Efficiency at maximum power output of quantum heat engines under finite-time operation

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We study the efficiency at maximum power, η_m, of irreversible quantum Carnot engines (QCEs) that perform finite-time cycles between a hot and a cold reservoir at temperatures T_h and T_c, respectively. For QCEs in the reversible limit (long cycle period, zero dissipation), η_m becomes identical to the Carnot efficiency $\eta_C = 1 - T_c/T_h$. For QCE cycles in which nonadiabatic dissipation and the time spent on two adiabats are included, the efficiency η_m at maximum power output is bounded from above by $\eta_C/(2-\eta_C)$ and from below by $\eta_C/2$. In the case of symmetric dissipation, the Curzon-Ahlborn efficiency $\eta_{\rm CA} = 1 - \sqrt{T_c/T_h}$ is recovered under the condition that the time allocation between the adiabats and the contact time with the reservoir satisfy a certain relation.

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I. INTRODUCTION

The concept of Carnot efficiency is of paramount importance in thermodynamics, since the Carnot cycle is the most efficient heat engine cycle allowed by physical laws. When the thermodynamic second law states that not all the supplied heat is applied to producing work, the Carnot efficiency presents the limiting value on the fraction of the heat which can be so used. Although the quasistatic Carnot cycle has the highest efficiency, it outputs zero power because it takes infinite time to output a finite amount of work. By contrast, Curzon and Ahlborn [1] considered a finite-time Carnot cycle under the assumption of endoreversibility that irreversible processes occur only through these heat exchanges, they obtained the efficiency $\eta_{\rm CA}$ at maximum power output as

$$\eta_{\rm CA} = 1 - \sqrt{\frac{T_c}{T_h}}, \tag{1}$$

where T_h and T_c are the temperatures of the hot and cold heat reservoirs, respectively. The Curzon-Ahlborn (CA) paper has triggered the development of research into finite-time thermodynamics [2–19]. The above $\eta_{\rm CA}$ is usually called the CA efficiency, describing the efficiency of several engine models [2,16,18,20] and of actual thermal plants [1,2,6–8,10] very well. The CA efficiency has been found to be a universal result in the case of low and asymmetric dissipation, by optimizing the power output with respect to time allocation when the time durations in adiabats and nonadiabatic phenomena were ignored [8].

Great efforts have been devoted to the study of quantum heat engines [4,5,10–12,21–29], beginning with the concept of the quantum heat engine introduced by Scovil and Schulz-DuBois [30]. Quantum heat engines differ from their classical counterparts mainly in the following three respects: (i) the working substance is composed of quantum matter such as spin systems [4,5,10,11,26], harmonic oscillator systems [5,10,12], two-level or multilevel systems [21–24], cavity quantum electrodynamics systems [24,27–29], etc. (ii) The state of the system is depicted by a quantum-mechanical operator, and the thermodynamic observables are associated with the expectation values of operators [10–12]. (iii) Quantum equations of motion are used to describe the time evolution of the observables in quantum heat engines, which can avoid the use of phenomenological heat transfer laws [10–12].

The previous literature discussed the heat engine models in the sudden limit in which the adiabatic process is a spontaneous switching and thus the time allocation on adiabats is negligible [8,16,17]. Thus it is significant to study more general models in which the "adiabatic" process (we take the two corresponding processes as two quantum "adiabats" throughout the paper) takes finite time as well as becoming nonadiabatic [31]. During a quantum adiabatic process, the variation of the eigenspectrum (quantum state) of the system must be so slow that the quantum adiabatic theorem [22–24,32] can apply. Otherwise, nonadiabatic dissipation (e.g., inner friction [11,12,26]) occurs because of rapid change in the energy level structure of the quantum system. In particular, nonadiabatic dissipation has been found to have a profound influence on the performance of quantum heat engines [11,12,26]. The inclusion of nonadiabatic dissipation is therefore essential for more realistic models of quantum heat engines.

In this paper, we study the efficiency at maximum power output of quantum Carnot engines (QCEs) performing finite-time cycles, in which the times taken by any adiabat and nonadiabatic dissipation are considered. We assume that the external parameter affecting the energy spectrum varies at a small but fixed speed which, however, may not be slow enough and thus might cause nonadiabatic phenomena. We derive the cycle period which consists of times spent both on the two quantum isotherms and on the two quantum adiabats. We show that the efficiency at maximum power output converges to an upper and a lower bound in the limits of extremely asymmetric dissipation. Based on the low-dissipation assumption that the irreversible entropy production in a thermodynamic process is inversely proportional to the time required to complete that process, our approach, similar to that of classical thermodynamics, predicts that the CA efficiency is an exact and universal property for QCEs operating under the conditions that the dissipation is symmetric and the time allocation between the

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adiabats and the contact time with the reservoir satisfy a certain relation.

II. EFFICIENCY AT MAXIMUM POWER OUTPUT

We consider a quantum system whose Schrödinger equation is given by $H|u_n\rangle = E_n|u_n\rangle$, where H, $|u_n\rangle$, and E_n are the Hamiltonian of the system, its nth eigenstate, and its eigenenergy, respectively. The internal energy U reads $U = \sum_n E_n P_n$, where P_n is the mean occupation probability of the nth eigenstate and obeys the canonical distribution $P_n = \frac{1}{Z}e^{-E_n/k_BT}$ in equilibrium, with the canonical partition function $Z = \sum_n e^{-E_n/k_BT}$. The derivation of U leads to the first law of quantum thermodynamics $dU = \sum_n E_n dP_n + \sum_n P_n dE_n$. Analogous to the classical thermodynamic first law, the first law of thermodynamics in quantum-mechanical systems is [23–25] $dU = đQ + đW$, in which $đQ = \sum_n E_n dP_n$ and $đW = \sum_n P_n dE_n$ depict the heat exchange and work done, respectively, during a thermodynamic process. Note that $\sum_n E_n dP_n$ is associated with the heat exchange because $đQ = TdS$ with the entropy $S = -k_B P_n \ln P_n$. Motivated by the definition of the generalized force F for a classical system, we define analogously the force for a quantum system as $F = \sum_n P_n \frac{\partial E_n(X)}{\partial X}$, where X is the external parameter (the generalized coordinate corresponding to the force F) [23–25,33]. Here the force F and generalized coordinate X are state variables [25,33] and quantum versions of the classical pressure P_r and volume V, respectively.

The generalized force $F(X)$ alters the generalized coordinate X that affects the eigenspectrum of the system during a thermodynamic process. The quantum Carnot cycle $1 \to 2 \to 3 \to 4 \to 1$ is drawn in the (F,X) plane, which is the quantum version of the classical (P_r,V) plane [see Fig. 1(a)]. During two quantum isothermal processes $1 \to 2$ and $3 \to 4$, the working substance is coupled to a hot and a cold heat reservoir at constant temperatures T_h and T_c, respectively. We apply $đQ = TdS$ directly to the calculation of the heat exchange $đQ$ in any quantum isothermal process. Let $S(X_i)$ and X_i be the entropies and the external parameters at the instants i with $i = 1,2,3,4$. The heat amount Q_h absorbed from the hot reservoir and the heat amount Q_c released to the cold reservoir are, respectively, $Q_h = T_h[S(X_2) - S(X_1)]$ and $Q_c = T_c|[S(X_4) - S(X_3)]|$. When the Carnot cycle is reversible, the system couples to the heat reservoir for a sufficiently long time until the system remains infinitesimally close to equilibrium throughout the cycle, and in order for the adiabatic theory to remain valid, the time scale of the change of the quantum state must be much larger than that of the dynamical one, $\sim E/\hbar$ [22,23,32]. For the reversible cycle where $S(X_2) = S(X_3)$ and $S(X_1) = S(X_4)$, we recover the Carnot efficiency $\eta_C = 1 - \frac{T_c}{T_h}$, which is independent of the properties of the quantum working substance. We denote by t_{12} (t_{34}) the time durations during which the system is coupled to the hot (cold) reservoir during a cycle. In the branch $2 \to 3$ ($4 \to 1$), the working substance is decoupled from the hot (cold) reservoir, and the entropy changes from S_2 to S_3 (S_4 to S_1) during a period t_{23} (t_{41}), as shown in Fig. 1(b), where $S_i \equiv S(X_i)$ with $i = 1,2,3,4$.

Let us consider a QCE under finite-time operation. Finite-time cycles move the system away from equilibrium, leading to irreversibility of the engine. Although the system need not be close to equilibrium during the isothermal process, it remains in an equilibrium state with the heat reservoir at special instants i with $i = 1,2,3,4$. Under such a circumstance, the thermodynamic quantities of the system—in particular the entropy—are well defined at these instants. During the processes $2 \to 3$ and $4 \to 1$, the rapid change (compared with the time scale $\sim E/\hbar$) in the energy level structure of the system results in quantum nonadiabatic phenomena. We emphasize that in any quantum thermodynamic process in which low dissipation exists, the system relaxation is assumed to be fast compared to the time of the process [8]. During a quantum isothermal (adiabatic) process, the entropy production caused by weak dissipation can be written as Σ_h/t_{12} or Σ_c/t_{34} (Σ_a/t_{23} or Σ_a/t_{41}), since the reversible regime is approached in the infinite-time duration. Thus, the real heat exchanges Q_h and Q_c are $Q_h = T_h\Delta S - T_h\frac{\Sigma_h}{t_{12}}$ and $Q_c = T_c\Delta S + T_c\frac{\Sigma_c}{t_{34}} + T_c(\frac{\Sigma_a}{t_{23}} + \frac{\Sigma_a}{t_{41}})$, with $\Delta S = S(X_2) - S(X_1) = |[S(X_4) - S(X_3)]| - (\frac{\Sigma_a}{t_{23}} + \frac{\Sigma_a}{t_{41}})$. According to the first law of the thermodynamics, the work W per cycle is determined by

$$W = Q_h - Q_c = (T_h - T_c)\Delta S - T_h\frac{\Sigma_h}{t_{12}} - T_c\frac{\Sigma_c}{t_{34}} - T_c\left(\frac{\Sigma_a}{t_{23}} + \frac{\Sigma_a}{t_{41}}\right). \quad (2)$$

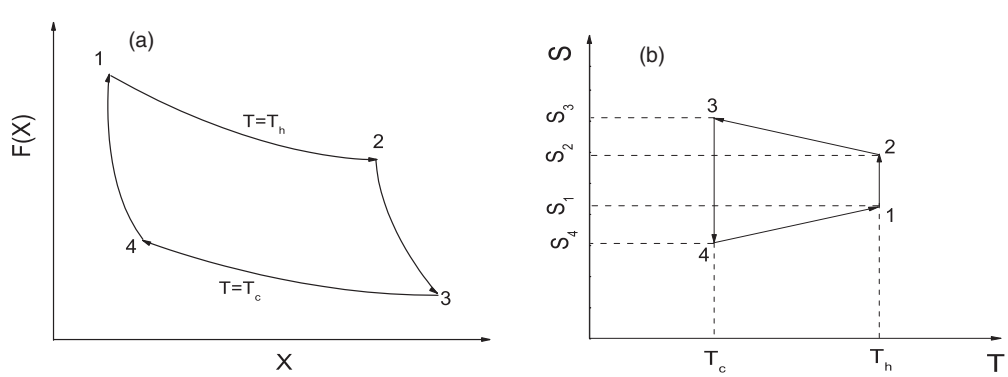

FIG. 1. Schematic diagram of an irreversible QCE cycle in the plane of the external parameter X and force $F(X)$ (a) and of the temperature T and entropy S (b). The values of the external parameter X and of the entropy S at the four special instants are indicated.

Let $v(t)$ and τ be the speed of the change of X and the cycle period, respectively. Then we have

$$X_0 = (X_2 - X_1) + (X_3 - X_2) + (X_3 - X_4) + (X_4 - X_1) = 2(X_3 - X_1) = \int_0^\tau v(t)dt = \bar{v}\tau, \tag{3}$$

where $\bar{v}$ is the average speed of the change of X. The displacement of X after a single cycle is zero and thus X is a state variable, although the total change per cycle X_0 is not equal to zero. The times spent on two isothermal processes can be expressed as $t_{12} = (X_2 - X_1)/\bar{v}$ and $t_{34} = (X_3 - X_4)/\bar{v}$, respectively, while the times of the two adiabats are $t_{23} = (X_3 - X_2)/\bar{v}$ and $t_{41} = (X_4 - X_1)/\bar{v}$. Therefore, the power output $P = W/\tau$ and the efficiency $\eta = W/Q_h$ are

$$P = \frac{\bar{v}}{2(X_3 - X_1)}\left[(T_h - T_c)\Delta S - \frac{\bar{v}T_h\Sigma_h}{(X_2 - X_1)} - \frac{\bar{v}T_c\Sigma_c}{(X_3 - X_4)} - \frac{\bar{v}T_c\Sigma_a}{(X_3 - X_2)} - \frac{\bar{v}T_c\Sigma_a}{(X_4 - X_1)}\right], \tag{4}$$

and

$$\eta = \frac{(T_h - T_c)\Delta S - \frac{\bar{v}T_h\Sigma_h}{(X_2-X_1)} - \frac{\bar{v}T_c\Sigma_c}{(X_3-X_4)} - \frac{\bar{v}T_c\Sigma_a}{(X_3-X_2)} - \frac{\bar{v}T_c\Sigma_a}{(X_4-X_1)}}{T_h\Delta S - \frac{\bar{v}T_h\Sigma_h}{(X_2-X_1)}}, \tag{5}$$

respectively. Here ΔS is a state variable determined only by the initial and final states of the isothermal process. The generalized coordinates X_i with $i = 1,2,3,4$, corresponding to the system volume V in classical thermodynamics, are state variables and independent of the detailed protocols. To specify the time allocation at maximum power output, the values of X_i as well as the average speed $\bar{v}$ should be optimized. We will optimize the power output P over the average speed $\bar{v}$ and the variables X_i to obtain the time allocation during a cycle and thus to determine the corresponding efficiency. We will assume, for simplicity, that the initial value of the external parameter is a constant, i.e., $X_1 = X_1^0$. The maximum power is therefore found by setting the derivatives of P with respect to the average speed $\bar{v}$ and X_i with $i = 2,3,4$ equal to zero.

The maximization conditions $\frac{\partial P}{\partial X_i}|_{X_i=X_i^m} = 0$ and $\frac{\partial P}{\partial \bar{v}}|_{\bar{v}=\bar{v}_m} = 0$ give the physical solution. The value of X_3^m is determined by the following equation:

$$(T_h - T_c)\Delta S = \bar{v}_m T_c\Sigma_a\left(\frac{1}{X_3^m - X_2^m} + \frac{1}{X_4^m - X_1^m}\right) + \frac{\bar{v}_m T_c\Sigma_c}{X_3^m - X_4^m} + \bar{v}_m T_c\left(X_3^m - X_1^0\right) \times \left[\frac{\Sigma_a}{\left(X_3^m - X_2^m\right)^2} + \frac{\Sigma_c}{\left(X_3^m - X_4^m\right)^2}\right] + \frac{\bar{v}_m T_h\Sigma_h}{X_2^m - X_1^0}, \tag{6}$$

where

$$\bar{v}_m = \frac{1}{2}\frac{(T_h - T_c)\Delta S}{\left(\frac{1}{X_3^m-X_2^m} + \frac{1}{X_4^m-X_1^0}\right)T_c\Sigma_a + \frac{1}{X_3^m-X_4^m}T_c\Sigma_c + \frac{1}{X_2^m-X_1^0}T_h\Sigma_h}, \tag{7}$$

$$X_2^m = \frac{\left|T_h\Sigma_h X_3^m - T_c\Sigma_a X_1^0\right| + \left(X_3^m - X_1^0\right)\sqrt{T_c T_h \Sigma_a \Sigma_h}}{\left|T_h\Sigma_h - T_c\Sigma_a\right|}, \tag{8}$$

and

$$X_4^m = \frac{\left|\Sigma_a X_3^m - \Sigma_c X_1^0\right| + \left(X_3^m - X_1^0\right)\sqrt{\Sigma_a\Sigma_c}}{\left|\Sigma_a - \Sigma_c\right|}. \tag{9}$$

From Eqs. (6) and (7), we find that the optimal value of X_3^m with a fixed value of X_1^0 is independent of the value of the state variable ΔS, as expected. Substitution of Eqs. (7), (8), and (9) into Eq. (6) leads to the fundamental optimal relationship between X_3^m and X_1^0 at maximum power output. Under the assumption that the value of X_1 is fixed at the start of the engine cycle, Eq. (6) can be evaluated numerically for given values of entropy production Σ_a, Σ_c, and Σ_h in the specific processes and of temperatures T_h and T_c. Once we have obtained the optimal relationship between X_1^0 and X_3^m at maximum power output P, we can then determine the optimal values of X_i^m with $i = 2,4$, and the average speed $\bar{v}_m$ by Eqs. (7), (8), and (9).

Substituting Eq. (7) into Eq. (5), we find the expression for the efficiency at maximum power as follows:

$$\eta_m = \frac{1 - \frac{T_c}{T_h}}{2 - \frac{(T_h - T_c)}{\left(X_2^m - X_1^0\right)\left[\left(\frac{T_c}{X_3^m - X_2^m} + \frac{T_c}{X_4^m - X_1^0}\right)\frac{\Sigma_a}{\Sigma_h} + \frac{T_c}{X_3^m - X_4^m}\frac{\Sigma_c}{\Sigma_h} + \frac{T_h}{X_2^m - X_1^0}\right]}}. \tag{10}$$

Equation (10) together with Eqs. (6), (7), (8), and (9), as one of our main results, conveys the following physical features:

(1) The nonadiabatic dissipation is neglected, i.e., $\Sigma_a \to 0$. In such a case, the limits $\frac{\Sigma_c}{\Sigma_h} \to 0$ and $\frac{\Sigma_c}{\Sigma_h} \to \infty$ lead to the result that the efficiency η_m at maximum power approaches the upper bound $\eta_+ \equiv \frac{\eta_C}{2-\eta_C}$ and the lower bound $\eta_- \equiv \frac{\eta_C}{2}$, respectively. That is, the efficiency η_m at maximum power satisfies the following condition:

$$\frac{\eta_C}{2} \equiv \eta_- \leqslant \eta_m \leqslant \eta_+ \equiv \frac{\eta_C}{2 - \eta_C}. \tag{11}$$

In Fig. 2 we plot the efficiency (10) as a function of η_c, comparing $\eta_{\rm CA}$ with the upper and lower bounds (11). The lower and upper bounds, which are reached in the completely asymmetric limits $\frac{\Sigma_c}{\Sigma_h} \to 0$ and $\frac{\Sigma_c}{\Sigma_h} \to \infty$, are identical to the corresponding ones derived in different approaches [8]. However, unlike in the previous literature in which the durations of the two adiabats are ignored, the times spent on the two adiabats in the quantum Carnot cycle are taken into account. In the case of symmetric dissipation ($\sum_c = \sum_h$), we can recover the CA efficiency $\eta_m = \eta_{\rm CA} = 1 - \sqrt{\frac{T_h}{T_c}}$ by using Eq. (10), if $\frac{X_2^m - X_1^0}{X_3^m - X_4^m} = \sqrt{\frac{T_h}{T_c}}$, i.e., the time allocation to the hot and cold processes at maximum power output satisfies the following relation:

$$\frac{t_{12}}{t_{34}} = \sqrt{\frac{T_h}{T_c}}, \tag{12}$$

we can recover the CA efficiency $\eta_m = \eta_{\rm CA} = 1 - \sqrt{\frac{T_c}{T_h}}$ by using Eq. (10). The result of consideration of symmetric dissipation agrees with that obtained by optimizing the power

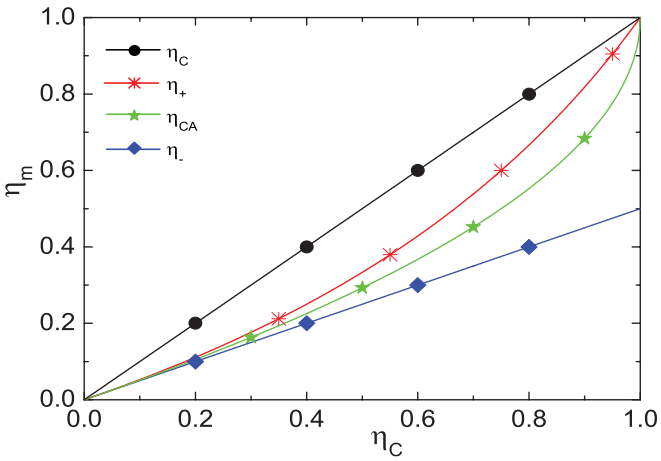

FIG. 2. (Color online) Efficiency η_m at maximum power as a function of the Carnot value η_C. The upper and lower bounds of the efficiency, η_+ and η_- given in Eq. (11), are represented by a red and a blue solid line with asterisks and squares, respectively. The CA efficiency η_{CA} is denoted by a green solid line with five-pointed stars, and the Carnot efficiency η_C by a black solid line with circles.

output with respect to the times of the two isothermal processes [8,34].

(2) There is nonadiabatic dissipation, and the dissipation in at least one quantum isothermal process is not considered, namely, $\Sigma_a \neq 0$, and $\Sigma_h \to 0$ (or $\Sigma_c \to 0$). From Eq. (10) we find in this case

$$\eta_m = \eta_- = \frac{\eta_C}{2}, \tag{13}$$

which is independent of the values of both Σ_a and Σ_c (or Σ_h). The lower bound, which is found in the case when nonadiabatic dissipation exists but dissipation vanishes in at least one isotherm, is particularly interesting. Although it coincides with a reported universal lower bound in Refs. [8,17], it is derived in the generalized engine model with nonadiabatic phenomena. Physically, even in the case when the two isothermal processes are reversible, the inclusion of an arbitrary low dissipation in the adiabats reduces the efficiency at maximum power output to half the Carnot value, $\frac{\eta_C}{2}$.

(3) The dissipations in the four quantum thermodynamic processes are equal, i.e., $\Sigma_c/\Sigma_h = 1$ and $\Sigma_a/\Sigma_h = 1$. Let $R_x \equiv (X_2^m - X_1^0)(\frac{1}{X_3^m - X_2^m} + \frac{1}{X_4^m - X_1^0} + \frac{1}{X_3^m - X_4^m})$; in the limits $R_x \to 0$ and $R_x \to \infty$, the efficiency η_m corresponding to maximum power output converges to the upper bound $\eta_+ = \frac{\eta_C}{2-\eta_C}$ and to the lower bound $\eta_- = \frac{\eta_C}{2}$, respectively. Here the lower and upper bounds are equal to the corresponding ones in previous studies, but extended to irreversible QCEs in which the time spent on two adiabats and nonadiabatic dissipation are considered. According to Eq. (10), the CA efficiency $\eta_{CA} = 1 - \sqrt{\frac{T_c}{T_h}}$ is achieved when the times spent on the four quantum thermodynamic processes are distributed in such a way that

$$\frac{t_{12}(\tau - t_{12})}{t_{23}t_{34}t_{41}} = \sqrt{\frac{T_h}{T_c}}, \tag{14}$$

where $t_{12} = (X_2^m - X_1^0)/\bar{v}_m, t_{23} = (X_3^m - X_2^m)/\bar{v}_m, t_{34} = (X_3^m - X_4^m)/\bar{v}_m$, and $t_{41} = (X_4^m - X_1^0)/\bar{v}_m$.

III. CONCLUSIONS

In conclusion, we have determined the efficiency at maximum power for a QCE engine performing finite-time cycles. To correctly describe irreversible QCEs, the times spent on two adiabats and nonadiabatic phenomena have been taken into account. In the limits of extremely asymmetric dissipation ($\frac{\Sigma_c}{\Sigma_h} \to 0$ and $\frac{\Sigma_c}{\Sigma_h} \to \infty$, with $\frac{\Sigma_a}{\Sigma_h} \to 0$), the efficiency at maximum power output converges to an upper and a lower bound, coinciding with the results obtained previously in different approaches. When dissipation in any isothermal process vanishes, the efficiency at maximum power output is equal to the lower bound $\frac{\eta_C}{2}$. For the QCE with symmetric dissipation but without nonadiabatic dissipation ($\frac{\Sigma_c}{\Sigma_h} \to 1$, while $\frac{\Sigma_a}{\Sigma_h} \to 0$), we have derived the CA efficiency at maximum power output, only provided that the ratio of the times of contact with two heat reservoirs satisfies the relation given in Eq. (12). In the case of $\frac{\Sigma_c}{\Sigma_h} \to 1$ and $\frac{\Sigma_a}{\Sigma_h} \to 1$, we have also recovered the CA efficiency at maximum power output, if the time allocations of the four processes fulfill the condition in Eq. (14).

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[33] For example, we consider a very simple QCE model consisting of an ensemble of many identical noninteracting particles in a one-dimensional power-law potential with size L. The power-law potentials can be parametrized by a single-particle energy spectrum of the form $E_n(L) = E_g(L)n^\sigma$ [23], where $E_g(L) \equiv E_1(L)$ is the energy of the ground state, and σ is the index of the single-particle energy spectrum. Without loss of generality, the ground state energy is assumed to be proportional to $L^{-\alpha}$, i.e., $E_g(L) = \gamma L^{-\alpha}$, where γ is a constant for a given potential, and the index α is positive and depends on the form of the external potential.
[34] For the case with symmetric dissipation and without nonadiabatic dissipation, in [8] the CA efficiency was recovered by optimizing the power output with respect to time allocation, and thus the correponding time allocation to the hot and cold reservoirs was obtained.