Supersharp resonances in chaotic wave scattering

Marcel Novaes
Departamento de Física, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil
(Received 13 January 2012; published 5 March 2012)

Wave scattering in chaotic systems can be characterized by its spectrum of resonances, $z_n = E_n - i\frac{\Gamma_n}{2}$, where E_n is related to the energy and Γ_n is the decay rate or width of the resonance. If the corresponding ray dynamics is chaotic, a gap is believed to develop in the large-energy limit: almost all Γ_n become larger than some γ. However, rare cases with $\Gamma < \gamma$ may be present and actually dominate scattering events. We consider the statistical properties of these supersharp resonances. We find that their number does not follow the fractal Weyl law conjectured for the bulk of the spectrum. We also test, for a simple model, the universal predictions of random matrix theory for density of states inside the gap and the hereby derived probability distribution of gap size.

DOI: 10.1103/PhysRevE.85.036202 PACS number(s): 05.45.Mt, 03.65.Sq, 05.60.Gg

I. INTRODUCTION

Scattering of waves in complex media is a vast area of research, from oceanography and seismology through acoustics and optics, all the way to the probability amplitude waves of quantum mechanics [1–3]. We focus our attention on the important class of systems for which the complexity is not due to the presence of randomness or impurities, but rather because the corresponding ray dynamics is chaotic. The presence of multiple scattering leads to very complicated cross sections, with strongly overlapping resonances that, although deterministic, have apparently random positions and widths. It is not uncommon that, in a given situation only the sharpest resonances are relevant, with the others providing an approximately uniform background. Recent experiments include the areas of lasers [4,5], microwaves [6], nanostructures [7], and graphene nanoribbons [8].

For concreteness of terminology, we consider quantum mechanical systems, but our results are general. We denote resonances by $E - i\frac{\Gamma}{2}$ and call Γ the width or the decay rate. In the large-energy limit a gap develops in the resonance spectrum: most resonances have their widths larger than γ, which is the average decay rate of the corresponding classical (ray) dynamics. This was noticed long ago [9], and more recently there have been attempts to prove it rigorously [10]. We are interested in the rare case of states inside the gap (i.e., with $\Gamma < \gamma$), which we call supersharp resonances.

The distribution of typical resonances in chaotic systems is conjectured to follow the so-called fractal Weyl law [11]: their number grows with E as a power law whose exponent is related to the fractal dimension d of the classical repeller, the set of rays which remain trapped in the scattering region for infinite times, both in the future and in the past [12]. In a numerical experiment with a simple model, we find that the number of supersharp resonances, denote it by $\mathcal{N}_{SSR}$, does not follow this law. It does grow with E according to a power law, but the exponent seems to be insensitive to γ or the dimension of the repeller.

We also investigate the dependence on energy of the width of the sharpest (and usually most important) resonance. Let this be denoted Γ_0. As E grows, it is expected to converge to γ. We use their exponentials as alternative variables more suited to our modeling. We find that the distance $e^{-\Gamma_0} - e^{-\gamma}$ decreases with E according to a power law, whose exponent is well approximated by $d - \alpha$.

A very fruitful approach to chaotic scattering of waves is random matrix theory (RMT) [13], in which the system's propagator (the Green's function of the wave equation) is replaced by a random matrix whose spectral properties are studied statistically. The RMT prediction for the density of resonance states inside the gap was derived in [14]. In this work we derive the RMT prediction for the probability distribution of $e^{-\Gamma_0} - e^{-\gamma}$ and compare both these predictions to numerical results in a specific system.

II. WEYL LAWS

Waves in chaotic systems can be modeled by the so-called "quantum maps." These are $N \times N$ unitary matrices where $N \sim 1/\hbar$, and the large-energy limit $E \to \infty$ is replaced by the limit of large dimension, $N \to \infty$. This approach has been used to study transport properties of semiconductor quantum dots [15], entanglement production [16], the fractal Weyl law [17], fractal wave functions [18,19], and proximity effects due to superconductors [20], among other phenomena. Scattering can be introduced by means of projectors. The propagator becomes a subunitary matrix of dimension $M < N$, and its spectrum comprises $N - M$ zero eigenvalues and M resonances of the form $z_n = e^{-i(E_n - i\Gamma_n/2)}$.

As our dynamical model we use the kicked rotator. Its classical dynamics is determined by canonical equations of motion in discrete time,

$$q_{t+1} = q_t + p_t + \frac{K}{4\pi}\sin(2\pi q_t) \pmod 1, \tag{1}$$

$$p_{t+1} = p_t + \frac{K}{4\pi}[\sin(2\pi q_t) + \sin(2\pi q_{t+1})] \pmod 1. \tag{2}$$

This system is known to be fully chaotic for $K > 7$, with Lyapunov exponent $\lambda \approx \ln(K/2)$. Its quantization [20,21] yields an N-dimensional unitary matrix U. We set equal to zero a fraction of $1 - \mu$ of its columns, corresponding to a "hole" in phase space. Here μ corresponds to the fraction of rays which escape the scattering region per unit time, so $\mu = e^{-\gamma}$. We do not consider very small values of μ, which would correspond to widely open systems.

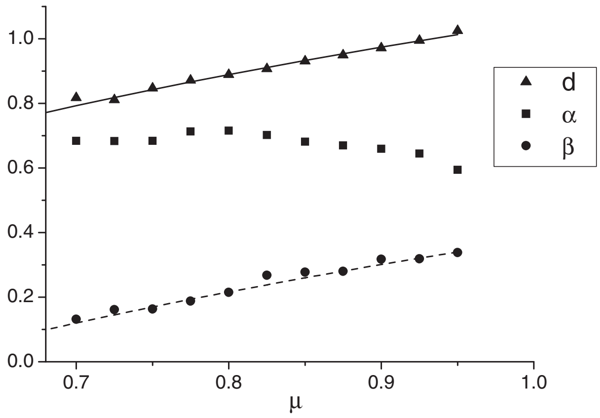

FIG. 1. Scaling exponents as functions of μ for open kicked rotator. d, α, and β are related to the fractal Weyl law, the number of supersharp resonances and the width of the gap, respectively. We see that α is approximately independent of the size of the opening. The solid line is a best linear fit to d, while the dashed line is $d - \langle\alpha\rangle$, where $\langle\alpha\rangle \approx 0.66$ is the average value of α.

We find numerically that $\mathcal{N}_{SSR} \sim N^{\alpha}$, while $e^{-\Gamma_0} - e^{-\gamma} \sim N^{-\beta}$. The exponents are plotted as functions of μ in Fig. 1, together with d, the numerically determined exponent in the Weyl law. We see that, somewhat surprisingly, in this range α is approximately constant, (i.e., insensitive to the dimension of the repeller). The supersharp resonances do not follow the fractal Weyl law. On the other hand, the exponent β has approximately the same slope as d. We find the relation $d - \beta = \alpha$ to be approximately fulfilled, which is to be expected since it says that the number of supersharp resonances, $\sim N^{\alpha}$, is proportional to the total number of resonances, $\sim N^{d}$, times the width of the gap, $\sim N^{-\beta}$.

III. RANDOM MATRIX THEORY APPROACH

A. Average density of supersharp resonances

We now turn to a RMT treatment of the problem. RMT for quantum maps amounts to taking matrices uniformly distributed in the unitary group. In [22], an ensemble of truncated unitary matrices was introduced as a model for scattering, and it was shown that, as $N,M \to \infty$ with $\mu = M/N$ held fixed, the probability density of $r = e^{-\Gamma/2}$ converges to

$$P_{\infty}(r) = \frac{1-\mu}{\mu}\frac{2r}{(1-r^2)^2}, \tag{3}$$

if $r^2 < \mu$, and to 0 otherwise. The expression (3) was tested numerically for a chaotic quantum map in [21] and found to be an accurate description of the bulk of the spectrum, provided a rescaling was performed to incorporate the fractal Weyl law.

Let us start with the density of supersharp resonances. This calculation can be found in [14]; we sketch it here for completeness. The probability density for r is known exactly [22], and when $M = \mu N$ and $N \gg 1$ it can be approximated to

$$P(r) \approx \frac{P_{\infty}(r)}{2}\left[1 + \operatorname{erf}\left(\eta\sqrt{\mu}\frac{(\mu - r^2)}{r}\right)\right], \tag{4}$$

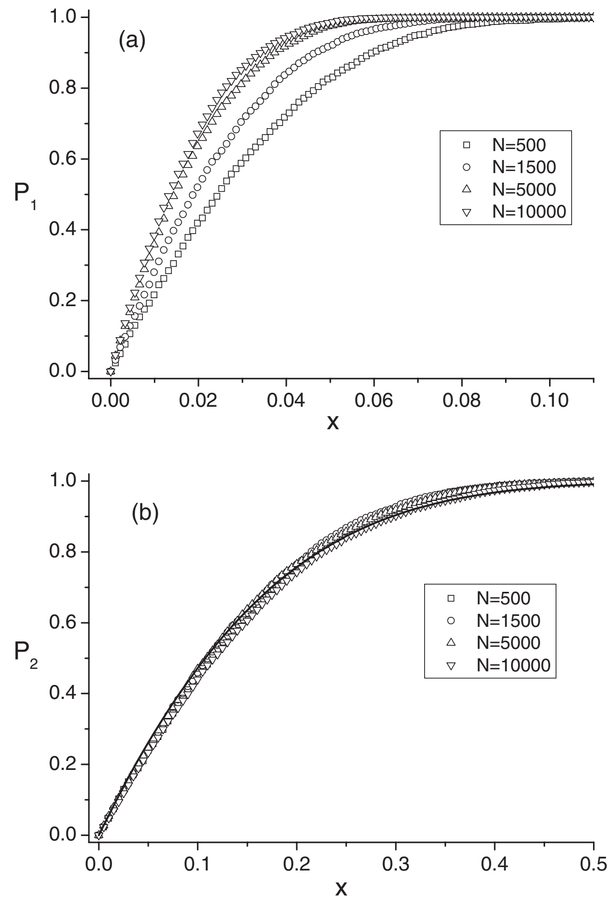

FIG. 2. (Top panel) P_1 is the probability that there are eigenvalues inside the gap whose modulus squared is smaller than $\mu + x$. (Lower panel) P_2 is the probability that there are eigenvalues inside the gap whose modulus squared is smaller than $\mu + x/\widetilde{\eta}$, where $\widetilde{\eta}$ is defined in the text. Results are for the open kicked rotator, at several dimensions, for $\mu = 0.8$. Solid line is a rescaled RMT prediction.

where erf is the error function and

$$\eta = \sqrt{\frac{N}{2\mu(1-\mu)}} \tag{5}$$

is now the large number. Clearly, (4) will be different from $P_{\infty}(r)$ only if $\mu - r^2$ is of the order of $1/\eta$. After setting $r^2 = \mu + \epsilon/\eta$, the probability distribution for the variable ϵ becomes

$$\widetilde{P}(\epsilon) = \sqrt{\pi}[1 - \operatorname{erf}(\epsilon)]. \tag{6}$$

This is the density of states inside the gap. It is only nonzero in a small region that shrinks as $N^{-1/2}$ in the asymptotic limit. Its integral provides the probability that $r^2 - \mu$ be less than some value x. This we denote by $\Pr(r^2 - \mu < x)$.

In Fig. 2(a) we see $\Pr(r^2 - \mu < x)$ for the open kicked rotator, at various dimensions for $\mu = 0.8$. We see that, as N grows, the values of r become more localized around μ. In Fig. 2(b) we introduce a scaled variable $\widetilde{\eta}(r^2 - \mu)$, but with $\widetilde{\eta}$ different from η in that it involves the actual exponent β instead of the RMT prediction $1/2$:

$$\widetilde{\eta} = \left(\frac{N}{2\mu(1-\mu)}\right)^{\beta}. \tag{7}$$

All curves fall on top of each other, indicating that this is the correct scaling. Moreover, the shape of the curve agrees with (6).

B. Distribution of sharpest resonance

Let us now consider the probability distribution of the largest eigenvalue of the propagator, which corresponds to the sharpest resonance and whose modulus we denote by R. Largest eigenvalue distributions are important in different areas of mathematics [23] and physics [24], and have even been measured [4]. A similar calculation to the one below can be found in [25]. Let $\mathcal{P}(\{z\})$ denote the joint probability density function for all eigenvalues. When integrated over all variables from 0 to x, it gives the probability that the modulus of all eigenvalues is smaller than x. Therefore, the probability that the modulus of the largest eigenvalue be smaller than x is

$$\Pr(R < x) = \int \mathcal{P}(\{z\}) \prod_{i=1}^{M} \Theta(|z_i| - x) d^2 z_i. \tag{8}$$

The joint probability distribution function of the eigenvalues is [22]

$$\mathcal{P}(\{z\}) \propto \prod_{i<j}^{1\ldots M} |z_i - z_j|^2 \prod_{i=1}^{M} (1 - |z_i|^2)^{N-M-1}. \tag{9}$$

It is a usual trick to write $\prod_{i<j} |z_i - z_j|^2$ in terms of the Vandermonde determinant $|\det A|^2$, where $A_{ij} = z_j^{i-1}$. This can be shown to be equal to $M! \det B$, where $B_{ij} = z_i^{j-1}(z_i^*)^{i-1}$. Each element of the matrix B depends on a single variable, and the integration decouples. The angular part of the integrals diagonalizes the matrix and, if $M = \mu N$ and $N \gg 1$, the result becomes

$$\Pr(R < x) \propto \prod_{j=0}^{M} \int_0^x (1 - y)^{N(1-\mu)} y^j dy. \tag{10}$$

This result is exact, but a bit cumbersome. Approximating the integrand by a Gaussian function we arrive at $\Pr(R < x) \propto \prod_{\ell=0}^{M} \{\frac{1}{2} + \frac{1}{2}\mathrm{erf}[\eta(x^2 - \mu + \frac{\ell(1-\mu)}{N})]\}$. This result can be further simplified by exponentiating the product into a sum, setting the scale as $\ell = 2\mu\eta\xi$, and approximating the sum by an integral. We get

$$\Pr(R^2 - \mu < x) \propto \exp\left\{2\mu\eta \int_0^\infty d\xi \mathcal{L}(\eta x^2 + \xi)\right\}, \tag{11}$$

where we have defined the function

$$\mathcal{L}(z) = \ln\left(\frac{1 + \mathrm{erf}(z)}{2}\right). \tag{12}$$

One interesting question that can be answered at this point is, how likely is it that a true gap will exist at μ for a finite value of N? The probability that all eigenvalues are smaller than μ is simply given by the exponential in (11) with $x = 0$. It is thus proportional to $e^{-c\sqrt{N}}$ for some constant c.

Notice that (11) has some similarity with the Tracy-Widom distribution [26] of the largest eigenvalue of Gaussian ensembles of RMT, in the sense that it involves the exponential of the integral of a function that satisfies a nonlinear differential equation, $\mathcal{L}''(z) = -2z\mathcal{L}'(z) - (\mathcal{L}'(z))^2$. It is not a Painlevé transcendent, however.

Returning to the calculation, let us change variable to $\delta = \eta x^2$ and obtain $\exp\{2\mu\eta \int_\delta^\infty \mathcal{L}(z) dz\}$. Clearly, this function does not converge as $\eta \to \infty$. Assuming δ to be large, we use $\mathcal{L}(z) \approx -e^{-z^2}/(2\sqrt{\pi} z)$ and integrate by parts to get $\exp\{-2\mu\eta e^{-\delta^2}/(4\sqrt{\pi}\delta^2)\}$. In order to have a finite limit, we must have $\eta e^{-\delta^2}/\delta^2 = O(1)$. This implies that $\delta^2 = y + W(\eta)$, where W is the Lambert function, which for large η can be approximated as $W(\eta) \approx \ln\eta - \ln\ln\eta$. Therefore, if instead of R we consider the variable

$$\rho = \eta^2 (R^2 - \mu)^2 - \ln\eta + \ln\ln\eta, \tag{13}$$

then

$$\Pr(\rho < y) = \exp\left\{-\frac{\mu}{2\sqrt{\pi}} e^{-y}\right\}, \tag{14}$$

which is a modified Gumbel function. Therefore, the distribution of the slightly awkward variable ρ (see also [25]) has a limit as $\eta \to \infty$, but this limit is approached very slowly and finite-η calculations may present significant deviations.

In Fig. 3(a) we see $Q_1 = \Pr(R^2 - \mu < x)$ for the open kicked rotator, at various dimensions for $\mu = 0.8$. In Fig. 3(b)

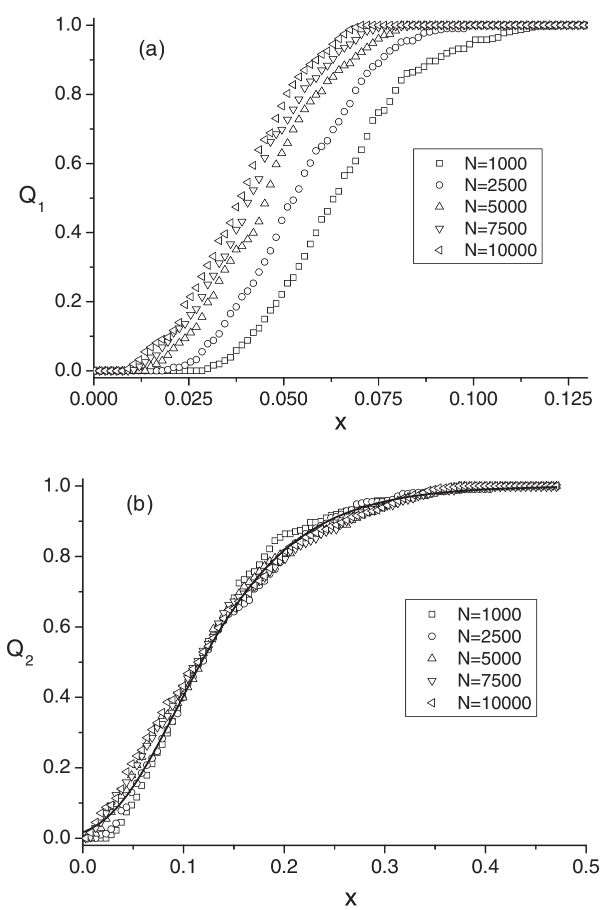

FIG. 3. (Top panel) Q_1 is the probability that R, the modulus of the largest eigenvalue, satisfies $R^2 - \mu < x$. (Lower panel) Q_2 is the probability that $\widetilde{\eta}^2(R^2 - \mu)^2 < x$. Results are for the open kicked rotator at several dimensions, for $\mu = 0.8$. Solid line is a rescaled RMT prediction.

we introduce the scaled variable $\widetilde{\eta}^2(R^2-\mu)^2$, where $\widetilde{\eta}$ is given by (7). As a result, all curves fall on top of each other, indicating that this is the correct scaling. The results agree very well with the function $\exp\{-e^{-ax+b}\}$, with fit values of a and b. Notice, however, that $\widetilde{\eta}^2(R^2-\mu)^2$ is always positive while $\exp\{-e^{-ax+b}\}$ is finite for negative arguments, so the agreement cannot be too good near the origin. Interestingly, we must not introduce the $\ln\widetilde{\eta}$ or $\ln\ln\widetilde{\eta}$ factors that appear in (13), as they would spoil the scaling. Why these factors are present in RMT but not in our dynamical model is not clear at present.

IV. CONCLUSIONS AND PERSPECTIVES

We close with some remarks on resonance eigenfunctions. These may be depicted in phase space by means of their Husimi function, $H_\psi(q,p)=|\langle q,p|\psi\rangle|^2$, where $|q,p\rangle$ is a coherent state. It was shown in [18] that these Husimi functions are supported on the backward trapped set, the unstable manifold of the repeller. How they are distributed on this support depends on their decay rate: states with larger Γ concentrate on the dynamical pre-images of the opening, while states with small Γ concentrate on the repeller. Semiclassically,

$$\int_{R_m} H_{\psi_n}(q,p)dqdp \approx |z_n|^{2m}(1-|z_n|^2), \tag{15}$$

where R_m is the mth pre-image of the opening. In principle, this relation would allow a state whose decay rate equals the classical decay rate to be uniformly distributed, because the area of R_m decays like $e^{-m\gamma}$. Therefore, supersharp resonances must show an increased degree of localization above uniformity. Indeed, since they can also be seen as superlong-lived states, one would expect them to be associated with periodic orbits (see for example [27–30]). This topic deserves further investigations.

In summary, we have introduced the concept of supersharp resonances in chaotic wave scattering as those with decay rates smaller than the classical escape rate. We have presented numerical evidence that these resonances do not follow the usual fractal Weyl law. We have numerically tested on a dynamical model a RMT prediction for the density of supersharp resonances. Finally, we have derived and tested on a dynamical model a RMT prediction of probability distribution of the sharpest resonance. The generality of the results mean that they may be experimentally addressed in a variety of settings. One important open question is the phase space morphology of the associated wave functions. Another line of research to be followed would be to investigate a possible relation between the exponents α and β to ray dynamics. In particular, it is not clear whether they depend on the Lyapounov exponent and whether they are universal or system specific.

ACKNOWLEDGMENTS

This work was supported by CNPq and FAPESP.

[1] *Wave Scattering in Complex Media: From Theory to Applications*, edited by B. A. van Tiggelen and S. E. Skipetrov (Springer, Berlin, 2003).
[2] *New Directions in Linear Acoustics and Vibration: Quantum Chaos, Random Matrix Theory and Complexity*, edited by M. Wright and R. Weaver (Cambridge University Press, Cambridge, 2010).
[3] Y. V. Fyodorov, T. Kottos, and H.-J. Stöckmann (eds.), J. Phys. A (Special edition) **38**(49), (2005).
[4] M. Fridman, R. Pugatch, M. Nixon, A. A. Friesem, and N. Davidson, e-print arXiv:1012.1282.
[5] C. Yan *et al.*, Appl. Phys. Lett. **94**, 251101 (2009).
[6] B. Dietz, T. Friedrich, H. L. Harney, M. Miski-Oglu, A. Richter, F. Schafer, and H. A. Weidenmuller, Phys. Rev. Lett. **98**, 074103 (2007).
[7] A. E. Miroshnichenko, S. Flach, and Y. S. Kivshar, Rev. Mod. Phys. **82**, 2257 (2010).
[8] C. Stampfer, J. Guttinger, S. Hellmuller, F. Molitor, K. Ensslin, and T. Ihn, Phys. Rev. Lett. **102**, 056403 (2009).
[9] P. Gaspard and S. A. Rice, J. Chem. Phys. **90**, 2242 (1989).
[10] S. Nonnenmacher and M. Zworski, Acta Math. **203**, 149 (2009).
[11] W. T. Lu, S. Sridhar, and M. Zworski, Phys. Rev. Lett. **91**, 154101 (2003).
[12] P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press, Cambridge, 2005).
[13] Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. **38**, 1918 (1997); JETP Lett. **72**, 422 (2000).
[14] B. A. Khoruzhenko and H.-J. Sommers, e-print arXiv:0911.5645.
[15] S. Rahav and P. W. Brouwer, Phys. Rev. B **73**, 035324 (2006).
[16] A. J. Scott and C. M. Caves, J. Phys. A **36**, 9553 (2003).
[17] S. Nonnenmacher and M. Zworski, Commun. Math. Phys. **269**, 311 (2007); D. L. Shepelyansky, Phys. Rev. E **77**, 015202(R) (2008).
[18] J. P. Keating, M. Novaes, S. D. Prado, and M. Sieber, Phys. Rev. Lett. **97**, 150406 (2006).
[19] G. Casati, G. Maspero, and D. L. Shepelyansky, Physica D **131**, 311 (1999).
[20] Ph. Jacquod, H. Schomerus, and C. W. J. Beenakker, Phys. Rev. Lett. **90**, 207004 (2003).
[21] H. Schomerus and J. Tworzydlo, Phys. Rev. Lett. **93**, 154102 (2004).
[22] K. Zyczkowski and H.-J. Sommers, J. Phys. A **33**, 2045 (2000).
[23] C. A. Tracy and H. Widom, *Proceedings of the ICM, Beijing* (2002), Vol. 1, pp. 587–596.
[24] S. N. Majumdar, O. Bohigas, and A. Lakshminarayan, J. Stat. Phys. **131**, 33 (2008).
[25] B. Rider, J. Phys. A **36**, 3401 (2003).
[26] C. A. Tracy and H. Widom, Commun. Math. Phys. **159**, 151 (1994).
[27] F. Borgonovi, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. A **43**, 4517 (1991).
[28] D. Wisniacki and G. G. Carlo, Phys. Rev. E **77**, 045201(R) (2008).
[29] M. Novaes, J. M. Pedrosa, D. Wisniacki, G. G. Carlo, and J. P. Keating, Phys. Rev. E **80**, 035202(R) (2009).
[30] J. M. Pedrosa, D. Wisniacki, G. G. Carlo, and M. Novaes (accepted by Phys. Rev. E).