Stochastic theory of quantum vortex on a sphere

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A stochastic theory is presented for a quantum vortex in superfluid films coated on a two-dimensional sphere S^2. The starting point is the canonical equation of motion (Kirchhoff equation) for a point vortex, which is derived using the time-dependent Landau-Ginzburg theory. The vortex equation, which is equivalent to the spin equation, turns out to be the Langevin equation in presence of random forces. This is converted to the Fokker-Planck (FP) equation for the distribution function of a point vortex by using a functional integral technique. The FP equation is analyzed with special emphasis on the role of the pinning potential. By considering a typical form of the pinning potential, we address two problems: (i) The one is concerning an interplay between strength of the pinning potential and effective temperature, which discriminates the weak and strong coupling scheme to determine the solutions of the FP equation. (ii) The other is concerning a small diffusion limit, for which an asymptotic analysis is given using the functional integral to lead a compact expression of the distribution function. An extension to the vortex in nonspherical geometry is briefly discussed for the case of vortex on a plane and a pseudosphere.

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I. INTRODUCTION

The study of vortex has been one of central subjects in classical fluid dynamics [1]. The idea of the classical vortex has been transcribed to the investigation of the modern quantum vortex, which covers the various fields: statistical mechanics of a point vortex in superfluid He4 [2–5], a vortex string, [6], the textural structure in He3 [7], and even the vortex in cosmological scale (e.g., [8]).

In conventional treatments so far developed [1–8], the vortex is considered to be an object defined on Euclidean flat space of two and/or three dimensions. However, it is possible that vortices occur in superfluids coated on curved spaces [9]. A simple and important case is the vortex on a two-dimensional sphere, e.g., [10–12], which has been studied in several contexts by starting with the purely classical hydrodynamical procedure. Moreover, it is remarkable that the quantum vortex on a sphere was realized by superfluid adsorbed in porous materials [13] with the aim at elucidating the Kosteritz-Thouless type transition in nonplaner geometry. Being inspired by these previous works, we address a specific aspect of the quantum vortex on a sphere (see below).

On the other hand, apart from usual dynamical and statistical mechanical aspects of a quantum vortex, it is highly probable that the vortex motion can be influenced by randomness caused by, e.g., temperature fluctuation and interaction with various sorts of impurities. For such cases, we need to treat the vortex motion within a framework of stochastic approach. Indeed, the stochastic theory of a vortex has been investigated in connection with the superconductivity, specifically, the high T_c superconductivity [14–16], in which the fluctuations play an important role as a cooperative effect with vortex *pinning*.

Having mind of the role of a random effect on the vortex motion, we here present a stochastic theory of a quantum vortex on a two-dimensional sphere S^2 and its variants. This attempt is expected to provide with a modest step toward a stochastic aspect of a quantum vortex on the curved space which has not been well recognized so far.

In order to develop the stochastic theory, the first step is to settle the canonical (Kirchhoff) equation of motion for a single vortex. Here we suppose a special circumstance so as to realize a single vortex configuration on the sphere (see the argument in Sec. II). The equation of motion of the center of a single vortex is derived by using the time-dependent Landau-Ginzburg (LG) theory [14,17] that is described by a complex order parameter for the superfluids. It is crucial that the canonical equation of motion has a close resemblance with the spin equation of motion, that is, the vortex charge quantization plays a role of the quantized spin magnitude.

Following the analogy with spin, the Langevin equation is introduced for the Brownian motion of a vortex under the random force arising from temperature fluctuations together with the effect of *dissipation*. The procedure is similar to the theory of single magnetic "domain" [18]. By adopting the assumption of a Gaussian white noise for the random fluctuations, the Langevin equation can be converted to the Fokker-Planck (FP) equation for the probability distribution of a point vortex. This can be most efficiently achieved by using the functional integral developed for the statistical theory of wave propagation in random media [19,20].

The FP equation thus derived enables us to analyze the stochastic behavior of a quantum vortex in the presence of a pinning potential [15,16]. Our concern is to explore the role of pinning in connection with random characteristics: the diffusion and dissipation. In order to get insight into this mechanism, it is efficient to consider a special and typical configuration for the vortex and pinning centers (for the details, see Sec. IV).

Besides the stochastic theory of a sphere vortex, the problem is raised as to how this can be extended beyond the spherical geometry, that is, an extension to a quantum vortex adsorbed on a nonspherical geometry. To examine this is another subject of the paper.

The paper is organized as follows: In the next section we derive the equation of motion for a single vortex on sphere starting with the LG equation. In particular the pinning potential is given in terms of the order parameter. Section III

deals with the Langevin and Fokker-Planck equations. In Sec. IV we analyze the FP equation with emphasis on the role of a pinning potential. In Sec. V we consider an extension to the case of nonspherical geometry. The last section is a summary.

II. EQUATION OF MOTION FOR THE QUANTUM VORTEX

We start with the time-dependent Landau-Ginzburg Lagrangian, that is written in terms of the complex order parameter ψ [14,17]:

$$L = \int \left[\frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \text{c.c.} \right) - \mathcal{H}(\psi, \psi^*) \right] d\sigma. \tag{1}$$

Here $\psi(\mathbf{r},t)$ is defined on a two-sphere S^2 with radius $|\mathbf{r}| = a$ and the integral $\int d\sigma$ is taken over S^2. The coordinate $\mathbf{r}$ is written in terms of the Cartesian form. $\mathcal{H}$ represents the Hamiltonian density which consists of the kinetic and potential energy:

$$\mathcal{H} = \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi + V(\psi^*, \psi), \tag{2}$$

where ∇ means the derivative with respect to the spherical coordinate and m means the mass of constituent particles (e.g., He4 atoms).

Now we consider how to incorporate the vortex configuration into the order parameter. In order to realize this, one should take into account a peculiar feature of the vortex on a sphere, that is, due to the spherical topology, there is no isolated single vortex on a two-dimensional sphere, namely, at least we have one pair of vortices with opposite charges to each other [13]. In this point, it may be possible to suppose the situation such that the one vortex is *strongly trapped* by a pinning center that is "virtually built," so it behaves as if it were inert. Consequently, the remaining vortex is active and behaves as it it were freely moving on the sphere. As a result of this dynamical mechanism, we can consider the single vortex configuration effectively.

Having settled the single vortex configuration, we can write the order parameter in a form such that the vortex center is incorporated:

$$\psi(\mathbf{r},t) = \psi(\mathbf{r} - \mathbf{R}(t)). \tag{3}$$

Here $\mathbf{R}(t)$ denotes the time-dependent vortex center satisfying $|\mathbf{R}| = a$. By this parametric form, the first term of the Lagrangian is calculated as

$$\frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \text{c.c.} \right) = -\frac{i\hbar}{2}(\psi^* \nabla_r \psi - \text{c.c.}) \frac{d\mathbf{R}}{dt}, \tag{4}$$

where use is made of the relation

$$\frac{\partial \psi}{\partial t} = \frac{d\mathbf{R}}{dt} \nabla_R \psi$$

together with the relation $\nabla_R = -\nabla_r$ (namely, the nabla with respect to the vortex center coordinate turns out to be the derivative with respect to field argument $\mathbf{r}$). Now we adopt the polar form of the order parameter:

$$\psi = \sqrt{\rho} \exp\left[i \frac{m}{\hbar} \alpha(\mathbf{r} - \mathbf{R}(t)) \right]. \tag{5}$$

Here $\rho = \rho(\mathbf{r} - \mathbf{R}(t))$ is the density, which vanishes at the vortex center $\mathbf{r} = \mathbf{R}$ and tends to a constant value: $\rho = \rho_0$ outside the coherent length. The first term of L (denoted by L_c that is called the canonical term) is calculated as

$$L_c = \int \mathbf{j} \cdot \frac{d\mathbf{R}}{dt} d\sigma \equiv \int (j_x \dot{X} + j_y \dot{Y} + j_z \dot{Z}) d\sigma, \tag{6}$$

where the *mass current* $\mathbf{j} = m\rho \nabla \alpha = m\rho \mathbf{v}$ is introduced. On the other hand, the Hamiltonian term becomes

$$H = \int \left[\frac{1}{2} m\rho \mathbf{v}^2 + \frac{1}{4\rho} (\nabla \rho)^2 + V(\rho) \right] d\sigma. \tag{7}$$

The first term means the fluid kinetic energy, and the second and third terms are the internal energy, which is quoted as $U(\rho)$. The explicit form will be given later.

The equation of motion for the vortex center is derived from the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}} \right) - \frac{\partial L}{\partial \mathbf{R}} = 0.$$

The contribution from the Hamiltonian is

$$\frac{\partial L}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}} \int \mathcal{H} d\sigma = \frac{\partial H}{\partial \mathbf{R}}.$$

On the other hand, the canonical term is manipulated as follows: By differentiating under the integral symbol, we have

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{X}} \right) = \int \frac{dj_x}{dt} d\sigma$$

together with $\frac{dj_x}{dt} = \frac{\partial j_x}{\partial X}\dot{X} + \frac{\partial j_x}{\partial Y}\dot{Y} + \frac{\partial j_x}{\partial Z}\dot{Z}$ and

$$\frac{\partial L_c}{\partial X} = \int \left(\frac{\partial j_x}{\partial X}\dot{X} + \frac{\partial j_y}{\partial X}\dot{Y} + \frac{\partial j_z}{\partial X}\dot{Z} \right) d\sigma.$$

Hence we obtain

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{\mathbf{R}}} \right) - \frac{\partial L_c}{\partial \mathbf{R}} = \int \{(\nabla \times \mathbf{j}) \times \dot{\mathbf{R}}\} d\sigma, \tag{8}$$

where use is made of $\frac{\partial \mathbf{j}}{\partial X} = -\frac{\partial \mathbf{j}}{\partial x}$. By evaluating the right hand side of (8) (see Appendix A), we arrive at the Kirchhoff equation of motion for the vortex center which is written by the unit vector $\hat{\mathbf{R}} = \mathbf{R}/a [= (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)]$:

$$\mu\Omega\left(\hat{\mathbf{R}} \times \frac{d\hat{\mathbf{R}}}{dt} \right) = \frac{\partial H}{\partial \hat{\mathbf{R}}} \tag{9}$$

Here $\Omega = m\rho_0 a^2$ and H is replaced by H/a. μ means the "vortex charge," which is introduced as a multiple factor in the velocity field (Appendix A). Equation (9) coincides with the equation of motion for the vortex on a sphere [11,12], which was derived in the framework of classical fluid dynamics. This equation has a simple dynamical meaning: the balance of two types of forces; the left hand side represents the Magnus force, and the right hand side is the gradient force coming from the Hamiltonian (that plays a role of potential energy in analogy with classical particle mechanics). The equation of motion can be written in an alternative form:

$$\mu\Omega \frac{d\hat{\mathbf{R}}}{dt} = -\hat{\mathbf{R}} \times \frac{\partial H}{\partial \hat{\mathbf{R}}}, \tag{10}$$

which can be rewritten in a form of a Hamiltonian equation of motion; namely, using the spherical vector basis

$$\frac{d\hat{\mathbf{R}}}{dt} = (0, \dot{\Theta}, \sin\Theta\dot{\Phi}),$$

the equation of motion is written in terms of the angular form:

$$\mu\Omega\dot{\Theta} = \frac{1}{\sin\Theta}\frac{\partial H}{\partial\Phi}, \quad \mu\Omega\dot{\Phi} = -\frac{1}{\sin\Theta}\frac{\partial H}{\partial\Theta}. \tag{11}$$

This is of the same form as the equation of motion for spin (see, e.g., [21,22]). The effective action corresponding to (11) is given as

$$S = \int [\mu\Omega(1+\cos\Theta)\dot{\Phi} - H]dt. \tag{12}$$

Quantization of vortex charge: We examine a special feature of the above form of action function, namely, the term of a line integral. As a particular case, we consider the integral along a closed loop, say C. By using the Stokes theorem, this can be written in a form of a surface integral, for which we have an ambiguity to choose the surfaces,

$$\int_C \mu\Omega(1+\cos\Theta)d\Phi = \begin{cases} -\mu\Omega\int_{\mathrm{S}}\sin\Theta d\Theta d\Phi \\ \mu\Omega\int_{\hat{\mathrm{S}}}\sin\Theta d\Theta d\Phi. \end{cases} \tag{13}$$

Here S and $\hat{\mathrm{S}}$ are the upper and lower surfaces, respectively, which are complements to each other; $\mathrm{S}+\hat{\mathrm{S}} = \mathrm{S}^2$. This ambiguity is related to the gauge choice [23]. Upon quantization, the ambiguity is removed by a relation

$$\exp\left[-\frac{i}{\hbar}\mu\Omega\int_{\mathrm{S}}\sin\Theta d\Theta d\Phi\right] = \exp\left[+\frac{i}{\hbar}\mu\Omega\int_{\hat{\mathrm{S}}}\sin\Theta d\Theta d\Phi\right];$$

then we obtain

$$\mu\Omega\int_{\mathrm{S}^2}\sin\Theta d\Theta\wedge d\Phi = 2n\pi\hbar(n = \text{integer}),$$

leading to the quantization condition:

$$4\pi\mu\Omega \equiv \mu M = 2n\pi\hbar, \tag{14}$$

where $M = 4\pi\Omega$, which means the mass of the superfluid on the sphere. Equation (14) is regarded as a counterpart of the Dirac monopole quantization.

Pinning potential as a Hamiltonian term: As is seen from (7), the first two terms, which represent the fluid kinetic energy and the gradient energy, are apparently independent of the position of the vortex center for the case of a single vortex because of the rotational symmetry. In order to obtain the term depending on the vortex center, we need to settle a coordinate that is relative to the vortex center. The term satisfying this criterion is the last one in Eq. (7), that is, $V(\rho)$. If there is an interaction with the *pinning center* built in the superfluid, we get

$$U = \int \psi^*(\mathbf{r})\psi(\mathbf{r})V(\mathbf{r})d\mathbf{r}. \tag{15}$$

Here the density profile is given as $|\psi(\mathbf{r})|^2 = \rho_0 + \tilde{\rho}(|\mathbf{r}-\mathbf{R}|)$, where ρ_0 is the uniform background term and $\tilde{\rho}$ describes the vortex profile. The form of the interaction $V(\mathbf{r})$ is rather complicated and we resort to fix it in a phenomenological way. It is plausible to assume that the $V(\mathbf{r})$ is localized near the pinning center, so the simplest choice is such that it is a contact form, namely, that may be given by the δ function with the strength V_0:

$$V(\mathbf{r}) = V_0\delta(\mathbf{r}-\mathbf{a}), \tag{16}$$

where $\mathbf{a}$ stands for the position of the pinning center. Thus we have $U = V_0\tilde{\rho}(\mathbf{R}-\mathbf{a})$ up to an additional constant term coming from the uniform background. If we choose $\mathbf{a} = (0,0,a)$ as the north pole, it turns out to be

$$U = V_0\tilde{\rho}(a\sqrt{1-\cos\Theta}), \tag{17}$$

where Θ denotes the angle between the vortex center and the north pole. In the case that there are several pinning centers located at $\mathbf{a}_k$, U is given by

$$U = \sum_{k=1}^{N} V_{0k}(\tilde{\rho}(|\mathbf{R}-\mathbf{a}_k|)). \tag{18}$$

The concrete form of (15) may be described by a Gaussian form with a central peak at the vortex center. This will be discussed later in the analysis of the Fokker-Planck equation.

III. THE LANGEVIN AND FOKKER-PLANCK EQUATIONS

In this section we consider the stochastic equation for the vortex motion by following the analogy with the spin equation of motion.

In the following treatment we have in mind the semiclassical approach, namely, the quantum effect is not taken account in the stochastic equation except for the quantized value of the vortex charge. That means the Planck constant never appears in the stochastic equation. Although it looks unsatisfactory to treat the quantum fluid in terms of the stochastic approach in a "classical sense," the present approach may be regarded as a leading order of the full quantum mechanical treatment [24]. Indeed the previous stochastic theory has been carried out in the similar spirit to such a concept (see, e.g., [14]).

A. Spin analogy and the functional integral

As is remarked in the previous section, the equation of motion (10) is of same form as the spin equation of motion. The vector $\hat{\mathbf{R}}$ plays a role for the spin vector, which we call the *pseudospin*. Here in order to emphasize the spin analogy, we introduce the quantity $\mathbf{J} = J\hat{\mathbf{R}}$ with $J \equiv \mu\Omega$, which corresponds to the magnitude of the spin. In terms of $\mathbf{J}$, the above equation is written as

$$\frac{d\mathbf{J}}{dt} = -\mathbf{J}\times\frac{\partial H}{\partial\mathbf{J}}. \tag{19}$$

The right hand side of the equation represents the torque, namely, $\frac{\partial H}{\partial\mathbf{J}}$ is nothing but a substitute for the "magnetic field" acting on the pseudospin, which may be called a *pseudomagnetic field*. From a physical point of view, the similarity between spin and vortex is not surprising, since the vortex can be regarded as a *region* where the angular momentum of the fluid is concentrated.

Furthermore, following the well known fact in spin theory, it is possible to include the effect of dissipation such that ∇H is replaced by $\nabla H + \eta\frac{d\mathbf{J}}{dt}$ in Eq. (19) with the dissipative

coefficient η, namely,

$$\frac{d\mathbf{J}}{dt} = -\mathbf{J} \times \left(\frac{\partial H}{\partial \mathbf{J}} + \eta \frac{d\mathbf{J}}{dt} \right). \tag{20}$$

By multiplying $\mathbf{J}$ to both sides, we get

$$\mathbf{J} \times \frac{d\mathbf{J}}{dt} - \eta J^2 \frac{d\mathbf{J}}{dt} = J^2 \frac{\partial H}{\partial \mathbf{J}}, \tag{21}$$

where we have used the relation $(\mathbf{J} \cdot \nabla H) = 0$. Having solved the equation for $\frac{d\mathbf{J}}{dt}$ from the above two equations, it follows that

$$\frac{d\mathbf{J}}{dt} + \mathbf{A}(\mathbf{J}) = 0, \tag{22}$$

with

$$\mathbf{A}(\mathbf{J}) = \frac{1}{1+\eta^2 J^2} \left[\mathbf{J} \times \frac{\partial H}{\partial \mathbf{J}} + \eta J^2 \frac{\partial H}{\partial \mathbf{J}} \right]. \tag{23}$$

Equation (22) is known as "Landau-Lifschitz equation" [25].

We now treat the Brownian motion of the point vortex, which is caused by random effects of several origins: the temperature fluctuations, and inevitably existing impurities. Here we follow an analogy between a single vortex and a ferromagnetic spin, which can be observed in a single ferromagnetic domain [18]. For the case of ferromagnetic spin, some simplification is adopted, namely, the random thermal fluctuations have a correlation time much shorter than the response time of the system. The response time of a single domain particle is of the same order as the reciprocal of the gyromagnetic resonance frequency. However, the case of the vortex on a sphere, it is not feasible to estimate the ratio between two characteristic times, so we adopt it as a working hypothesis.

In what follows we have in mind the case that the fluctuation effect gives rise to the randomness of the pseudomagnetic field, which is written as $\mathbf{b}$, so we replace $\frac{\partial H}{\partial \mathbf{J}} \to \frac{\partial H}{\partial \mathbf{J}} + \mathbf{b}$. Here $\mathbf{b}(t)$ is assumed to be the Gaussian white noise as a random magnetic field. Hence we have

$$\frac{d\mathbf{J}}{dt} + \mathbf{A}(\mathbf{J}) = \mathbf{c}(t), \tag{24}$$

where $\mathbf{c}(t)$ is written in terms of the random field $\mathbf{b}(t)$:

$$\mathbf{c}(t) = \frac{1}{1+\eta^2 J^2} (\mathbf{J} \times \mathbf{b} + \eta J^2 \mathbf{b}), \tag{25}$$

which is a combination of random force and torque. As a result of the randomly uncorrelated nature of the function $\mathbf{b}$, $\mathbf{c}$ can be also expected to have the Gaussian white noise, which is expressed as

$$\langle c_i(t) \rangle = 0, \quad \langle c_i(t) c_j(t+u) \rangle = h \delta_{ij} \delta(u),$$

with $\delta(u)$ the δ function. h means the diffusion constant. We note that the validity of the above form of white noise is not easy to justify and it is nothing else than the working hypothesis. The white noise is introduced to express that the random magnetic field is correlated on time scales much smaller than the characteristic response time of the pseudospin. It is assumed that $\langle \mathbf{c}^2 \rangle = 2h$, and its probability distribution may be given by the standard Gaussian functional form [19,20]:

$$P[\mathbf{c}(t)] = \exp\left[-\frac{1}{2h} \int_0^t \mathbf{c}^2(t) dt \right]. \tag{26}$$

Using this distribution, the propagator K, which is connected between two end points of pseudospin, is given by the functional integral:

$$K[\mathbf{J}(t)|\mathbf{J}(0)] = \int \prod_t \delta \left[\frac{d\mathbf{J}}{dt} + \mathbf{A}(\mathbf{J}(t)) - \mathbf{c}(t) \right] \times \exp\left[-\int \frac{\mathbf{c}^2(t)}{2h} dt \right] \mathcal{D}[\mathbf{J}] \mathcal{D}[\mathbf{c}(t)], \tag{27}$$

with δ being the Dirac δ functional. In order to carry out the Gaussian functional integral with respect to $\mathbf{c}(t)$, we need to evaluate the functional Jacobian factor, the details of which are briefly given in Appendix B and the result is

$$K[\mathbf{J}(t)|\mathbf{J}(0)] = \int \exp\left[-\frac{1}{2h} \int_0^t \left(\frac{d\mathbf{J}}{dt} + \mathbf{A}(\mathbf{J}) \right)^2 dt \right] \mathcal{D}[\mathbf{J}]. \tag{28}$$

Expanding the square term inside the exponential, one can write this functional integral in the familiar form of a path integral for a particle in the vector potential $\mathbf{A}$ together with the scalar potential $V = \frac{\mathbf{A}^2}{2}$. The parameter h just corresponds to the Planck constant. We can formally write the above functional integral by the quantum mechanical path integral; that is, by using the imaginary time $\tau = -it$, we obtain

$$K = \int \exp\left[\frac{i}{h} \int \left\{ \frac{1}{2} \left(\frac{d\mathbf{J}}{d\tau} \right)^2 + i\mathbf{A} \cdot \frac{d\mathbf{J}}{d\tau} - V \right\} d\tau \right] \mathcal{D}\mathbf{J}(t). \tag{29}$$

B. The Fokker-Planck equation

The derivation of the FP equation is carried out most directly by using the above path integral. If we introduce the "wave function" $\psi(\mathbf{J}, \tau)$, we have the integral equation:

$$\Psi(\mathbf{J}, \tau) = \int K[\mathbf{J}(\tau)|\mathbf{J}(0)] \Psi(\mathbf{J}, 0) d\mathbf{J}(0). \tag{30}$$

Following the standard procedure [26], we obtain the Schrödinger equation:

$$ih \frac{\partial \Psi}{\partial \tau} = \left[\frac{1}{2} (\mathbf{P} - i\mathbf{A})^2 + V \right] \Psi,$$

with $\mathbf{P} = -ih \frac{\partial}{\partial \mathbf{J}} \equiv -ih\nabla$. Returning to real time, namely, $i \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial t}$, and writing $\Psi \to P$, we arrive at the standard form of the FP equation [27]:

$$\frac{\partial P}{\partial t} = \frac{h}{2} \nabla^2 P + \nabla \cdot (\mathbf{A} P). \tag{31}$$

This can be rewritten as the continuity equation: $\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{s} = 0$, where $\mathbf{s}$ denotes the probability current $\mathbf{s} = -\frac{h}{2} \nabla P - \mathbf{A} P$,

and the components in polar coordinates (Θ,Φ) are

$$s_\theta = -\frac{h}{2J}\frac{\partial P}{\partial \Theta} - \frac{P}{1+\eta^2 J^2}\left(\eta J \frac{\partial H}{\partial \Theta} - \frac{1}{\sin\Theta}\frac{\partial H}{\partial \Phi}\right),$$
$$s_\phi = -\frac{h}{2J\sin\Theta}\frac{\partial P}{\partial \Phi} - \frac{P}{1+\eta^2 J^2}\left(\frac{\partial H}{\partial \Theta} + \frac{\eta J}{\sin\Theta}\frac{\partial H}{\partial \Phi}\right).$$

Thus the FP equation is written as

$$\frac{\partial P}{\partial t} = -\frac{1}{J\sin\Theta}\left\{\frac{\partial}{\partial\Theta}(\sin\theta s_\theta) + \frac{\partial s_\phi}{\partial\Phi}\right\}. \tag{32}$$

We here examine several general consequences from the FP equation.

(i) *The stationary distribution*. We consider $\frac{\partial P}{\partial t} = 0$, and we put ansatz of the Boltzmann distribution: $P(\mathbf{J}) = \exp[-\beta H]$ with β the inverse temperature $\beta = \frac{1}{k_B T}$. By substituting this into the right hand side of the FP equation, we get

$$\left(\frac{\beta h}{2} - \frac{\eta J^2}{1+\eta^2 J^2}\right)\{\nabla^2 H - \beta(\nabla H)^2\} = 0.$$

From this it follows the relation

$$\frac{\beta h}{2} - \frac{\eta J^2}{1+\eta^2 J^2} = 0. \tag{33}$$

This is the well known fluctuation-dissipation relation, which establishes the relation between the dissipation coefficient η and the diffusion (fluctuation) coefficient h.

(ii) *Evolution equation for average value* [28]. The FP equation enables us to derive the evolution equation for the average of functions on the sphere, which is given by $\langle F\rangle = \int F(\mathbf{J})P(\mathbf{J})d\mathbf{J}$. Using the derivative under integration together with partial integration, we obtain

$$\frac{\partial\langle F\rangle}{\partial t} = \frac{h}{2}\langle\nabla^2 F\rangle + \langle\nabla F\cdot\mathbf{A}\rangle. \tag{34}$$

As a simple example, we here take $F(\mathbf{J}) = J_i^2 (i = x,y,z)$ and choose $\mathbf{A} = -\gamma\mathbf{J}$, (which represents the relaxation effect); then we have the equation for $\langle F\rangle$,

$$\frac{d\langle J_i^2\rangle}{dt} = h - \gamma\langle J_i^2\rangle. \tag{35}$$

The solution becomes

$$\langle J_i^2\rangle = \frac{h}{\gamma}\left(1 - \exp\left[-\frac{\gamma}{h}t\right]\right). \tag{36}$$

Namely, this shows a typical relaxation behavior of $\langle J_i^2\rangle$ leading to the asymptotic value $\langle J_i^2(\infty)\rangle = \frac{h}{\gamma}$.

IV. ANALYSIS OF THE FOKKER-PLANCK EQUATION: THE ROLE OF THE PINNING POTENTIAL

The FP equation is of a peculiar form, that is, rather different from the usual diffusion equation (or Schrödinger type equation) which includes the potential term. So special techniques are required to deal with this. We now consider a typical example that can be treated by analytic as well as approximate ways.

General setting. In what follows, the argument is restricted to the case of one or two pinning centers (Fig. 1).

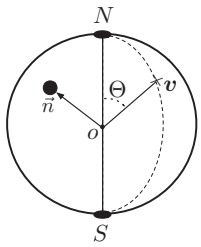

FIG. 1. Conceptual image of the vortex (marked by **v**) in the pinning potentials located at the north (N) and south (S) poles and a general point marked by **n**.

As the first case, let us consider the single pinning center that is located at the north pole: $\mathbf{a} = (0,0,a)$, for which the potential is given by the pinning potential (17). As the concrete form of the profile for $\tilde{\rho}$, we adopt the Gaussian form [29]: $\tilde{\rho} = \rho_0 \exp[-\frac{(\mathbf{r}-\mathbf{R})^2}{l}]$ with the vortex size l; hence we have

$$H(\Theta) = V_0 \exp[-a^2(1-\cos\Theta)/l] \equiv V_0\tilde{H}. \tag{37}$$

Next, the Hamiltonian (37) can be extrapolated to two pinning centers with the strengths V_0, V_0', which are assumed to be positive. The configuration of two centers is arranged such that the one center is located at the north and the other is the general point indicated by the direction cosine: $\mathbf{n} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. The Hamiltonian for these two pinning centers is thus written in terms of the two angles (Θ,Φ):

$$H(\Theta,\Phi) = V_0 \exp\left[-\frac{a^2}{l}\right]\left\{\exp\left[\frac{a^2}{l}\cos\Theta)\right] + r\exp\left[\frac{a^2}{l}\cos\Pi)\right]\right\}, \tag{38}$$

where the ratio $r = V_0'/V_0$ is introduced and

$$\cos\Pi = \mathbf{n}\cdot\hat{\mathbf{R}} = n_x\sin\Theta\cos\Phi + n_y\sin\Theta\sin\Phi + n_z\cos\Theta.$$

As a particular case, if the other center is located at the south pole, $\Pi = \pi$ ($\cos\pi = -1, \sin\pi = 0$), the Hamiltonian is reduced to

$$H(\Theta) = V_0\exp\left[-\frac{a^2}{l}\right]\left\{\exp\left[\frac{a^2}{l}\cos\Theta\right] + r\exp\left[-\frac{a^2}{l}\cos\Theta)\right]\right\} = V_0\tilde{H}(\Theta), \tag{39}$$

which includes (37) as a special case $r = 0$.

The Hamiltonians (37) and (39) are relevant for getting concrete forms of the FP equation in an analytic or approximate way (see below), since these depend on Θ alone. These two cases seem to be enough for analyzing the stochastic behavior of the vortex under the influence of pinning potentials.

Here we give a remark on the general configuration of two centers, (38). The classical equation of motion reads

$$\dot{\Theta} = \frac{1}{J(1+\eta^2J^2)\sin\Theta}\frac{\partial H}{\partial\Phi} + \frac{\eta}{1+\eta^2J^2}\frac{\partial H}{\partial\Theta}$$
$$\dot{\Phi} = -\frac{1}{J(1+\eta^2J^2)\sin\Theta}\frac{\partial H}{\partial\Theta} + \frac{\eta}{1+\eta^2J^2}\frac{1}{J\sin\Theta}\frac{\partial H}{\partial\Phi}. \tag{40}$$

To deal with this equation is not simple, because the Hamiltonian depends on two variables Θ,Φ. The standard way to analyze the behavior of an orbit is to search the path which connects the local minima that may be determined by the potential landscape given by $H(\Theta,\Phi)$. The procedure needs laborious manipulation and the details of this will be discussed elsewhere.

A. The case without dissipation: $\eta = 0$

Strictly speaking this is considered to be a limiting case: $\eta \sim 0$, for which the effective temperature $k_B T \sim \beta^{-1}$ is extremely large, by noting the fluctuation dissipation relation (33). This means that all the vortex states are equally probable, and there does not appear to be the temperature effect.

In order to treat this case, we use the functional integral (28), for which the "Lagrangian" is written in terms of the angular variables:

$$\mathcal{L} = \frac{1}{2}J^2\left[\dot{\Theta}^2 + \sin^2\Theta\left(\dot{\Phi} + \frac{V_0}{J\sin\Theta}\frac{d\tilde{H}}{d\Theta}\right)^2\right]. \quad (41)$$

The classical equation of motion (40) is written as

$$\dot{\Theta} = \frac{1}{J\sin\Theta}\frac{\partial H}{\partial\Phi} = 0, \quad \dot{\Phi} = -\frac{1}{J\sin\Theta}\frac{\partial H}{\partial\Theta}.$$

Hence the orbit becomes a circle with constant latitude: $\Theta = \Theta_0$, and this represents a rotation with the constant angular velocity:

$$\kappa \equiv \left[\frac{V_0}{J\sin\Theta}\frac{d\tilde{H}}{d\Theta}\right]_{\Theta=\Theta_0}. \quad (42)$$

Thus the functional integral is reduced to the form:

$$K = \int \exp\left[-\frac{1}{2h}\int_0^t \sin^2\Theta_0(\dot{\Phi}+\kappa)^2 dt\right]\mathcal{D}(\Phi), \quad (43)$$

which is a path integral for a *free particle* on a circle. The corresponding FP equation is given by the diffusion-type equation

$$h\frac{\partial P}{\partial t} = \frac{h^2}{2\sin^2\Theta_0}\frac{\partial^2 P}{\partial\Phi^2} - h\kappa\frac{\partial P}{\partial\Phi}. \quad (44)$$

In order to get the solutions of this equation, we put ansatz:

$$P(\Phi,t) = \sum_{n=-\infty}^{\infty}\exp[-\lambda_n t]f_n(\Phi), \quad (45)$$

where $f_n(\Phi)$ satisfies the eigenvalue equation

$$\frac{h}{2\sin^2\Theta_0}\frac{d^2 f_n}{d\Phi^2} - \kappa\frac{df_n}{d\Phi} = -\lambda_n f_n.$$

We put $f_n(\Phi) = \exp[in\Phi](n = \text{integers})$, and the eigenvalue is obtained as

$$\lambda_n = \frac{h}{2\sin^2\Theta_0}n^2 + in\kappa.$$

By substituting this into the (45), we have

$$P(\Phi,t) = \sum_{n=-\infty}^{\infty}\exp\left[-\left(\frac{h}{2\sin^2\Theta_0}n^2 + in\kappa\right)t\right]\exp[in\Phi], \quad (46)$$

which turns out to be the Θ function [30]. The expectation values for $\langle\mathbf{J}\rangle = \int_0^{2\pi}\mathbf{J}P(\Phi,t)d\Phi$ are calculated as

$$\begin{aligned}\langle J_x\rangle &= J\sin\Theta_0\exp\left[-\frac{ht}{2\sin^2\Theta_0}\right]\cos\kappa t,\\ \langle J_y\rangle &= J\sin\Theta_0\exp\left[-\frac{ht}{2\sin^2\Theta_0}\right]\sin\kappa t, \quad (47)\\ \langle J_z\rangle &= J\cos\Theta_0.\end{aligned}$$

This is a typical damped oscillation and similar to the diffusive behavior of spin in magnetic systems [31]. The feature that this does not show up the temperature effect may be a consequence of the high temperature limit.

B. The case with dissipation: $\eta \neq 0$

This case is complicated compared to the case without dissipation. We look for the distribution function as a function of Θ only, namely, $P \equiv P(\Theta,t)$. Assuming a separable form $P(\Theta,t) = P(\Theta)\exp[-\lambda t]$, then it follows the eigenvalue equation

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dP}{d\Theta}\right) - \frac{\epsilon}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{d\tilde{H}}{d\Theta}P\right) = -\lambda P, \quad (48)$$

where we introduce the parameter

$$\epsilon = \frac{2V_0\eta J^2}{h(1+\eta^2 J^2)}. \quad (49)$$

The parameter ϵ plays a significant role. That is, as a consequence of the fluctuation-dissipation relation (33), ϵ can be expressed as $\epsilon = V_0\beta$, which gives the ratio between potential strength and temperature. This fact suggests that ϵ can be regarded as an expansion parameter to treat approximate schemes. We have two cases that will be given below.

(a) We first consider the case that $\epsilon \ll 1$ ($V_0\beta \ll 1$) is satisfied, which means the temperature is high enough so as to satisfy the condition $V_0 \ll k_B T$. In terms of quantum mechanical terminology, this case may correspond to *weak coupling*. It can be treated by perturbation procedure. By expanding P and λ in terms of a power series with respect to ϵ:

$$P(\Theta) = P^{(0)}(\Theta) + P^{(1)}(\Theta) + \cdots, \quad \lambda = \lambda_0 + \lambda_1 + \cdots,$$

and substituting this into the above eigenvalue equation (48), then we have the zeroth order:

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dP^{(0)}}{d\Theta}\right) = -\lambda_0 P^{(0)}, \quad (50)$$

and the first order:

$$\begin{aligned}&\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dP^{(1)}}{d\Theta}\right) - \frac{\epsilon}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{d\tilde{H}}{d\Theta}\right)P^{(0)}\\ &\quad = -\lambda_1 P^{(0)} - \lambda_0 P^{(1)}. \quad (51)\end{aligned}$$

The zeroth order gives the Legendre polynomial: $P^0 = P_l(\Theta)$, and the corresponding eigenvalue is $\lambda_0 = l(l+1)$ with $l = 1\cdots$. Here we note that there is an ambiguity to choose which

$P_l(\Theta)$ should be chosen. For example, we arbitrarily pick up a particular l, namely, $P^0 = P_l(\Theta)$; then the first order term is calculated to

$$\lambda_1 = \epsilon \int P_l(\Theta)\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{d\tilde{H}}{d\Theta}\right)P_l(\Theta)d\Omega / \times \int P_l^2(\Theta)d\Omega \tag{52}$$

($d\Omega = \sin\Theta d\Theta d\Phi$). $P^{(1)}$ can be expanded in terms of $P_k(\Theta)$: $P^{(1)} = \sum_k C_k P_k(\Theta)$; then we obtain the coefficients C_k:

$$C_k = \frac{V_{kl}}{\lambda_0(l) - \lambda_0(k)}, \tag{53}$$

with the matrix element:

$$V_{kl} \equiv \epsilon \int P_k(\Theta)\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{d\tilde{H}}{d\Theta}\right)P_l(\Theta)d\Omega$$

for $l \neq k$. Using this, the second order shift of the eigenvalue is evaluated as

$$\lambda_2 = \epsilon^2 \sum_{k\neq l} \frac{V_{kl}^2}{\lambda_0(l) - \lambda_0(k)}. \tag{54}$$

The still higher order may be obtained by continuing the usual perturbation scheme.

Here a remark is given for the above perturbation procedure. We have no rule to fix the initial unperturbed states given by the Legendre polynomial; namely, it is not known which $P_l(\Theta)$ should be chosen, as we have seen in the above. This fact means that the distribution of a vortex may be *unstable* in the sense of thermodynamics. In other words, the strength of pinning is too weak to keep the distribution stable. This implies that the weak coupling limit is irrelevant to determine the solution of the FP equation.

(b) We next consider the case of $\epsilon \sim 1$ ($V_0\beta \sim 1$), which means $V_0 \simeq k_B T$. In the quantum mechanical sense, this regime may be regarded as *strong coupling*. For this case we need to use nonperturbation procedure. In order to carry out this, we rewrite the FP equation in terms of the variable $\xi = \cos\Theta$:

$$\frac{d}{d\xi}\left[(1-\xi^2)\left\{\frac{dP}{d\xi} + \epsilon\frac{d\tilde{H}}{d\xi}P\right\}\right] = -\lambda P.$$

By using the relation

$$\exp[-\epsilon\tilde{H}]\frac{d}{d\xi}\exp[\epsilon\tilde{H}] = \frac{d}{d\xi} + \epsilon\frac{d\tilde{H}}{d\xi},$$

it follows that

$$\frac{d}{d\xi}\left\{(1-\xi^2)\exp[-\epsilon\tilde{H}]\frac{d}{d\xi}(\exp[\epsilon\tilde{H}]P)\right\} = -\lambda P.$$

Putting $Q = \exp[\epsilon\tilde{H}]P$, we have

$$\frac{d}{d\xi}\left\{(1-\xi^2)\exp[-\epsilon\tilde{H}]\frac{dQ}{d\xi}\right\} = -\lambda\exp[-\epsilon\tilde{H}]Q. \tag{55}$$

This equation is translated to the variational problem, namely, by introducing the "action" function:

$$I[Q] = \int_{-1}^{1}(1-\xi^2)\exp[-\epsilon\tilde{H}]\left(\frac{dQ}{d\xi}\right)^2 d\xi, \tag{56}$$

together with the constraint coming from the normalization:

$$\int_{-1}^{1}\exp[-\epsilon\tilde{H}]Q^2 d\xi \equiv N[Q] = 1. \tag{57}$$

Then the "Euler-Lagrange equation" $\delta(I[Q] - \lambda N[Q]) = 0$ recovers the above eigenvalue equation with λ being the Lagrangian multiplier.

We see that there is a trivial solution $Q_0 =$ constant with zero eigenvalue $\lambda_0 = 0$. The corresponding solution becomes

$$P(\Theta) = \exp[-V_0\beta\tilde{H}]. \tag{58}$$

By starting from (58) as the lowest solution, we have an algorithm to get the subsequent solutions: $Q_n(n = 0,\ldots)$ by noting the orthogonality condition:

$$\int_{-1}^{1}\exp[-\epsilon\tilde{H}]Q\,Q_n d\xi = 0. \tag{59}$$

We examine how to construct the second lowest state. This may be achieved by choosing a trial function given by a quadratic function $Q_1(\xi) = A + B\xi + C\xi^2$. Substituting this into (59), we have a linear relation between A,B,C, by which C can be expressed by A,B. Next (57) gives a quadratic equation for A,B. Finally (56) gives another quadratic form for A,B. Then, the problem is reduced to minimization of (56) written in terms of the quadratic form under the constraint (57) expressed by the quadratic equation. Having solved these, we arrive at the solution of Q_1.

This procedure may be carried out step by step, and we can have a series of the stationary solution of the FP equations. Having obtained a series of the eigenfunctions $\{Q_n(\Theta)\}$, one can construct the time-dependent solution for the FP equation by taking the linear combination:

$$P(\Theta,t) = \exp[-\epsilon\tilde{H}]\sum_{n=0} c_n Q_n(\xi)\exp[-\lambda_n t].$$

If we are concerned with the long time behavior, it may be enough to keep the lowest two terms: $n = 0,1$ only:

$$P(\Theta,t) = \exp[-V_0\beta\tilde{H}(\Theta)]\{c_0 + c_1 Q_1(\xi)\exp[-\lambda_1 t]\}. \tag{60}$$

Here we discuss the meaning of the solution in the strong coupling regime. Equation $P(\Theta) = \exp[-V_0\beta\tilde{H}]$ gives the distribution representing the equilibrium state with temperature β. This result is quite different from the perturbational procedure that has been treated in the case (a), in which there may be no thermally stable states. To have the equilibrium state (58) means that the stable state of "thermal equilibrium " is pronounced in this strong coupling regime.

C. The small diffusion limit

We have another asymptotic limit for which the diffusion constant h is regarded as small. We here look for an alternative way to obtain an approximate form of the solution of the FP equation. For this case, we have $h\epsilon = \frac{V_0\eta J^2}{1+\eta^2 J^2}$. In particular, we consider the case that ϵ is extremely large by setting this $h\epsilon \sim 1$.

In the limit of $h \simeq 0$, the functional integral is treated by the stationary phase approximation; namely, we get

$$K_{cl} \simeq \exp\left[-\frac{1}{h}S_{cl}\right], \tag{61}$$

where S_{cl} denotes the *classical action* that satisfies an extreme condition $\delta S = 0$, which is given by $S_{cl} = \int \mathcal{L} dt$ with the Lagrangian

$$\mathcal{L} = \frac{J^2}{2}\left\{\left(\dot{\Theta} - h\epsilon\frac{d\tilde{H}}{d\Theta}\right)^2 + \sin^2\Theta\left(\dot{\Phi} + \frac{h\epsilon}{J\eta\sin\Theta}\frac{d\tilde{H}}{d\Theta}\right)^2\right\}. \tag{62}$$

Besides K_{sc}, we have a term coming from the second variation about the extreme path, that is written in terms of the Gaussian functional integral, but this will be omitted here. We note a peculiar feature of the Lagrangian: The variable Φ does not appear in the Lagrangian, namely, Φ is cyclic coordinate and the "momentum' that is conjugate to Φ becomes a constant of motion:

$$J^2\sin^2\Theta\left(\dot{\Phi} + \frac{h\epsilon}{J\eta\sin\Theta}\frac{d\tilde{H}}{d\Theta}\right) = C.$$

Hence, following the well known procedure in analytical dynamics, we have the "Rouse function' [32]: $R = C\dot{\Phi} - \mathcal{L}$, in which Φ variable is eliminated to give

$$R = -\frac{J^2}{2}\left(\dot{\Theta} - h\epsilon\frac{d\tilde{H}}{d\Theta}\right)^2 + \frac{C^2}{2J^2\sin^2\Theta} - \frac{Ch\epsilon}{J\eta\sin\Theta}\frac{d\tilde{H}}{d\Theta}, \tag{63}$$

from which the equation of motion for Θ is derived from the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial\dot{\Theta}}\right) - \frac{\partial R}{\partial\Theta} = 0.$$

By substituting the solution (classical orbit) of the equation of motion into $\mathcal{L}$, we have

$$K_{cl} = \exp\left[-\frac{1}{h}\int_0^t\left\{\frac{J^2}{2}\left(\frac{d\Theta}{dt} - h\epsilon\frac{d\tilde{H}}{d\Theta}\right)^2 + \frac{C^2}{2J^2\sin^2\Theta}\right\}dt\right]. \tag{64}$$

We look for a further reduced form of K_{cl}, that is, we consider the case such that C/J is small. If we regard this as a small parameter, we can omit the last two terms in Eq. (63), so the equation of motion becomes a simple form ("instanton' type equation):

$$\frac{d\Theta}{dt} - h\epsilon\frac{d\tilde{H}}{d\Theta} = 0. \tag{65}$$

Noting that the first integral in K_{cl} vanishes as a result of (65), we have

$$K_{cl} = \exp\left[-\frac{1}{h}\int_0^t\frac{C^2}{2J^2\sin^2\Theta}dt\right],$$

which is rewritten in terms of the Θ variable using the relation $dt = \frac{1}{h\epsilon}(\frac{d\tilde{H}}{d\Theta})^{-1}d\Theta$:

$$K_{cl} = \exp\left[-\frac{C^2}{2J^2h^2\epsilon}\int\left(\sin^2\Theta\frac{d\tilde{H}}{d\Theta}\right)^{-1}d\Theta\right].$$

Substituting the explicit form of the potential (39) [or (37)], we get

$$K_{cl}(x_f,x_i) = \exp\left[-M\int_{x_f}^{x_i}F(x)dx\right],$$
$$F(x) = \left\{\exp\left[\frac{a^2}{l}x\right] - r\exp\left[\frac{-a^2}{l}x\right]\right\}^{-1}\frac{1}{(1-x^2)^2},$$
$$M = \frac{C^2}{2J^2h^2\epsilon}\frac{l}{a^2}\exp\left[\frac{a^2}{l}\right], \tag{66}$$

where we have used the change of variable: $x = \cos\Theta$ and put $x_f = \cos[\Theta(t)]$ and $x_i = \cos[\Theta(0)]$. Finally, the distribution function is obtained by putting the initial distribution as a form $P(x_i) = \delta(x_i - x_0)$ (the δ function):

$$P(x_f) = \int K_{cl}(x_f,x_i)P(x_i)dx_i = \exp\left[-M\int_{x_f}^{x_0}F(x)dx\right]. \tag{67}$$

$P(x_f)$ has been calculated numerically, as is shown in Fig. 2. This is carried out for the special case of $r = 0$, the potential coming from the single pinning center located at the north pole. The parameters are chosen in the following: We take $\frac{C^2}{2J^2h^2\epsilon} \sim 10^{-3}$, which means that $\frac{C^2}{2J^2h\epsilon} \sim 10^{-3}h$, while the value a^2/l take several values depending on the vortex size l with radius a being fixed. We choose three values: $a^2/l = 10,50,100$. From the result depicted in Fig. 2, we see that the vortex appears to be localized in the upper hemisphere ($0 < \Theta < \pi/2$) and the probability drops to zero upon departing from the north pole ($x = 1$). The three curves indicate that the feature of falling down to zero of $P(x_f)$ depends on the size of the vortex. That is, the probability distribution of the vortex with smaller size becomes zero rapidly in the upper sphere.

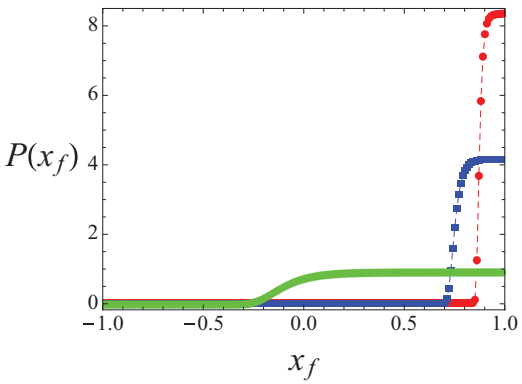

FIG. 2. (Color online) Numerical plot for $P(x_f)$: The red circles, blue squares, and upper green line represent $a^2/l = 100,50,10$, respectively.

V. EXTENSION TO THE VORTEX IN NONSPHERICAL GEOMETRY

The stochastic formulation developed in the previous sections is based on a simple structure of spherical geometry. It is desirable if this could be extended to the case of the vortex adsorbed on the surface with similar simple geometrical structure. In what follows we shall briefly discuss the case of two-dimensional flat space as well as the pseudosphere (Lobachevsky plane).

A. The case of the planer vortex

The vortex in the plane may be smoothly extrapolated to the vortex in spherical geometry. This feature was implied experimentally [13], so it is desirable to examine the stochastic equation in a similar manner to the spherical case. Let the vortex center coordinate be $\mathbf{R} = (X, Y)$. The equation of motion of the vortex center may be obtained as a limiting case of the infinite radius, $a \to \infty$, or more precisely, when the vortex size is regarded to be much smaller than the radius of sphere. Without repeating the detailed manipulation, we have the equation of motion:

$$m\rho_0\mu\left(\mathbf{k} \times \frac{d\mathbf{R}}{dt}\right) = \frac{\partial H}{\partial \mathbf{R}}, \tag{68}$$

where $\mathbf{k}$ is the unit vector perpendicular to (x, y) plane. By multiplying $\mathbf{k}$ to both sides and noting the relation $(\mathbf{k} \cdot \nabla H) = 0$, we have

$$m\rho_0\mu\frac{d\mathbf{R}}{dt} = -\mathbf{k} \times \frac{\partial H}{\partial \mathbf{R}}. \tag{69}$$

By replacing the nabla term in Eqs. (68) and (69) by the term including the dissipation, namely, $\nabla H \to \nabla H + \eta\dot{\mathbf{R}}$ $[\nabla = (\frac{\partial}{\partial X}, \frac{\partial}{\partial Y})]$ and solving these for $\dot{\mathbf{R}}$, we obtain

$$\frac{d\mathbf{R}}{dt} + \mathbf{B} = 0, \tag{70}$$

where the force $\mathbf{B}$ is given by a combination of the torque and the gradient term:

$$\mathbf{B} \equiv \frac{1}{(m\rho_0\mu)^2 + \eta^2}\left(m\rho_0\mu\mathbf{k} \times \frac{\partial H}{\partial \mathbf{R}} + \eta\frac{\partial H}{\partial \mathbf{R}}\right).$$

By adding the random force $\mathbf{c}$, (70) turns out to be the Langevin equation: $\frac{d\mathbf{R}}{dt} + \mathbf{B} = \mathbf{c}$. By assuming that $\mathbf{c}$ has the Gaussian white noise and following the same procedure as the case of a spherical vortex, we obtain the FP equation,

$$\frac{\partial P}{\partial t} = \frac{h}{2}\nabla^2 P + \nabla \cdot (\mathbf{B}P). \tag{71}$$

The FP equation (71) is essentially same as the one that has been used in the case of the superconductivity vortex [14]. The more detailed and intricate treatment was given for the actual superconductors in connection with high T_c superconductivity [15,16]. Our treatment here would provide a refined way to explore the stochastic behavior of a planer vortex. The details of the argument will be given elsewhere.

B. The case of the vortex on the pseudosphere

Next we consider the vortex on the pseudosphere (which is denoted by PS^2, the surface with constant negative curvature). The superfluid adsorbed on this pseudosphere may be realized in actual situations as a local part of the surface of complicated shapes.

Here an explanation is prepared for some geometrical characteristics of PS^2: (i) The point of the surface is described by the equation in terms of the rectilinear coordinates, $x^2 + y^2 - z^2 = -a^2$, which is parametrically written as

$$x = a\sinh\theta\cos\phi, \quad y = a\sinh\theta\sin\phi, \quad z = a\cosh\theta$$

[note that this is formally obtained by replacing θ by $i\theta$ ($i = \sqrt{-1}$) in the polar coordinate of sphere], and we introduce the vortex center:

$$\hat{\mathbf{R}} \equiv \mathbf{R}/a = (\sinh\Theta\cos\Phi, \sinh\Theta\sin\Phi, \cosh\Theta).$$

(ii) The scalar product and vector product (which is denoted by $*$) are, respectively, defined by [33]

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 - a_3b_3,$$
$$\mathbf{a} * \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_2b_3, -(a_1b_2 - a_2b_1)).$$

Note that the 3rd term in the scalar product has a minus sign compared with the spherical case. As for the vector product, the third component has a minus sign. The triple product is given as

$$\mathbf{a} * (\mathbf{b} * \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}.$$

Now, following the same procedures as the one for the spherical vortex developed in Sec. II, we obtain the equation of motion for a *pseudospin*:

$$\frac{d\mathbf{K}}{dt} = -\mathbf{K} * \frac{\partial H}{\partial \mathbf{K}}. \tag{72}$$

Here the pseudospin is defined: $\mathbf{K} = K\hat{\mathbf{R}}$ with $K \equiv \mu\Omega$ and $\Omega = m\rho a^2$. This has the same form as the one used in a different context [34]. For the case of the vortex on PS^2, there is no counterpart of the topological quantization, because PS^2 is trivial in a topological sense, namely, this is isomorphic to two-dimensional flat Euclidean space. The norm of $\mathbf{K}$ is given by $\mathbf{K}^2 = -K^2$ and K is not quantized, which is a consequence of the nonexistence of a topological invariant for PS^2. If the dissipation effect is included, Eq.(72) turns out to be

$$\frac{d\mathbf{K}}{dt} = -\mathbf{K} * \left(\frac{\partial H}{\partial \mathbf{K}} + \eta\frac{d\mathbf{K}}{dt}\right). \tag{73}$$

Multiplying $*\mathbf{K}$ to (73) and noting the vector product formula, we get

$$\mathbf{K} * \frac{d\mathbf{K}}{dt} = K^2\frac{\partial H}{\partial \mathbf{K}} + \eta K^2\frac{d\mathbf{K}}{dt}. \tag{74}$$

From (73) and (74), the equation of motion for $\mathbf{K}$ is derived as follows:

$$\frac{d\mathbf{K}}{dt} + \tilde{A}(\mathbf{K}) = 0,$$
$$\tilde{A}(\mathbf{K}) \equiv \frac{1}{1 + \eta^2K^2}\left(\mathbf{K} * \frac{\partial H}{\partial \mathbf{K}} + \eta K^2\frac{\partial H}{\partial \mathbf{K}}\right). \tag{75}$$

The Langevin equation is thus written as

$$\frac{d\mathbf{K}}{dt} + \tilde{A}(\mathbf{K}) = \tilde{\mathbf{c}}.$$

Here the random "torque' $\tilde{c}$ has the norm $\tilde{\mathbf{c}}^2 = c_1^2 + c_2^2 - c_3^2$, where the minus sign of the third term is a consequence of non-Euclidean metric on the pseudosphere. By adopting the Gaussian white noise and following the same procedure as the spherical vortex case, we arrive at the FP equation:

$$\frac{\partial P}{\partial t} = -\frac{1}{K\sinh\Theta}\left\{\frac{\partial}{\partial\Theta}(\sinh\theta s_\theta) + \frac{\partial s_\phi}{\partial\Phi}\right\}, \tag{76}$$

where the current is

$$s_\theta = -\frac{h}{2K}\frac{\partial P}{\partial\Theta} - \frac{P}{1+\eta^2K^2}\left(\eta K\frac{\partial H}{\partial\Theta} - \frac{1}{\sinh\Theta}\frac{\partial H}{\partial\Phi}\right),$$
$$s_\phi = -\frac{h}{2K\sinh\Theta}\frac{\partial P}{\partial\Phi} - \frac{P}{1+\eta^2K^2}\left(\frac{\partial H}{\partial\Theta} + \frac{\eta K}{\sinh\Theta}\frac{\partial H}{\partial\Phi}\right).$$

It is possible to develop an approximate scheme in the same way as the spherical vortex. This may be realized by finding the pinning potential that is obtained by replacing $\cos\Theta \to \cosh\Theta$ in Eq. (17). For the case corresponding to the small ϵ, the perturbational calculation is possible, for which the zeroth order (unperturbed) distribution function satisfies

$$\frac{1}{\sinh\Theta}\frac{\partial}{\partial\Theta}\left(\sinh\Theta\frac{\partial P}{\partial\Theta}\right) = \lambda P,$$

and the eigenvalue is given by $\lambda = \rho(\rho+1)$ (ρ is real) together with the eigenfunction written in terms of the integral representation [35]:

$$P_\rho(\cosh\Theta) = \frac{1}{2\pi}\int_0^{2\pi}(\cosh\Theta + \sinh\Theta\cos\xi)^\rho d\xi.$$

Furthermore, for the nonperturbation case, it may be possible to construct the similar equation with (55), which is simply obtained by replacing $\xi = \cos\Theta$ by $\xi = \cosh\Theta$. Then we can apply the variation principle for $I[Q]$, by which the approximation solutions for the FP equation may be derived.

VI. SUMMARY

We have developed a stochastic theory of the quantum vortex on the two-dimensional sphere as well as its variants of nonspherical geometry. This attempt may shed light on a new facet of stochastic aspects of the vortex which have not been well recognized so far. The main consequence is the Fokker-Planck equation using the Langevin equation for the Brownian motion of the vortex on the sphere by adopting the functional integral method [19,20]. The basic point of our theory is to follow an analogy between the vortex on the sphere and spin, more specifically, the magnetic system of a single domain [18]. The analogy seems natural, because the vortex carries angular momentum of the fluid in a form of a "lump' which is a small region where the angular momentum is concentrated. The randomness comes from several origins. We have in mind the conventional assumption: temperature fluctuations. For the case of mesoscopic size, the fluctuation of temperature may be more appreciable than macroscopic systems in the relative sense.

In developing the stochastic theory, we essentially adopt the semiclassical approach. Except for the quantized value of the vortex charge, there are no quantum effects in neither vortex dynamics nor statistical mechanical quantities. This means that the Planck constant is not reflected in the fluctuation-dissipation relation.

We have analyzed the FP equation in order to examine the role of the pinning potential. This was carried out by choosing the special form of the potential that depends on the latitude angle Θ: that is derived by the Gaussian profile. Our findings are as follows: (i) The first is the case without dissipation, for which the analytically closed form is obtained for the probability distribution function in terms of the Θ function, which implies the diffusional behavior (Sec. IV A). This may be utilized for a limiting case with infinitesimal dissipation. (ii) The second is the case with dissipation (Sec. IV B), for which we could find a criterion for determining the weak and strong coupling approximation. These can be treated by perturbative and nonperturbative procedures [the cases (a) and (b), respectively]. The latter is relevant to keep the ground state stable in a thermodynamic sense, whereas the former is not and the vortex distribution is unstable to be formed in a robust manner. (iii) We have also examined another aspect of the small diffusion limit (Sec. IV C), in which we could get a simple form of the probability distribution by applying an analogous technique with the semiclassical limit for the functional integral. The result shows the the probability distribution of the vortex center depends on the size of the vortex profile.

Besides the stochastic theory for the vortex on a sphere, an extension was given for the vortex in nonspherical geometry: the case of the planer and pseudosphere. These could be obtained as a natural continuation of the sphere vortex.

We expect that the present approach provides with a clue to study the stochastic aspect of the quantum vortex defined on the complicated two-dimensional manifold.

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APPENDIX A: CALCULATION OF $\nabla \times \mathbf{j}$

We rewrite the surface integral of the right hand side of (8) by using the Stokes theorem:

$$\int_S(\nabla\times\mathbf{j})d\sigma = \int_C m\rho\mathbf{v}d\mathbf{s}, \tag{A1}$$

with C being boundary of S. If C is chosen such that it is sufficiently remote from the vortex center, $\rho \to \rho_0$, it follows that

$$\int_C m\rho\mathbf{v}d\mathbf{s} = m\rho_0\int_C \mathbf{v}\cdot d\mathbf{s} \equiv m\rho_0\int_S \omega d\sigma,$$

with the vorticity $\omega = \nabla\times\mathbf{v}$. Next we give an explicit form for the velocity field, which is given by

$$\mathbf{v} = \mu(\hat{r}\times\nabla f), \tag{A2}$$

where the *vortex charge* μ is introduced and f is

$$f = \log|\mathbf{r} - \mathbf{R}(t)| = \log a + \log|\hat{\mathbf{r}} - \hat{\mathbf{R}}(t)|. \quad \text{(A3)}$$

Here $\hat{\mathbf{r}} = \mathbf{r}/a, \hat{\mathbf{R}} = \mathbf{R}/a$, which is written as

$$\hat{R} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta),$$
$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Thus the velocity field becomes

$$\mathbf{v} = \mu\left(0, -\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}, \frac{\partial f}{\partial\theta}\right). \quad \text{(A4)}$$

Hence the vorticity is calculated as $\omega = \nabla\times\mathbf{v} = \mu(\nabla^2 f)\hat{r}$ with the Laplacian on the two-sphere:

$$\nabla^2 f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = \delta(\hat{\mathbf{r}} - \hat{\mathbf{R}}(t)),$$

where we note $\nabla^2\log|\hat{\mathbf{r}} - \hat{\mathbf{R}}(t)| = \delta(\hat{\mathbf{r}} - \hat{\mathbf{R}}(t))$. From the above expression the vorticity for the spherical vortex directs to the normal of the sphere. So we get

$$\int_S(\nabla\times\mathbf{j})d\sigma = m\mu\rho_0\int\delta(\hat{r} - \hat{\mathbf{R}})\hat{r}d\sigma = m\mu\rho_0\hat{\mathbf{R}}.$$

By substituting this into the Lagrangian equation of motion, we arrive at the Kirchhoff equation.

Here we give a sketch of the similar calculation for the vortex on pseudosphere PS^2, which is used for the derivation of the equation of motion in Sec. V B. The procedure of calculation is parallel to the case of S^2; we need only the following change:

$$\nabla f = \left(0, \frac{\partial f}{\partial\theta}, \frac{1}{\sinh\theta}\frac{\partial f}{\partial\phi}\right).$$

That is, the velocity field is given by replacing $\sin\theta$ with $\sinh\theta$, and the vortex becomes $\omega = \nabla\times\mathbf{v} = (\nabla^2 f)\hat{r}$ with the Laplacian on the pseudosphere

$$\nabla^2 f = \frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\left(\sinh\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sinh^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

Hence we can obtain the equation of motion in the same form as the spherical case.

APPENDIX B: REDUCTION OF FUNCTIONAL INTEGRAL

We show how to reduce the Gaussian functional integral. We introduce the notation : $\mathbf{G} = \frac{d\mathbf{J}}{dt} + \mathbf{A}(\mathbf{J})$. First we use a functional identity:

$$\det\left[\frac{\delta\mathbf{G}(t)}{\delta\mathbf{J}(t)}\right]_{G=c}\int\prod_t\delta[\mathbf{G}(\mathbf{J}) - \mathbf{c}]D[\mathbf{J}(t)] = 1, \quad \text{(B1)}$$

then we have the functional integral including the the Jacobian factor [36,37]:

$$K = \int\det\left[\frac{\delta\mathbf{G}(t)}{\delta\mathbf{J}(t)}\right]_{\mathbf{G}=\mathbf{c}}\int\prod_t\delta[\mathbf{G}(t) - \mathbf{c}(t)]D[\mathbf{J}(t)]$$
$$\times\exp\left[-\frac{1}{2h}\int\mathbf{c}^2(t)dt\right]D[\mathbf{c}(t)]. \quad \text{(B2)}$$

Here we can remove the constraint that attaches to the determinant, namely, $\mathbf{G} = \mathbf{c}$. Because of the well known relation for the δ function $c(x)\delta(x-a) = c(a)\delta(x-a)$, the determinant factor does not depend on $\mathbf{c}(t)$ any more. Furthermore, using an integral identity $\delta[g(x)] = \int\exp[i\lambda g(x)]d\lambda$, the above is written as a functional integration over three variables $\mathbf{J},\mathbf{c},\lambda$: Thus we have

$$K = \int\exp\left[i\int\lambda(t)\{\mathbf{G}(t) - \mathbf{c}(t)\}dt\right]$$
$$\times\exp\left[-\frac{1}{2h}\int\mathbf{c}^2(t)dt\right]D[\lambda(t)]$$
$$\times\det\left[\frac{\delta\mathbf{G}(t)}{\delta\mathbf{J}(t)}\right]D[\mathbf{c}(t)]D[\mathbf{J}(t)]. \quad \text{(B3)}$$

By accomplishing two Gaussian functional integrals for $\mathbf{c}$ and $\lambda(t)$, we obtain (28).

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