior up to $t = 7000$. Such changes of the control parameter (from 7 to 28 or the other way around) lead to the transition of the unperturbed system from a chaotic regime to a stable fixed point and from a fixed point to a chaotic regime, respectively. Correspondingly, we consider the first case as prognosis "from complex to simple" and the second one as prognosis "from simple to complex."

On the upper panels of Fig. 5 the time series used for model construction are shown in black. "Future" behavior as produced by the original system, which is subject to prognosis, is gray (green online). The qualitative changes that one can see in the time-series presented are concerned with the time which the system spends in the vicinity of the maximum of its invariant measure until it switches to the other one.

We now assume that the model constructed in the form (1) maps the vector $\mathbf{U}_i = (y_{i-d+1}, \ldots, y_i)$ to the vector $\mathbf{U}_{i+1} = (y_{i-d+2}, \ldots, y_{i+1})$, where $d$ is the dimension of the "phase"

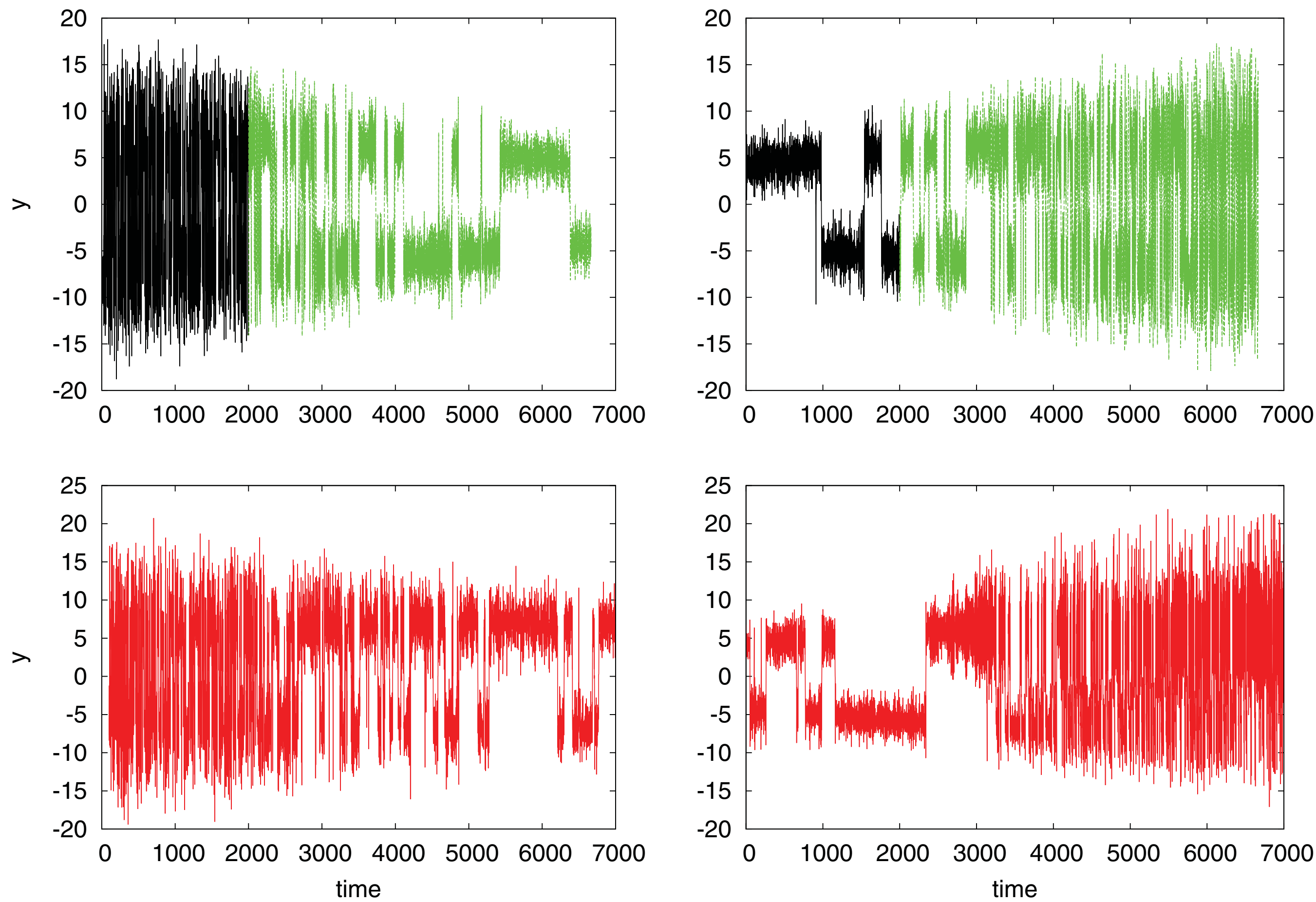


FIG. 5. (Color online) Prognosis of the behavior of stochastic Lorenz system (14). Left: "from complex to simple," right: "from simple to complex." $X$ axis: time $t$, $Y$ axis: variable $y$ of the system (14). Upper panel: black trace—the observed time series (see explanations in the text), grey (green online) trace—future behavior to be predicted. Lower panel: the dynamics of the most probable stochastic model constructed from the observed "black" time series $t \in (0,2000)$ and extrapolated to the future $t \in (2000,7000)$.

space reconstructed by the method of delayed coordinates [16]. Taking into account the trivial expressions for the first $d-1$ components of the resulting vector we can represent the model in the following form:

$$\begin{aligned} U^1_{n+1} &= U^2_n, \\ &\dots \\ U^{d-1}_{n+1} &= U^d_n, \\ U^d_{n+1} &= f(\mathbf{U}_n,t_n) + g(\mathbf{U}_n,t_n)\zeta_n. \end{aligned} \tag{15}$$

where $f(\mathbf{U},t) = \mathbf{A}^1_d(\mathbf{U},t)$, $g(\mathbf{U},t) = \mathbf{A}^1_d(\mathbf{U},t)$.

The low panels of Fig. 5 show the evolution of the most probable models [i.e., corresponding to the maximum of posterior PDF (2)] constructed from black time series with $d = 3$. In both cases the model constructed demonstrates the behavior qualitatively similar to the original system.

Comparing the actual and predicted dependencies of the average interval between switches on time (Fig. 6), we can say that the prognosis of complex to simple seems to be quantitatively more accurate than simple to complex.

Figure 7 shows evolution of the invariant measure with slow time predicted by the model (second and fourth columns) in comparison with the invariant measure of the system (first and third columns). The upper panels correspond to the observable behavior, and the lower panels to a time moment in the future as remote as the observation duration. It is worth noting that in the case of the prognosis from simple to complex (the right pair in Fig. 7), although the predicted evolution of the invariant measure is qualitatively correct, the symmetry in the model appears to be broken (compare Fig. 7 B3 and B4) as a consequence of less accurate reconstruction.

Thus, the model quite accurately reproduces the invariant measure of the system extrapolated to the future for times comparable to or even exceeding the observation duration. This works for both the prognosis from complex to simple (left panels of Figs. 5–7) and (although to less extent) the prognosis from simple to complex (right panels of those figures).

## VIII. DISCUSSION AND CONCLUSION

We developed a Bayesian approach to global reconstruction of a stochastic system from observed time series. An efficient form of a stochastic model of an evolution operator was proposed. In this form an evolution operator is represented as a superposition of deterministic and stochastic parts, and the latter is treated as a multivariate Gaussian noise with

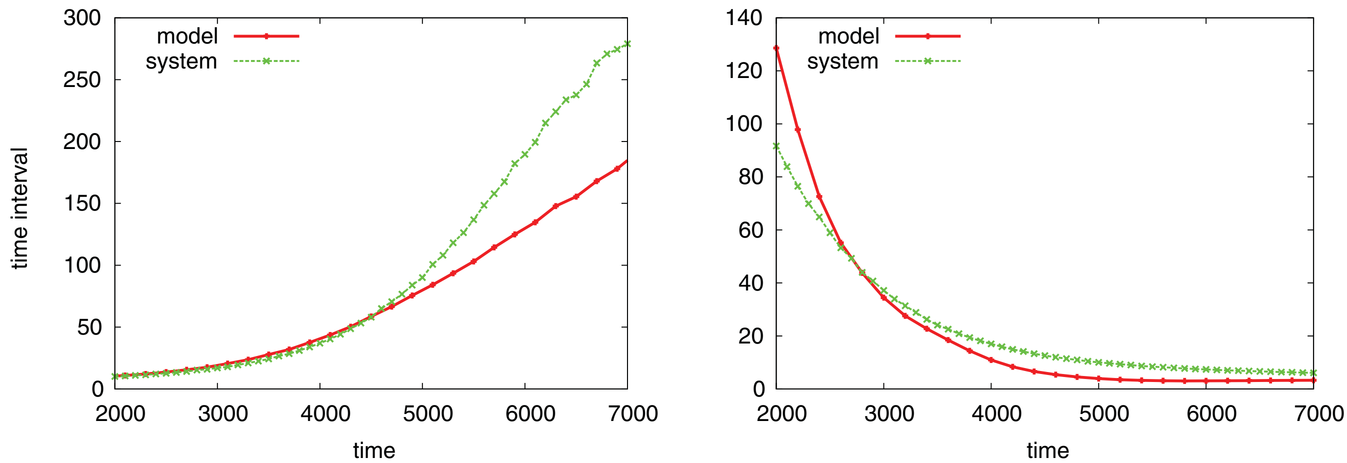


FIG. 6. (Color online) Average interval of time between switches as a function of time corresponding to Fig. 5. Gray (green online)—for system (14), black (red online)—as predicted by the model constructed from "black" time series.

a covariation matrix dependent on the state of phase space. This representation is based on the hypothesis that the noise inhomogeneity is inevitably concerned with nonlinearity of the underlying system and it has the most profound effect. The approach was implemented algorithmically using artificial neural networks. The capabilities of the approach were illustrated on model examples.

One of the very important conclusions of the asymptotic stability and bifurcation theory of RDS (see [14] for review) was that qualitative changes of invariant measure (the so called P-bifurcations) are not always bifurcations in a strict sense, that is, concerned with appearance of new or disappearance of old solutions and/or changes in their stability (D-bifurcations), and vice versa. Due to this fact the P-bifurcation-based approach (which is actually much more traditional and intuitive) was criticized. In this work we illustrate our approach using examples which are based on P-bifurcations for the sake of clearness. If an investigator is interested in exploring D-bifurcations in the system for some reason (as defined in Ref. [14]) he/she should just use an appropriate classifier, although "the theory of stochastic bifurcations is sill in its infancy" [14] and effective methods of RDS bifurcations investigation are still to be developed.

Two more aspects are worthy of special notice. In some works concerned with reconstruction of deterministic dynamical systems it was demonstrated that the situation changed cardinally when measurement noise was present in the data [9,25]. Like it was done for deterministic systems [9], the approach developed in the current work may be readily generalized to the case of noise measurements. The Bayesian approach allows estimating the most probable dispersion of measurement noise, thus giving an answer to the question as to what (and to what extent) the cause of data noise is: random actions on the system or inaccurate measurements. Consequently, independent checking of the hypothesis of the deterministic nature of the modeled system is of no need. Detailed analysis of "stochastic" reconstruction from noisy time series will be given elsewhere.

Another important application of the proposed approach is the possibility of modeling too high-dimensional deterministic

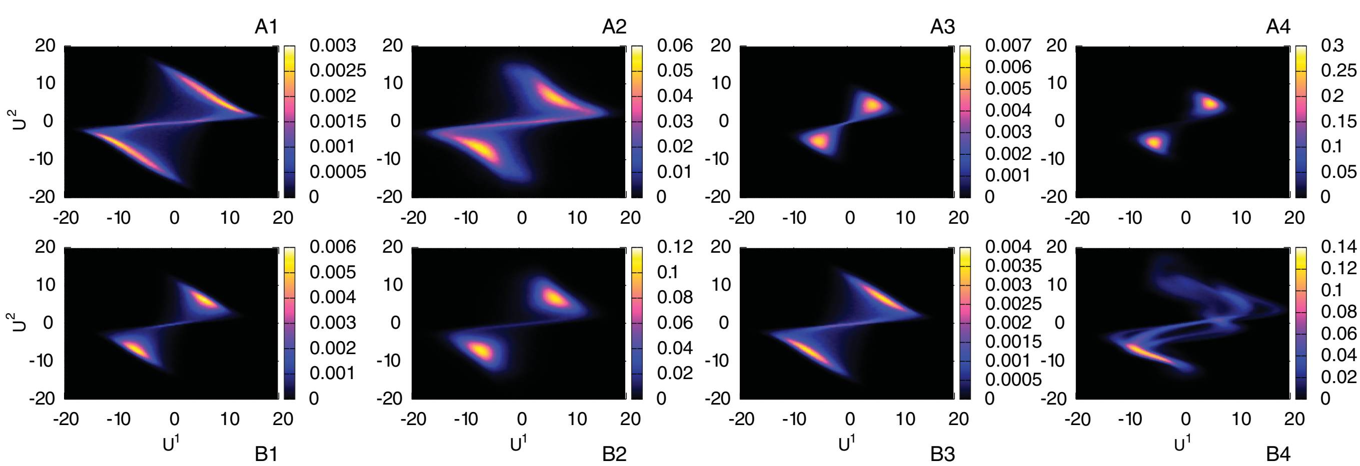


FIG. 7. (Color online) Projections of invariant measures of the original system (14) and of the model to the plane $(U^1, U^2)$ at two different slow time slices. Upper panels: $t = 1000$, lower panels: $t = 4000$. First and third columns (panels A1, B1 and A3, B3) for the system; second and fourth columns (panels A2, B2 and A4, B4) for the models constructed from black time series shown on the left and right panels of Fig. 5, respectively.

systems (in terms of reconstruction following Takens, see the Introduction) by means of low-dimensional stochastic models. Particulary, such an approach can be useful for coping with the problem of limited length of a time series that prevents construction of a high-dimensional deterministic model, as well as with the problem of robustness of the model in relation to the reconstructed behavior. These aspects will also be considered elsewhere.

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