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How to obtain extreme multistability in coupled dynamical systems

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We present a method for designing an appropriate coupling scheme for two dynamical systems in order to realize extreme multistability. We achieve the coexistence of infinitely many attractors for a given set of parameters by using the concept of partial synchronization based on Lyapunov function stability. We show that the method is very general and allows a great flexibility in choosing the coupling. Furthermore, we demonstrate its applicability in different models, such as the Rössler system and a chemical oscillator. Finally we show that extreme multistability is robust with respect to parameter mismatch and, hence, a very general phenomenon in coupled systems.

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The study of the coexistence of a large number of asymptotic stable states in a dynamical system for a given set of parameters and their control is an important topic of research in nonlinear science. This phenomenon, called multistability, can be observed in a large variety of systems in many areas of science [1], namely, nonlinear optics [2], population dynamics [3], neuroscience [4], climate dynamics [5], condensed-matter physics [6], laser physics [7], and electronic oscillators [8]. Three classes of systems have been studied where multistability is commonly found: weakly dissipative systems [9,10], systems involving a delay [11], and coupled systems [12]. A common property of these systems is their large sensitivity to initial conditions because of the complex structure of the basins of attraction of the different attractors [13]. Besides extended theoretical studies on the coexistence of a multitude of attractors and its consequences for the dynamics of the system, experimental observations have revealed multistability in optical systems [2] in which the intrinsic delay plays the key role.

On the other hand, an extreme kind of multistability, where the number of coexisting attractors is infinite, has been reported in a coupled system by Sun *et al*. [14]. In that system a particular choice of the coupling plays the key role. Their work [14] was inspired by experimental observations of chemical reactions in which the outcome is uncertain despite the special care taken to ensure always the same experimental conditions [15]. This type of multistability has been explored in coupled systems [14,16] where the temporal evolution of the coupled system has a strong dependence on initial conditions while the uncoupled system does not show any multistability. The appearance of infinitely many attractors in those coupled systems is related to generalized synchronization [17]. Moreover, it is important to note that the coupling needed to obtain this extreme kind of multistability is rather unusual and in any case nonlinear. Recently it has been shown that the reason for the emergence of infinitely many attractors lies in the appearance of a conserved quantity in the long-term limit [18]. In particular cases, this conserved quantity can be considered as a unique emergent parameter which governs the dynamics of the system in the long-term limit and, hence, allows for a reduction of the dimension of the system. Furthermore, it was conjectured that at least two variables need to be coupled to obtain infinitely many attractors. However, in these studies of extreme multistability in coupled systems, it was not possible to find a systematic definition of the coupling type. In the example systems, such as the Lorenz system or the coupled chemical oscillators, the coupling was particularly chosen to create an infinity of attractors. The question arises if there is a general principle of designing the coupling, leading to infinitely many attractors in coupled systems. In this Rapid Communication, we address this issue of finding such a general principle of defining the coupling for extreme multistability. The coupling is defined in a systematic way by using the principle of partial synchronization based on the Lyapunov function stability [19]. The method is very general in its applicability to any dynamical system and allows many alternative design options. As a result, it provides flexibility in the physical realization of extreme multistability in dynamical systems. This phenomenon is also found to be robust with respect to parameter mismatch.

To demonstrate the general applicability of our method we use two examples: the Rössler oscillator as well as a chemical autocatalator model [18]. Before outlining the design principle of the coupling we briefly recall the emergence of extreme multistability in the first studied example, the coupled Lorenz systems [14]. Two Lorenz systems described by x_i and y_i $(i = 1,2,3)$ are coupled in a highly nonlinear way. They exhibit infinitely many coexisting attractors, but do not possess multistability when uncoupled. In the coupled state the systems show partial synchronization in the sense that they are completely synchronized in two of the variables, while the third pair of the variables keep a certain distance one from another. This property can be expressed in terms of error variables $e_i = x_i - y_i, i = 1,2,3$, measuring the distance from the synchronization manifold of complete synchronization. The error dynamics of the coupled Lorenz system was given there by $\dot{e}_1 = \sigma e_2$, $\dot{e}_2 = -e_2 - x_1 e_3$, and $\dot{e}_3 = x_1 e_2 - b e_3$. It was shown that this error dynamics possesses a fixed point $(e_1^*, e_2^*, e_3^*) = (e_1^*, 0, 0)$, where e_1^* obeys a constant value K which depends on the initial condition. The stability of this fixed point can be derived using a Lyapunov function $V = e_2^2 + e_3^2$ [14]. The constant K turns out to be a conserved quantity according to the analysis in Ref. [18] and can take any real value from $-\infty$ to ∞. To each of these values belongs

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one attractor which resides in a synchronization manifold characterized by a particular value of the conserved quantity K. Consequently, the whole state space is sliced into infinitely many synchronization manifolds, each of them given by the value of K. This resembles the situation of conservative systems where for each value of the conserved quantity another marginal stable dynamics is given. However, it is important to note that the conserved quantity in the case of extreme multistability is not necessarily given from the beginning but evolves temporally and emerges in the long-term limit only. Based on the observations above we formulate one precondition for the emergence of extreme multistability: *The coexistence of infinitely many attractors in an m-dimensional coupled system will be possible if* $m-1$ *of the variables of the two systems are completely synchronized and one of them obeys a constant difference between them.* Therefore we develop a method that makes the $m-1$ variables of two coupled systems completely synchronize. In order to achieve this goal we employ the stability theory using a Lyapunov function. Let an m-dimensional system be governed by the equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{F}(\mathbf{x})$ is the dynamics of the system; $\mathbf{F}: \mathbf{R^m} \to \mathbf{R^m}$ and $\mathbf{x} = [x_i; i = 1,2,\ldots,m]$. Consider now the same dynamical system but with different initial conditions and denote its dynamical variables by y so that $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$, $\mathbf{y} = [y_i; i = 1,2,\ldots,m]$. When these two systems are coupled, they can synchronize completely under certain conditions [19]. This synchronized dynamics takes place on a synchronization manifold which is defined by $\mathbf{x} = \mathbf{y}$ and the deviation from the synchronization manifold is described by the error $\mathbf{e} = \mathbf{x} - \mathbf{y}$. The time evolution of these deviations is given by the error dynamics $\dot{\mathbf{e}} = \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})$, $\mathbf{G}: \mathbf{R^m} \to \mathbf{R^m}$. As stated above, one way to achieve extreme multistability is that $m-1$ of the dynamical variables synchronize completely while one of them keeps a certain distance K. To fulfill these conditions the error dynamics has to obey a specific form, which we denote by $\tilde{\mathbf{G}}(\mathbf{x},\mathbf{y})$. This desired error dynamics is the basis for designing specific controllers $\mathbf{u_1}(\mathbf{x},\mathbf{y})$ and $\mathbf{u_2}(\mathbf{x},\mathbf{y})$ for coupling two dynamical systems in such a way that extreme multistability originates. Now the desired governing equation $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{y}} = \tilde{\mathbf{G}}(\mathbf{x},\mathbf{y})$ will imply that $\tilde{\mathbf{G}}(\mathbf{x},\mathbf{y}) - \mathbf{F}(\mathbf{x}) + \mathbf{F}(\mathbf{y}) = \mathbf{u_1}(\mathbf{x},\mathbf{y}) - \mathbf{u_2}(\mathbf{x},\mathbf{y})$. This last equation enables us to find the proper construction of the controllers $\mathbf{u_{1,2}}$ which yield the appropriate coupling terms of the two systems, and ensures that they exhibit extreme multistability.

To illustrate the construction of coupled systems possessing infinitely many attractors, we start with the Rössler system as a generic example. The parameters of the uncoupled Rössler system are chosen for a chaotic regime. A coupling scheme based on Lyapunov function stability yielding complete synchronization between identical Rössler systems was described earlier [19]. By contrast, we attempt to realize partial synchronization using a bidirectional coupling that will induce in the coupled system a different property: the coexistence of infinitely many attractors dependent on the initial conditions.

Consider two Rössler systems coupled through bidirectional controllers u_{1i} and u_{2i}, $i = 1,2,3$:

$$\dot{x}_1 = -x_2 - x_3 + u_{11}, \tag{1a}$$

$$\dot{x}_2 = x_1 + a x_2 + u_{12}, \tag{1b}$$

$$\dot{x}_3 = b + x_3(x_1 - c) + u_{13}, \tag{1c}$$

$$\dot{y}_1 = -y_2 - y_3 + u_{21}, \tag{1d}$$

$$\dot{y}_2 = y_1 + a y_2 + u_{22}, \tag{1e}$$

$$\dot{y}_3 = b + y_3(y_1 - c) + u_{23}. \tag{1f}$$

The dynamics of the synchronization errors $\dot{\mathbf{e}} = \mathbf{G}(\mathbf{x},\mathbf{y})$ are derived as $\dot{e_1} = -e_2 - e_3$, $\dot{e_2} = e_1 + a e_2$, and $\dot{e_3} = x_3(x_1 - c) - y_3(y_1 - c)$. To realize infinitely many coexisting attractors, we have to fulfill a goal error dynamics $\dot{\mathbf{e}} = \tilde{\mathbf{G}}(\mathbf{x},\mathbf{y})$. As a simplest goal error dynamics $\tilde{\mathbf{G}}$ we can choose $\dot{e_1} = -e_1$, $\dot{e_2} = e_1$, and $\dot{e_3} = -c e_3$. This error dynamics possesses a fixed point $(e_1^*,e_2^*,e_3^*) = (0,e_2^*,0)$, where e_2^* can take any real value from $-\infty$ to ∞ depending on the initial conditions. As we approach an attractor, we find a constant $K = e_2^*$ which characterizes a synchronization manifold and which can be considered as the conserved quantity emerging in the long-term limit. The corresponding Lyapunov function $V = e_1^2 + e_3^2 \geqslant 0$ obeys the dynamics $\dot{V} = 2e_1\dot{e}_1 + 2e_3\dot{e}_3 = -2(e_1^2 + c e_3^2) < 0$ to make this fixed point stable. To achieve this desired error dynamics, the components of the controllers are to be selected as $u_{11} - u_{21} = e_2 + e_3 - e_1$, $u_{12} - u_{22} = -a e_2$, and $u_{13} - u_{23} = -x_1 x_3 + y_1 y_3$. We can easily choose $u_{11} = 0$, $u_{22} = 0$, and $u_{23} = 0$, and thereby three equations [Eqs. (1a), (1e), and (1f)] remain unchanged, whereas the other three will be changed by the additive controllers u_{21}, u_{12}, and u_{13}. The modified coupled Rössler system becomes $\dot{x}_1 = -x_2 - x_3$, $\dot{x}_2 = x_1 + a y_2$, $\dot{x}_3 = b - c x_3 + y_1 y_3$, $\dot{y}_1 = x_1 - y_1 - x_2 - x_3$, $\dot{y}_2 = y_1 + a y_2$, and $\dot{y}_3 = b + y_3(y_1 - c)$.

To demonstrate the coexistence of infinitely many attractors we fix the parameters of the two uncoupled Rössler systems to $a = 0.2$, $b = 0.2$, and $c = 5.7$, and vary a single initial condition y_{01} and determine the final state of the coupled system by simulation. All the other initial conditions except y_{01} are fixed. Figure 1 shows the coexistence of different types of attractors in state space, such as period-3, period-4, period-8, period 5, and chaos, which appear for different values of y_{01}, and this list is not exhaustive. Extreme multistability under the variation of initial conditions is further illustrated in Figs. 2(a) and 2(b). While Fig. 2(a) plots the largest Lyapunov exponent in color coding, Fig. 2(b) depicts the corresponding attractors along the black line drawn in Fig. 2(a). We find several periodic and chaotic attractors for different initial conditions and the whole picture resembles a bifurcation diagram, though we emphasize that the abscissa is not a control parameter but an initial condition of the coupled

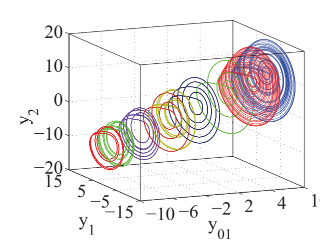

FIG. 1. (Color online) Multiple coexisting attractors for different initial conditions: period 3 ($y_{01} = -2.04$), period-4 ($y_{01} = 4.6$), period-8 ($y_{01} = -7.9$), and chaos ($y_{01} = 7.6$). Many other possible dynamics are also shown. Other initial conditions are fixed ($x_{01} = -0.1$, $x_{02} = 0.01$, $x_{03} = 0.3$, $y_{02} = 0.2$, $y_{03} = 2.0$).

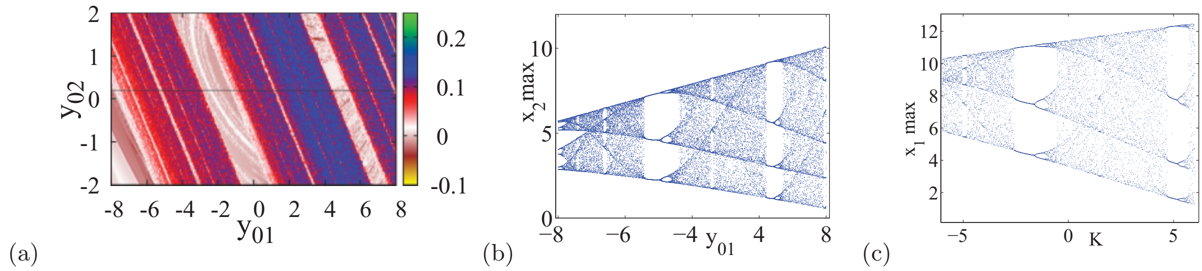

FIG. 2. (Color online) (a) Largest Lyapunov exponent λ_{max} of the coexisting states of the coupled Rössler system for different y_{01} and y_{02}. Figure 1 shows details of initial conditions. (b) Maxima of x_2 as a function of initial condition y_{01}. (c) Maxima of x_1 plotted against "parameter" K of the reduced system (5).

system. The emergence of a conserved quantity K in the long-term limit offers the opportunity to reduce the dimension of the system under investigation. As mentioned above, two of the variables of the coupled system synchronize completely while the systems keep a constant distance from each other in the third variable when an attractor is approached. One can use this fact to describe the long-term dynamics of the $2m$-dimensional coupled system by an m-dimensional one having an additional parameter $K = e_2^*$ which encodes the synchronization manifold (cf. Ref. [18]). The corresponding reduced system for our example reads $\dot{x}_1 = -x_2 - x_3$, $\dot{x}_2 = x_1 + a(x_2 + K)$, and $\dot{x}_3 = b + x_3(x_1 - c)$. This reduced system is only valid as $t \to \infty$. Figure 2(c) shows the qualitative behavior of the reduced system as the parameter K is varied. Now it is indeed a bifurcation diagram since the conserved quantity K, which emerges in the full system in the long-term limit, enters the reduced system as a parameter, and a closer look shows close agreement with Fig. 2(b).

We emphasize here that the choice of the controllers is not unique but rather flexible. We can always set m components of the $2m$ controllers to zero (except for setting $u_{1i} : i = 1,2,3$ or $u_{2j} : j = 1,2,3$ to zero at the same time), while the other m elements have to be nonzero. Moreover, the choice of the Lyapunov function is also not unique and several other such functions are possible to consider. This indicates that the method introduced here is rather general and offers a large flexibility in designing an appropriate coupling.

Next we present the example of the autocatalator model which is a three-variable extension [20] of the two-variable autocatalator model [21] that is widely used to describe complex chemical oscillations and chaos. A precursor chemical substance A is transformed into a product B via three different intermediates. While the precursor chemical substance is held fixed, the intermediates follow the equations $\dot{x}_1 = \mu(\kappa + x_3) - x_1(1 + x_2^2)$, $\sigma\dot{x}_2 = x_1(1 + x_2^2) - x_2$, and $\delta\dot{x}_3 = x_2 - x_3$, where $\sigma = 5 \times 10^{-3}$, $\delta = 2 \times 10^{-2}$, and $\kappa = 65$ are reaction constants and μ is related to the amount of precursor A and serves as a bifurcation parameter. For $\mu = 0.145$ the single autocatalator exhibits periodic motion. It was shown in Ref. [18] that two bidirectionally coupled autocatalators exhibit extreme multistability. The error dynamics was given there by $\dot{e}_1 = \mu e_3 - (1 + x_2^2)e_1$, $\sigma\dot{e}_2 = (1 + x_2^2)e_1$, and $\delta\dot{e}_3 = -e_3$, and depends on the state variables in a complex manner. The corresponding Lyapunov function $V(e_1,e_2,e_3) = \frac{(1+x_2^2)\delta+1}{2(\delta\mu)^2}e_1^2 + \frac{1}{\delta\mu}e_1e_3 + e_3^2$ was also rather complicated. Here we show that one can choose a much simpler Lyapunov function, leading again to extreme multistability. We require the error dynamics $\tilde{\mathbf{G}}$ to be $\dot{e}_1 = \mu e_3$, $\sigma\dot{e}_2 = -e_2$, and $\delta\dot{e}_3 = -e_3$. The fixed point corresponding to a synchronization manifold is then given by (e_1^*,0,0) and proven to be stable using the simple Lyapunov function $V = e_2^2 + e_3^2$. Following a similar procedure as above and choosing the vanishing controller components as $u_{11} = 0$, $u_{22} = 0$, and $u_{13} = 0$, we arrive at the following coupled autocatalator system:

$$\dot{x}_1 = \mu(k + x_3) - x_1\left(1 + x_2^2\right),$$
$$\sigma\dot{x}_2 = -x_2 + y_1\left(1 + y_2^2\right), \quad \delta\dot{x}_3 = x_2 - x_3,$$
$$\dot{y}_1 = \mu(k + y_3) - x_1\left(1 + x_2^2\right),$$
$$\sigma\dot{y}_2 = -y_2 + y_1\left(1 + y_2^2\right), \quad \delta\dot{y}_3 = x_2 - y_3.$$

The corresponding dynamics depending on the initial conditions is depicted in Fig. 3(a), together with the three largest Lyapunov exponents in Fig. 3(b). They clearly reveal the existence of multiple attractors.

In the real world, no two oscillators can be identical. Hence we check the robustness of the extreme multistability in the presence of parameter mismatch in two coupled systems. This is important for the physical realization of the phenomenon. The Lyapunov-function-based controller design has an inherent property of nullifying the effect of parameter mismatch [19] and it thereby provides a robustness of the extreme multistability to parameter mismatch in the coupled systems. To reveal this property, we take two mismatched Rössler oscillators and redefine the error dynamics by $\dot{e}_1 = -e_1$, $\dot{e}_3 = -ce_3$, and $\dot{e}_2 = (a_2 - a_1)e_1$, where $(a_2 - a_1) = \delta_a$ is the detuning parameter of the coupled system. The Lyapunov function $V = e_1^2 + e_3^2$ indicates that e_3 and e_1 tend to zero for $t \to \infty$,

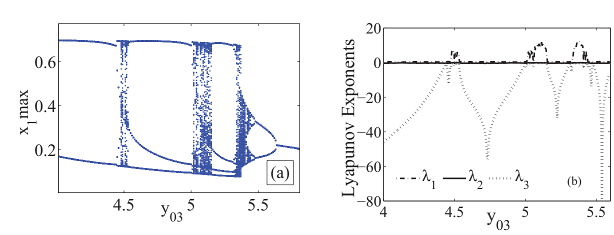

FIG. 3. (Color online) (a) Maxima of x_1 of coupled autocatalator as a function of y_{03}. Initial conditions: $x_{01} = 0.01$, $x_{02} = 0.1$, $x_{03} = 0.1$, $y_{01} = 0.0$, and $y_{02} = 0.0$. (b) Three largest Lyapunov exponents plotted with y_{03}.

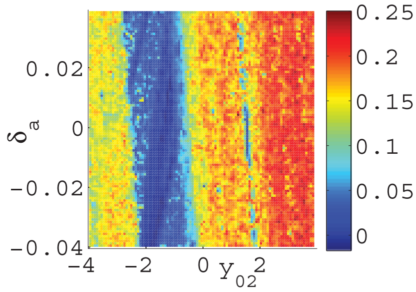

FIG. 4. (Color online) Largest Lyapunov exponent $\lambda_{\max}$ for $\pm 20\%$ detuning of parameter a. The initial condition y_{02} is varied from -4 to 4 when all others are fixed: $x_{01} = -0.1$, $x_{02} = 0.01$, $x_{03} = 0.3$, $y_{01} = -7.6$, $y_{03} = 2.0$, and $a = 0.2$. $\delta_a = 0$ represents identical oscillators. Color codes correspond to different $\lambda_{\max}$ values.

while e_2 becomes constant (K) for any real values of δ_a. Keeping a_2 fixed at 0.2 and varying a_1 from 0.16 to 0.24 ($\pm 20\%$ detuning) extreme multistability is still maintained, as shown in Fig. 4. The same applies for the detuning in other parameters.

In summary, we proposed a systematic method of designing the coupling for two m-dimensional dynamical systems governed by the same dynamics to exhibit infinitely many coexisting attractors when the isolated systems do not possess multistability. The coupling establishes partial synchronization in the two systems, leading to a conserved quantity that characterizes a synchronization manifold. The value of the conserved quantity is determined by the initial condition. This coupling scheme is very general since many different choices of Lyapunov functions are possible to consider and it incorporates a great flexibility in the design so that one can always find an optimal coupling for the physical realization of extreme multistability. Robustness of the extreme multistability with respect to parameter mismatch is also ensured. We successfully tested the coupling for extreme multistability in few systems, especially a magnetized plasma model to be reported in the future.

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