

Universality of the Ising and the $S = 1$ model on Archimedean lattices: A Monte Carlo determination

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The Ising models $S = 1/2$ and $S = 1$ are studied by efficient Monte Carlo schemes on the (3,4,6,4) and the (3,3,3,3,6) Archimedean lattices. The algorithms used, a hybrid Metropolis-Wolff algorithm and a parallel tempering protocol, are briefly described and compared with the simple Metropolis algorithm. Accurate Monte Carlo data are produced at the exact critical temperatures of the Ising model for these lattices. Their finite-size analysis provide, with high accuracy, all critical exponents which, as expected, are the same with the well-known 2D Ising model exact values. A detailed finite-size scaling analysis of our Monte Carlo data for the $S = 1$ model on the same lattices provides very clear evidence that this model obeys, also very well, the 2D Ising model critical exponents. As a result, we find that recent Monte Carlo simulations and attempts to define effective dimensionality for the $S = 1$ model on these lattices are misleading. Accurate estimates are obtained for the critical amplitudes of the logarithmic expansions of the specific heat for both models on the two Archimedean lattices.

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I. INTRODUCTION

The Ising model, and several of its generalizations, have been of central importance in the development of the theory of phase transitions and the formulation of the universality hypothesis [1,2]. According to this hypothesis, all critical systems with the same dimensionality, the same symmetry of the ordered phase, and the same number of order parameters are expected to share the same set of critical exponents. For the 2D Ising model (square and some other lattices) all critical exponents are known exactly [3–6]. These exponents are expected to be obeyed by the Ising model on all 2D lattices and also by all other models, which according to the above hypothesis are expected to belong to the same universality class.

In several cases, this expectation has been verified either by exact analytic solutions [6], or, with impressive accuracy [7–10], using Monte Carlo (MC) simulations and the theory of finite-size scaling (FSS) [11–13]. Furthermore, the issue of explicit finite-size expansion of the main thermodynamic functions, or their accurate numerical estimation, has been considered in several cases and is of substantial significance. Often this provides a tool to improve accuracy of the MC estimation of critical exponents, especially in cases complicated by the presence of logarithmic corrections [14]. In particular, the studies of critical amplitudes of the specific heat expansions [15–20] and the studies of universal and nonuniversal features of certain combinations of critical amplitudes of the order parameter [20–23] are very interesting topics. One of the well-known generalizations of the Ising model is the $S = 1$ model studied in this paper. Here, we will present a careful MC study verifying, with high accuracy, the universality hypothesis of the two models, and we will also report results related to the logarithmic expansions of the specific heat, for two different 2D lattices known as Archimedean lattices. Our motivation is also to test a recent MC study [24], that erroneously resulted in an attempt to define and estimate effective dimensionality for the $S = 1$ model on these lattices.

An Archimedean lattice is a graph of a regular tiling of the plane whose all corners are equivalent and are shared by the same set of polygons. Thus, we may denote each Archimedean lattice by a set of integers (p_1, p_2, \dots) , indicating, in cyclic order, the polygons meeting at a given vertex. As an example, the square lattice is the Archimedean lattice denoted by (4^4) . There exists 11 2D Archimedean lattices. In addition, they have dual lattices, three of which are Archimedean, the other eight are entitled Laves lattices [25]. Suding and Ziff presented precise thresholds for site percolation on eight Archimedean lattices determined by the hull-walk gradient-percolation simulation method [26]. Rigorous bounds for the bond percolation critical probability are determined for three Archimedean lattices by Wierman [27]. In addition, Scullard and Ziff showed that the exact determination of the bond percolation threshold for the Martini lattice can be used to provide approximations to the Kagome and $(3,12^2)$ lattices [28]. As illustrated by Suding and Ziff, the Archimedean lattices can be transformed to a square array of $N = L^2$ vertices and then apply periodic boundary conditions. For the computer simulation of the two models studied in this paper, we also apply periodic boundary conditions (PBC) and use their representation as illustrated in Fig. 3 of their paper [26]. The two Archimedean lattices (ALs), used in our study, are the $(3^4,6)$ and $(3,4,6,4)$ lattices illustrated in Fig. 1.

In 2005, Malarz *et al.* evaluated critical temperatures for the ferro-paramagnetic transition in the Ising model for five Archimedean lattices, utilizing Monte Carlo (MC) simulations [29]. Then $(3,4,6,4)$ and $(3^4,6)$ Archimedean lattices of the majority-vote model with noise are considered and studied through MC simulations and the order/disorder phase transition is observed in this study [30]. Later, Krawczyk *et al.* investigated the magnetic properties of a 2D periodic structure, which is topologically equivalent to the Archimedean $(3,12^2)$ lattice with Ising spins and antiferromagnetic nearest-neighbor interaction [31]. In 2010 Codello determined the exact Curie temperature for all 2D Archimedean lattices by making use

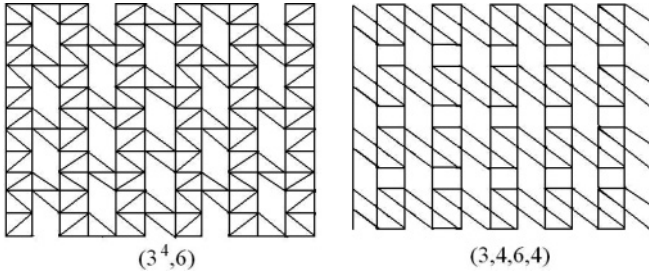


FIG. 1. Structure of the $(3^4,6)$ and $(3,4,6,4)$ Archimedean lattices (ALs). For the Monte Carlo simulations we used periodic boundary conditions.

of the Feynman-Vdovichenko combinatorial approach to the two-dimensional Ising model [32]. Recently, Lima *et al.* studied the critical properties of the Ising $S = 1/2$ and $S = 1$ models on $(3,4,6,4)$ and $(3^4,6)$ Archimedean lattices with MC simulations [24]. In this study they claimed that the Ising $S = 1$ model on $(3,4,6,4)$ and $(3^4,6)$ AL exhibits a second-order phase transition with critical exponents, which do not fall into universality class of the square lattice Ising $S = 1/2$ model.

The outline of the paper is as follows. In Sec. II, after the definition of the models, we present details of the MC schemes used in this paper and compare their performance with the simple Metropolis algorithm by illustrating the behavior of moving averages for the order parameter of the systems. In Sec. III, we present accurate MC data obtained at the exact critical temperatures [32] for the Ising model on the $(3,4,6,4)$ and $(3^4,6)$ ALs. Their FSS behavior yields with high accuracy all the expected critical properties of the universality class of the 2D Ising model. The logarithmic expansions of the corresponding specific heat on the two ALs are also studied. In Sec. IV, we estimate the critical temperatures and the corresponding critical properties of the $S = 1$ model on the $(3,4,6,4)$ and $(3^4,6)$ ALs with periodic boundary conditions. Estimates are given for the corresponding critical amplitudes of the specific heat expansions. This is carried out by observing the FSS behavior of several finite-size anomalies of the AL systems. Furthermore, the finite-size scaling behavior at the accurately estimated critical temperatures is discussed. In the conclusions, summarized in Sec. IV, we comment on previous observations concerning the critical behavior of the $S = 1$ model on these lattices.

II. DEFINITION OF THE MODELS, MONTE CARLO SCHEMES, AND FSS APPROACH

The (zero-field) Ising model is defined by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j, \quad (1)$$

with spin variables s_i taking on the values -1 or $+1$. As usual, $\langle ij \rangle$ indicates summation over all nearest-neighbor pairs of sites, and $J > 0$ for ferromagnetic exchange interaction. There is a variety of possible generalizations of the Ising model. Keeping only nearest-neighbor interactions one can generalize to a $S = 1$ model, including up to five interaction constants [33]. This is a rich model describing several phase transitions, critical and multicritical phenomena with a wide range of

physical applications. Special cases of this most general model are the well-known and extensively studied Blume-Capel (BC) model [34,35] and also the Blume-Emery-Griffiths (BEG) model [36]. For our purposes, it suffices to introduce only the above-mentioned generalization known as the BC model [34,35]. It is defined by introducing spin variables s_i that take on the values $-1, 0$, or $+1$, and a crystal field coupling Δ , so that the Hamiltonian is given by

$$H = -J \sum_{\langle ij \rangle} s_i s_j + \Delta \sum_i s_i^2. \quad (2)$$

This model is of particular importance for the theory of phase transitions and critical phenomena, since as is well known its phase diagram consists of a segment of continuous Ising-like transitions at high temperatures and for low values of the crystal field, which ends at a tricritical point, where it is joined with a second segment of first-order transitions between (Δ_t, T_t) and $(\Delta_0, T = 0)$. The BC model has been analyzed, besides the original mean-field theory [34,35], by a variety of approximations and numerical approaches, in both 2D and 3D. These include the real space renormalization group [37], MC simulations [38], and MC renormalization-group calculations [39], ϵ -expansion renormalization groups [40], high- and low-temperature series calculations [41], a phenomenological FSS analysis using a strip geometry [42,43], and MC simulations [44–49]. In particular, the 2D (mainly the square BC model) has been extensively studied and there is no doubt today that the continuous Ising-like transitions, along its second-order segment, obey the same critical properties with the 2D Ising model. Recently, a similar universality has been shown for its random-bond version [44]. The $S = 1$ model, studied in this paper, is the 2D BC model at zero crystal field and, therefore, it belongs to the same universality class with the 2D Ising model. Of course, this universality should be expected to hold also for all Archimedean lattices.

It is well known that the accuracy of MC data may be decisive for a successful FSS estimation of critical properties. Over the years, the numerical estimation of critical exponents has been a nontrivial exercise, even for the simpler models, such as the Ising model. An importance sampling approach, close to a second-order phase transition, requires appropriate use of cluster algorithms that can efficiently overcome the well-known effects of critical slowing down. Wolff-type algorithms [50–52] are easy to implement and very efficient close to the critical point. The Wolff algorithm will be implemented, in the present paper, to simulate both the Ising and the $S = 1$ model. However, for the $S = 1$ model, the Wolff algorithm cannot be used alone, because Wolff steps act only on the nonzero spin values. A suggested practice is now a hybrid algorithm along the lines followed by Ref. [53]. Since, we wanted to use a unified code for both models the hybrid approach was tested and implemented for both models. An elementary Monte Carlo step of this scheme consist of a number of Wolff steps (typically five Wolff steps) followed by a Metropolis sweep of the lattice. The combination with the Metropolis lattice sweep is dictated by the fact that the Wolff steps act only on the nonzero spin values.

Thus, we have simulated both the Ising model and the $S = 1$ model on the two ALs by implementing the same

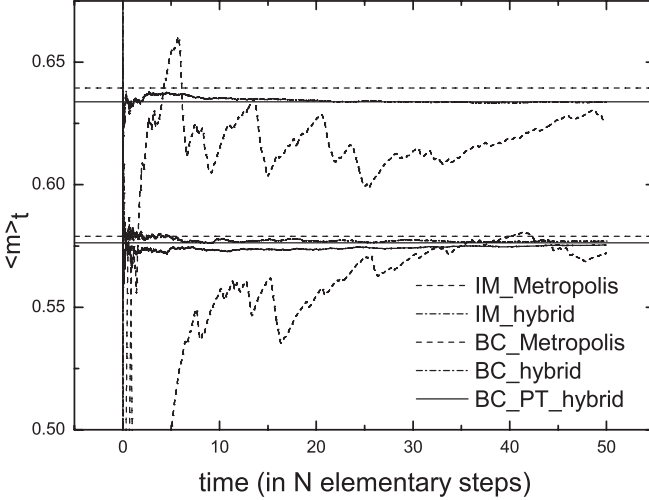


FIG. 2. Behavior of moving averages for the order parameter close to the critical temperatures for both the Ising model (IM upper part of the graph) and the $S = 1$ model (BC) on the same (3,4,6,4) AL with $L = 48$. The dashed and continuous straight lines give averages over 20 independent runs for the Metropolis algorithm (dashed lines) and the hybrid and PT-hybrid approach (continuous lines). The Metropolis algorithm has a very slow equilibration. For the time unit the elementary steps as defined in the text.

hybrid approach described above. For the Ising model case, we carried out simulations only at the exactly known critical temperatures [32], whereas for the $S = 1$ model we generated MC data to cover several finite-size anomalies. In this case, the hybrid approach was carried over to a certain temperature range depending on the lattice size. Furthermore, for this case, we found it convenient and of comparable accuracy to implement a parallel tempering (PT) protocol, based on temperature sequences corresponding to an exchange rate 0.5. This PT approach is very close to the practice suggested recently in Ref. [54]. The temperature sequences were generated by short preliminary runs. Using such runs and a simple histogram method [50,51], the energy probability density functions can be obtained and from these the appropriate sequences of temperatures can be easily determined [54].

The superiority of the hybrid approach, over a simple Metropolis scheme [55], is illustrated in Fig. 2. This figure is constructed by using moving averages for the order parameter $\langle m \rangle_t$ close to the corresponding critical temperatures for both the Ising model (denoted in the panel as IM) and the $S = 1$ model (denoted in the panel as BC) on the same (3,4,6,4) AL of linear size $L = 48$ and $N = L^2$ vertices. As can be seen from this illustration, the Metropolis algorithm suffers from very strong fluctuations. It follows a very slow approach to equilibrium and only with the help of heavy sampling (the dashed lines give the average over 20 independent Metropolis runs) its results are of reasonable accuracy. On the other hand, the hybrid approach converges very fast to equilibrium and produces accurate results even in only one single run. Here it should be noted that, both the simple hybrid approach and its combination with parallel tempering (in a convenient temperature range) give in 20 independent runs the same

results (with comparable accuracy) and these are indicated by the continuous straight lines in the panel of Fig. 2.

Let us discuss now the FSS tools used in this paper for the estimation of critical properties of the systems. In order to estimate the critical temperature, we follow the practice of simultaneous fitting approach of several pseudocritical temperatures [44]. From the MC data, several pseudocritical temperatures are estimated, corresponding to finite-size anomalies, and then a simultaneous fitting is attempted to the expected power-law shift behavior $T = T_c + bL^{-1/\nu}$. The traditionally used specific heat and magnetic susceptibility peaks, as well as, the peaks corresponding to the following logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature $K = 1/T$ [8],

$$\frac{\partial \ln \langle M^n \rangle}{\partial K} = \frac{\langle M^n H \rangle}{\langle M^n \rangle} - \langle H \rangle, \quad (3)$$

and the peak corresponding to the absolute order-parameter derivative

$$\frac{\partial \langle |M| \rangle}{\partial K} = \langle |M| H \rangle - \langle |M| \rangle \langle H \rangle, \quad (4)$$

will be implemented for a simultaneous fitting attempt of the corresponding pseudocritical temperatures. Furthermore, the behavior of the crossing temperatures of the fourth-order Binder cumulants [56], and their asymptotic trend, has been observed and utilized for a safe estimation of the critical temperatures.

The above described simultaneous fitting approach provides also an estimate of the correlation length exponent ν . An alternative estimation of this exponent is obtained from the behavior of the maxima of the logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature, since these scale as $L^{1/\nu}$ with the system size [8]. If the exponent ν has been estimated, then the behavior of the values of the peaks corresponding to the absolute order-parameter derivative, which scale as $L^{(1-\beta)/\nu}$ with the system size [8], gives one route for the estimation of the exponent ratio β/ν . Further, knowing the exact critical temperatures, or very good estimates of them, we can utilize the behavior of the order parameter at the critical temperatures for the traditional and effective estimation of the exponent ratio β/ν . Summarizing, our FSS approach utilizes, besides the traditionally used specific heat and magnetic susceptibility maxima, the above four additional finite-size anomalies for the accurate estimation of the critical temperature and critical exponents.

III. THE ISING MODEL ($S = 1/2$) ON THE (3,4,6,4) AND (3⁴,6) LATTICES

This section presents the FSS analysis for the Ising model on the two AL. The analysis is carried out only at the exactly known critical temperatures. For the (3,4,6,4) lattice the exact critical temperature is $T_c = 2.1433 \dots$ [32], and for each lattice size ($L = 18, 24, 30, 48, 54, 60, \dots, 138, 144, 150$) we carried out 20 independent runs of the hybrid Metropolis-Wolff algorithm at this temperature. The same number of independent runs was carried out for the (3⁴,6) lattice at

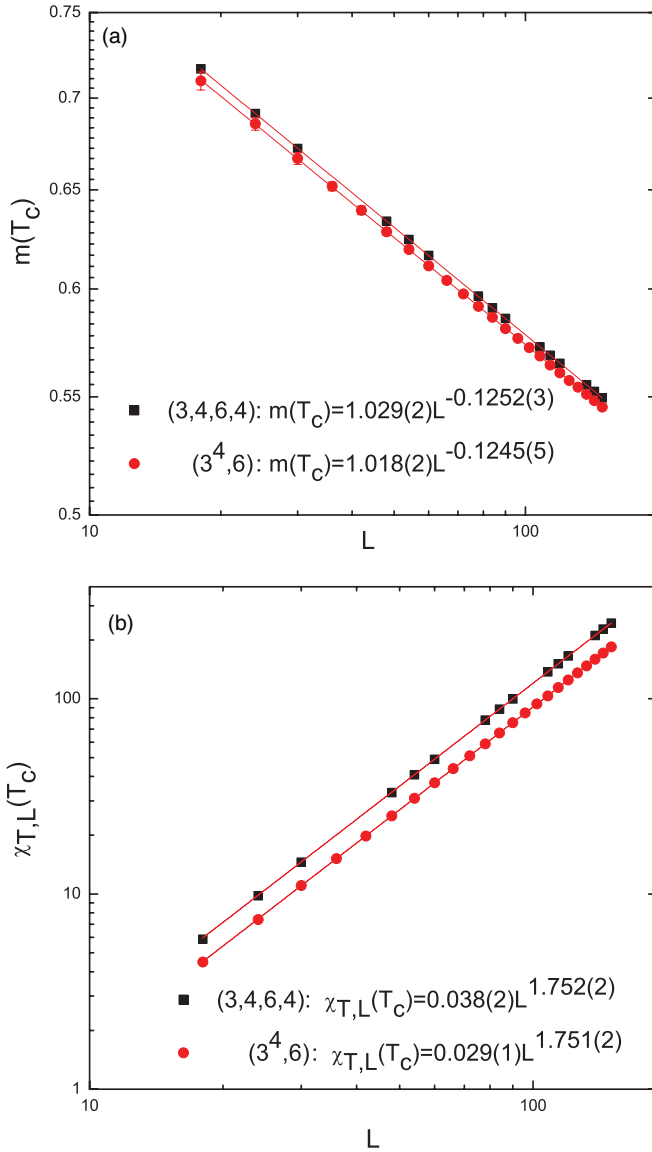


FIG. 3. (Color online) (a) Finite-size behavior of the order parameter at the exact critical temperatures, illustrated in a logarithmic scale, for both ALs. In the panel we show a simple power law estimation for the exponent ratio β/ν . (b) The same for the FSS behavior of the susceptibility. In the panel we show a simple power law estimation for the exponent ratio γ/ν .

the corresponding exact critical temperature ($T_c = 2.7858 \dots$ [32]). In this case, we have used a more dense sequence of lattice sizes (a six-step sequence): $L = 18, 24, 30, 36, \dots, 150$. We also give an indication of our practice for the number of sweeps used in our final runs. For each independent run we used for averaging $3 \cdot 10^5$ elementary Monte carlo steps (five Wolff-steps followed by a Metropolis sweep) of the hybrid scheme for the lattice with linear size $L = 108$ and $5 \cdot 10^5$ for the lattice of size $L = 150$. Equilibration periods were approximately a third of the corresponding averaging time.

We start the presentation of our FSS attempts by illustrating the behavior of the order parameter at the exact critical temperatures on the two AL. This behavior is illustrated, in a logarithmic scale, in Fig. 3(a). In the panel of this figure,

we show a simple power-law estimation for the exponent ratio β/ν . This simple estimation gives an accuracy to the third significant figure of the exact critical exponent ratio $\beta/\nu = 0.125$. We point out that the fitting parameters are not sensitive to fitting lattice-range used ($L = 18-150$) and are almost identical if we do not include, in the fitting attempts, the statistical errors shown in the panel. The errors bars shown were calculated by the jackknife method for each run. The estimated errors for 20 independent runs are shown in Fig. 3(a) and were used in the fitting attempt. For all other (diverging) thermodynamic parameters, such as the susceptibility, the corresponding jackknife errors are again very small, smaller than the symbol sizes, and are therefore omitted in the sequel. We continue by presenting now the estimation of the exponent ratio that characterizes the divergence of the susceptibility at the critical temperature. This behavior is illustrated, again in a logarithmic scale, in Fig. 3(b). In the panel of this figure we show a simple power-law estimation for the exponent ratio γ/ν . For both lattices the simple power law gives again an accuracy to the third significant figure of the exact critical exponent ratio $\gamma/\nu = 1.75$.

The critical exponent of the correlation length can be estimated from the behavior of the logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature. As pointed out earlier, these scale as $L^{1/\nu}$ with the system size [8] and their behavior provides an alternative route for the estimation of the correlation length critical exponent. Their behavior is illustrated in Figs. 4(a) and 4(b), respectively, for the two ALs. Our practice here is to use a simultaneous fitting attempt to a simple power law for the three cases $n = 1, 2, 4$ in each lattice. In Figs. 4(a) and 4(b) we show that a simple power-law estimation provides the estimates $1/\nu = 1.02(2)$ for both lattices. However, we point out that the fitting attempts are significantly improved here if we include suitable correction terms. One possibility is to include a constant term that does not, of course, affect the divergences at the critical temperatures. As shown in Figs. 4(a) and 4(b) the estimates are now $1/\nu = 1.001(2)$ for the (3,4,6,4) and $1/\nu = 0.998(2)$ for the (3⁴,6) AL, giving strong and clear evidence of $\nu = 1$.

We close this section by discussing the FSS of the specific heat at the exact critical temperatures for the two ALs. From the work of Ferdinand and Fisher [15] we know the characteristic specific heat expansion for the square lattice Ising model. Details of the size expansions on this lattice, but also on some other 2D lattices (plane triangular and honeycomb lattices) have been published in a number of papers [15–19]. This is an interesting topic, and one should expect that similar expansions are to be obeyed for all ALs. In Fig. 5 we illustrate the expected logarithmic divergence of the specific heat at the exact critical temperatures for the two ALs. Our fitting attempts have been restricted to the leading behavior $C_L(T_c) = B_c + A_0 \ln(L)$, which avoids expected higher order (L^{-1}, L^{-2}, \dots) correction terms [19]. This practice sidesteps problems of competition between small, but unavoidable, statistical errors and small correction terms. In Fig. 5 we show the estimated critical amplitudes A_0 for the two ALs studied here and also the constant B_c contributions. The estimation has been done by using the full size range $L = 18-150$. However, as mentioned, because of statistical errors and small higher-order

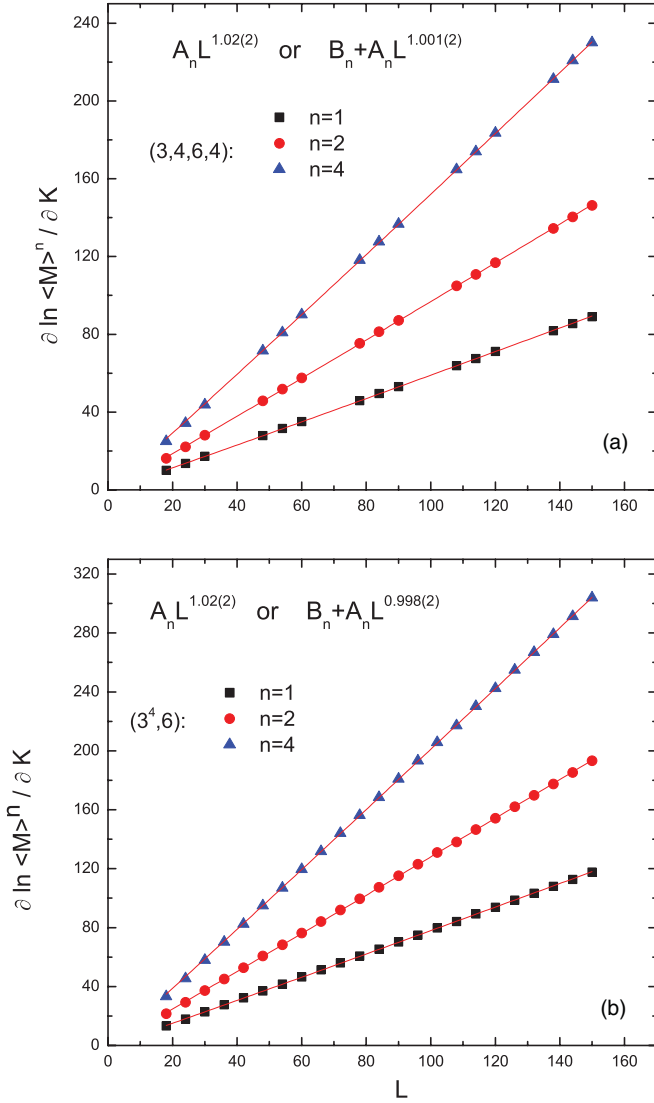


FIG. 4. (Color online) (a) FSS behavior of the logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature for the $(3,4,6,4)$ AL. Simultaneous fitting attempts to a simple power law and to a power law with a constant correction term are shown in the panel (a). (b) The same as described for (a) but for the $(3^4,6)$ lattice.

corrections these estimations show some sensitivity to the size-range used. Observing the asymptotic trend, our best estimates for the critical amplitudes are $A_0 = 0.450(8)$ for the $(3,4,6,4)$ lattice and $A_0 = 0.464(8)$ for the $(3^4,6)$ lattice. The constant contributions are more sensitive and our moderate estimates are of the order of $B_c = 0.15(3)$ for both ALs. For comparison the leading behavior for the square lattice Ising model, $C_L(T_c) = 0.138149\dots + 0.494538\dots \ln(L)$, is illustrated in the same figure by the dashed line. We point out that the small distance between the estimates for the critical amplitudes of the two ALs should not be taken as a sign indicating a possible equality. For instance a similar situation can be found between the square and plane triangular lattices with amplitudes $A_0 = 0.494538\dots$ and $A_0 = 0.499069\dots$, respectively [18]. In conclusion, as expected, all critical properties of the 2D Ising model, critical exponents and critical

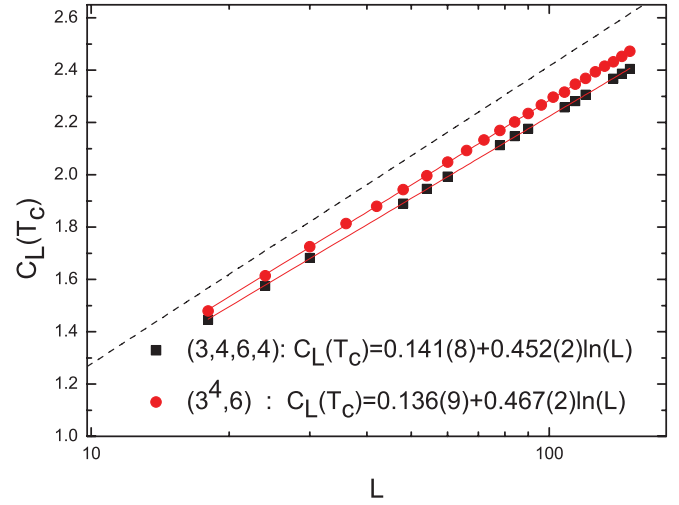


FIG. 5. (Color online) FSS behavior of the specific heat at the exact critical temperatures for the two ALs, illustrated in semilogarithmic scale. This behavior is described in detail in the text and is compared here with the dashed line, which describes the FSS behavior of the specific heat, at the exact critical temperature, of the Ising model on the square lattice with periodic boundary conditions.

expansions, are well obeyed on the two Archimedean lattices studied here.

IV. THE $S = 1$ MODEL ON THE $(3,4,6,4)$ AND $(3^4,6)$ LATTICES

This section presents the critical properties of the $S = 1$ model on the $(3,4,6,4)$ and $(3^4,6)$ ALs. The MC data were generated by the combination of the hybrid approach with the PT protocol, described in Sec. II, and we have averaged over five independent runs, in the appropriate temperature ranges, and use linear sizes $L = 30, 36, 48, 54, 60, 66, 72, 78, 84, 90, 96, 108, 120$ for the $(3,4,6,4)$ lattice and $L = 36, 48, 60, 72, 84, 96, 108, 120$ for the $(3^4,6)$ lattice. As discussed in Sec. II, the second-order transition of this model between the ferromagnetic and paramagnetic phases is expected to be in the universality class of the simple 2D Ising model. We will verify this expectation and contrast our findings with those in the report of Ref. [24].

Figure 6 presents the shift behavior of several pseudocritical temperatures for the $S = 1$ model on the two ALs [Figs. 6(a) and 6(b)]. These temperatures correspond to the peaks of the following six quantities: specific heat, magnetic susceptibility, inverse temperature derivative of the absolute order parameter, and inverse temperature logarithmic derivatives of the first, second, and fourth powers of the order parameter. The data are fitted, in the corresponding size ranges ($L = 30 - 120$ and $L = 36 - 120$) to the expected power-law behavior $T = T_c + bL^{-1/\nu}$ and the resulting estimates of the critical temperatures and $1/\nu$ are given in the panels. To some degree these estimates are sensitive to the size range used and to statistical errors. However, by varying the size ranges and also observing the asymptotic trend of the crossing temperatures of the fourth-order Binder cumulants we have with confidence estimated that the critical temperatures are $T_c = 1.62115(55)$ for the $(3,4,6,4)$ lattice and $T_c = 2.08605(15)$ for the $(3^4,6)$

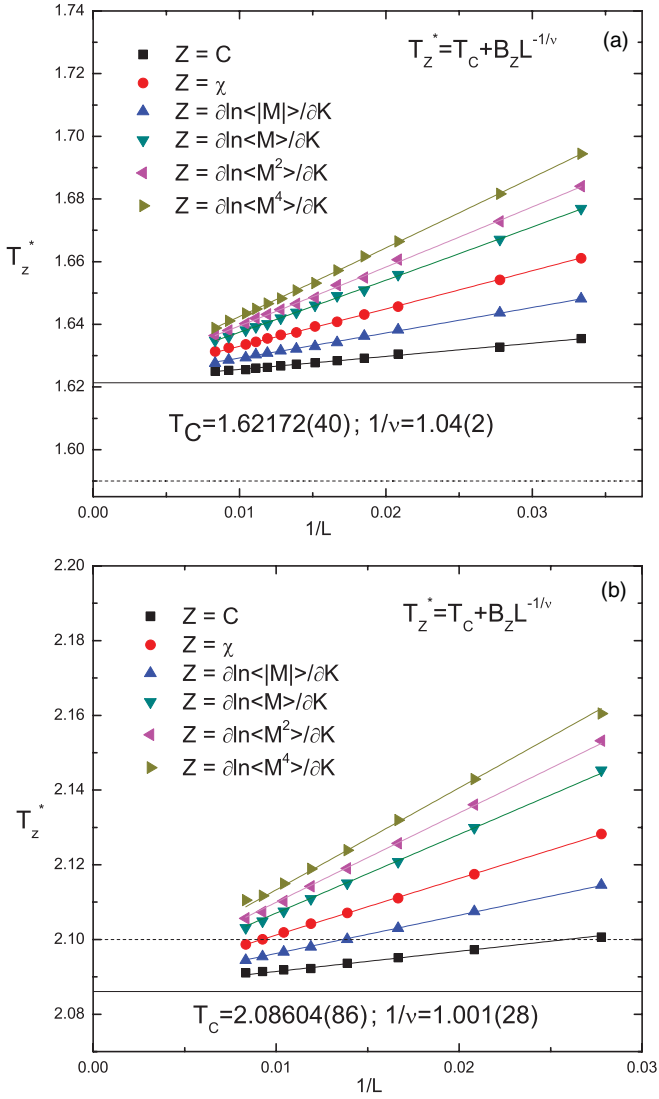


FIG. 6. (Color online) FSS behavior of the six pseudocritical temperatures defined in the text for the two ALs [panels (a) and (b) for the $(3,4,6,4)$ and $(3^4,6)$, respectively] for the $S = 1$ model. Corresponding estimates for the critical temperatures and $1/\nu$, of the illustrated fitting attempts, are given in the panels (a) and (b) and further discussed in the text. Our final estimates of the critical temperatures (see the text) are shown by the continuous straight lines in the corresponding panels, whereas dashed straight lines indicate the estimates for the critical temperatures of Ref. [24].

lattice. These values are indicated by the continuous straight lines in the corresponding panels of Fig. 6. With dashed straight lines we have also indicated the respective estimates for the critical temperatures $T_c = 1.590(3)$ and $T_c = 2.100(3)$ of Ref. [24].

Their estimates for T_c are in serious errors and so are the values $1/\nu = 0.83(5)$ and $1/\nu = 0.94(5)$ for the correlation length exponent reported in Ref. [24]. On the other hand, our estimates for this exponent, as shown in Figs. 6(a) and 6(b), are clear indications of the universality mentioned above. A further verification for this is the behavior of the logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature. Their behavior is

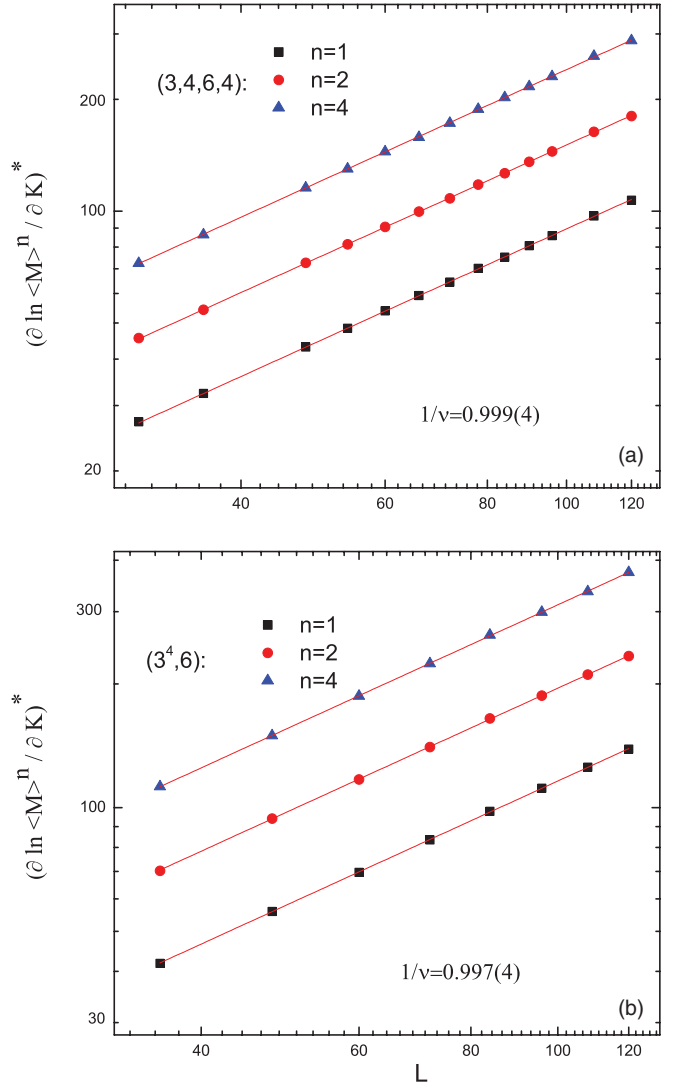


FIG. 7. (Color online) FSS behavior, illustrated in a logarithmic scale, of the peaks of the logarithmic derivatives of the powers $n = 1, 2, 4$ of the order parameter with respect to the inverse temperature for the two ALs for the $S = 1$ model. Corresponding estimates for the exponent $1/\nu$ are given in the panels by applying simultaneous fitting attempts to a simple power law.

illustrated in Figs. 7(a) and 7(b), respectively, for the two ALs. The estimates from the simultaneous fitting attempt are shown in the corresponding panels. Figures 8 and 9, present our estimations for the magnetic exponent ratios γ/ν and β/ν , obtained from the analysis of the corresponding finite-size anomalies (peaks). Again, they provide very strong verification of the expected universality. In conclusion, our results for the 2D BC model at $\Delta = 0$ are in full agreement with the universality arguments that place the BC model in the Ising universality class.

We now turn to the interesting topic of estimating the critical amplitudes of the logarithmic-specific heat expansions for the $S = 1$ model on the $(3,4,6,4)$ and $(3^4,6)$ AL. In Fig. 10 we plot the expected logarithmic divergences of the specific heat at the specific heat's pseudocritical temperatures for the two

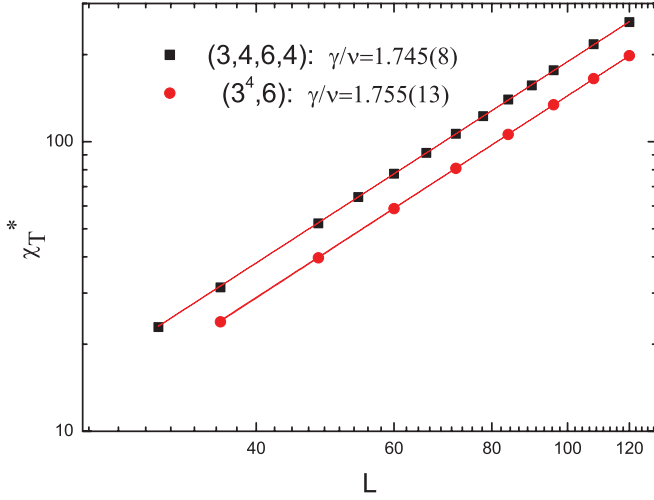


FIG. 8. (Color online) FSS behavior of the magnetic susceptibility maxima illustrated in a logarithmic scale for both ALs for the $S = 1$ model. In the panel we show simple power-law estimations for the exponent ratio γ/ν .

lattices. Again, and for similar reasons, our fitting attempts are restricted to the leading behavior $C^* = B^* + A_0 \ln(L)$. The estimates for the critical amplitudes, given in the panels, are very close to each other for the two ALs, but as mentioned earlier there are not any reasons to expect their equality. Since the estimate for the critical temperature [$T_c = 2.08605(15)$] for the $(3^4,6)$ AL appears to be accurate to at least five significant figures, we have undertaken for this lattice 20 independent runs using the hybrid approach only at this temperature. Figure 11 illustrates and contrast the expected logarithmic divergences of the specific heat at the specific heat's pseudocritical temperature and at the critical temperature $T_c = 2.08605(15)$ for the $S = 1$ model on the $(3^4,6)$ AL. It is notable here that the estimations in Fig. 11 for the critical amplitude are obtained by applying two independent fits, whereas a simultaneous fitting attempt gives $A_0 = 0.7103(46)$. This appears to be

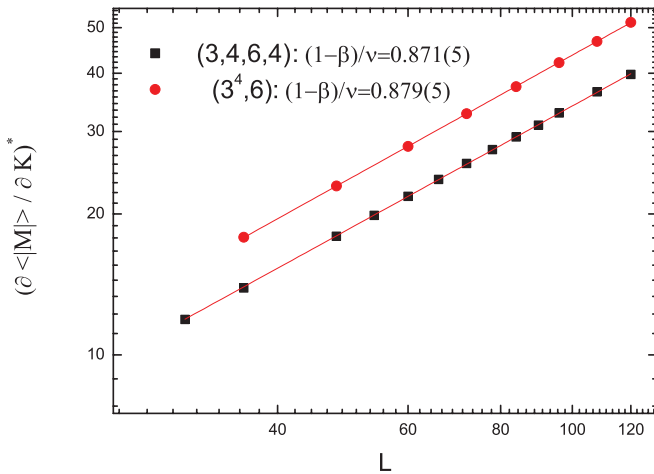


FIG. 9. (Color online) Estimations for the magnetic exponent ratio β/ν for both ALs for the $S = 1$ model, obtained from the analysis of the corresponding finite-size anomalies (absolute order-parameter derivative). In the panel we show simple power-law estimations for this exponent.

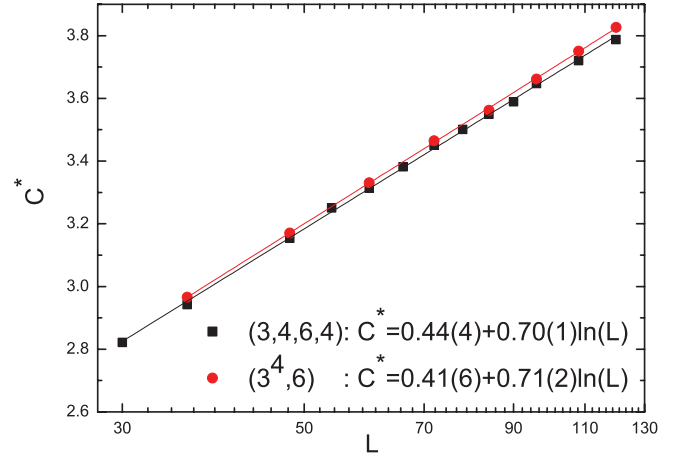


FIG. 10. (Color online) FSS behavior of the specific heat at the specific heat's pseudocritical temperatures (specific heat peaks) for the two ALs, for the $S = 1$ model, illustrated in semilogarithmic scale. This behavior is further discussed in text.

an accurate estimation and the values in the panels (which should be equal) are both within its error limits. This critical amplitude for the $S = 1$ model can be compared with the corresponding value $A_0 = 0.464(8)$ for the Ising model on the same $(3^4,6)$ Archimedean lattice. A similar project was carried out for the $(3,4,6,4)$ AL and our best estimate for the critical amplitude is $A_0 = 0.700(9)$, which now should be compared to the corresponding value $A_0 = 0.450(8)$ for the Ising model on the same AL. Let us close this section by pointing out that from the MC data, at the estimated critical temperatures for the $S = 1$ model on the two ALs, we also carried out the estimation of all critical exponents, by using the FSS tools mentioned in Sec. II. All obtained estimates were in excellent agreement verifying not only the expected universality but also the accuracy of the estimated critical temperatures.

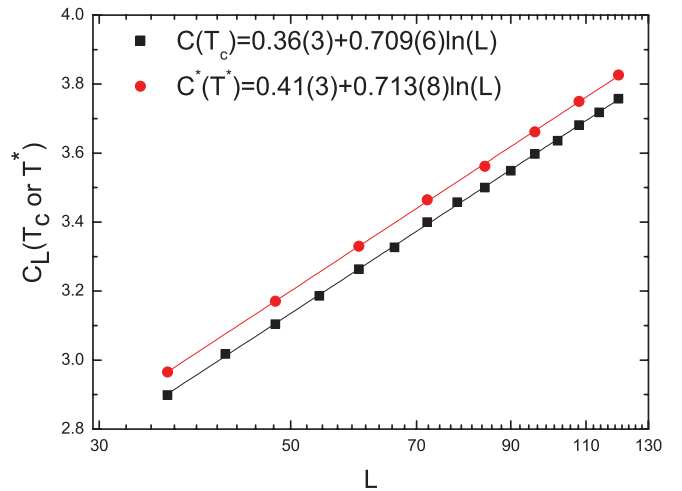


FIG. 11. (Color online) FSS behavior of the specific heat at the specific heat's pseudocritical temperature and at the estimated exact critical temperature for the $(3^4,6)$ AL, for the $S = 1$ model, illustrated in semilogarithmic scale. The critical amplitude is further discussed in text.

V. CONCLUSIONS

The Ising $S = 1/2$ and $S = 1$ models have been studied on two Archimedean lattices by an efficient Monte Carlo scheme, using a hybrid Wolff-Metropolis approach. The Ising model was analyzed by finite-size scaling at the exact critical temperatures. We verified, with high accuracy, all critical exponents of the well-known 2D Ising model exact values. For the $S = 1$ model, on the same lattices, we presented very clear evidence that this model obeys, also very well, the 2D Ising model critical exponents. Our results are in full agreement with the general universality arguments that place these models on all 2D lattices in the 2D Ising universality class. In conclusion, we have disclosed any questions raised

by the recent attempt [24] to estimate critical exponents on these lattices, for the $S = 1$ model, and define effective dimensionality. Their results, most likely, suffer from strong critical slowing down effects, due to the simple heat bath algorithm implemented by these authors. In addition, we have provided reliable results for the characteristic specific heat expansions on the two Archimedean lattices studied here for both models.

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