

Markov-switching multifractal models as another class of random-energy-like models in one-dimensional space

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We map the Markov-switching multifractal model (MSM) onto the random energy model (REM). The MSM is, like the REM, an exactly solvable model in one-dimensional space with nontrivial correlation functions. According to our results, four different statistical physics phases are possible in random walks with multifractal behavior. We also introduce the continuous branching version of the model, calculate the moments, and prove multiscaling behavior. Different phases have different multiscaling properties.

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I. INTRODUCTION

The random energy model (REM) introduced by Derrida [1–5] is one of the fundamental models of modern physics. Originally derived as a mean-field version of spin-glass models, it has subsequently been applied to describe some features of two-dimensional (2D) Liouville models [6,7] as well as the properties of quenched disorder in d -dimensional space with a logarithmic correlation function of energy disorder.

The logarithmic correlation is easy to organize both for models on a hierarchic tree and in two dimensions. The REM structure in the d -dimensional logarithmic correlation case has been proved in [8] using Bramson's results [9]. In [10,11] a REM-like model in one-dimensional (1D) space has been solved directly, using the generalization of Selberg integrals [11]. A mapping has been used in [12] to map the REM onto strings.

A REM can be formulated not only for the case of normal distributions of energies, which corresponds to the logarithmic correlation function of energies on a hierarchic tree, similar to logarithmic correlation functions in d -dimensional disorder case [8], but also for general distribution of energies [4,13]. It is an open problem to find solvable models with the nonlogarithmic correlation for energy disorder in 1D space. To describe the fluctuations in the financial market, [14,15] have constructed some dynamical models, the Markov-switching multifractal model (MSM). The MSM has time translational symmetry, contrary to cascade models, defined on hierarchic trees [7]. The connection of the 1D REM model [11] with the multifractal random-walk model [16–18] was found in [19]. In this paper we will prove that the dynamical models of [14,15] provide a 1D REM where the correlation function for energies ($\ln u_t$ in our case) has a general character instead of being logarithmic.

Let us give the definition of the MSM model, following [14,15,20]. The MSM was a generalization of the model from [21]. In the MSM model, one considers the sequence of variables r_t , where $t \geq 0$ describes a discrete moment of time:

$$r_t = x_t u_t, \quad (1)$$

x_t has a normal distribution,

$$\langle x_t^2 \rangle = J^2, \quad (2)$$

and u_t is defined at the moment t of time via a product of k components $M(t,l)$,

$$u_t = \prod_{l=1}^k M(t,l). \quad (3)$$

The variables $M(t,l)$ are random variables with some distribution.

Every moment of time, our random variables are replaced with new ones with a probability

$$\gamma_l = 1 - \exp[-ab^{(l-k)}], \quad (4)$$

where $a > 0$, $b > 1$, $1 \leq l \leq k$ are parameters of the model. The parameter b plays the role of the branching number in cascade models (models of random variables on the branches of hierarchic models), and k is the maximal number of hierarchy on the tree. An important difference is that now b is a real number, while in the case of hierarchic trees b should be an integer. Later we will formulate the continuous branching version of the model with a single relevant parameter V defined from the equation

$$b^k = e^V \equiv L. \quad (5)$$

We will use the notation L in Sec. II H and III B while investigating the multiscaling properties of the model.

The model is named the random-walk model because, from Eq. (1), it is equivalent to the random walks with an amplitude J^2 , when the time itself is a random variable; see [17,19] for a simple proof. The random variables $M(t,l)$ are described via a Markov process as any time period the transition probability depends only on the current state. There is a switching according to Eq. (4), which is why the authors of [15] define it as a “switching” model.

The distribution of $M(t,l)$ is chosen to ensure the constraint

$$\langle M(t,l) \rangle = 1, \quad (6)$$

where $\langle \rangle$ means an average.

We can take the lognormal distribution for $M(t,l)$ or normal distribution for ϵ_t^l , defined as

$$M(t,l) = \exp(\beta \epsilon_t^l), \quad (7)$$

where β is similar to the inverse temperature in statistical physics.

We consider a distribution for ϵ :

$$\rho(\epsilon) = \frac{\sqrt{k}}{\sqrt{2\pi V}} \exp\left[-\frac{k(\epsilon - \lambda)^2}{2V}\right], \quad (8)$$

$$\lambda = -\beta^2/2.$$

We have for $|t - t'| < e^V$ the following two-point correlation function for u_t :

$$\langle u_t u_{t'} \rangle \sim e^{\beta^2(2V - \ln|t - t'|/\ln b)}. \quad (9)$$

The correlation function for $\ln u_t$, $\ln u_{t'}$ is logarithmic, as in models discussed in [8,10,11,16].

The previous expression is derived by observing that u_t and $u_{t'}$ have identical $M(t,l)$ for $\ln|t - t'|/\ln b$ levels of hierarchy. The probability that $M(t,l)$ and $M(t',l)$ are identical is

$$\sim \exp\left[-|t - t'|e^{-ab^{l-k}}\right]. \quad (10)$$

Thus $M(t,l)$ and $M(t',l)$ are identical for l th level of hierarchy defined through the inequality

$$k - \frac{\ln|t - t'|}{\ln b} < l < k. \quad (11)$$

For the rest of the hierarchy levels $M(t,l)$ and $M(t',l)$ are different.

When

$$\frac{\ln|t - t'|}{V} \ll 1, \quad (12)$$

the majority of the hierarchies have the same $M(t,l)$ and $M(t',l)$. Alternatively, when

$$1 - \frac{\ln|t - t'|}{V} \ll 1, \quad (13)$$

$M(t,l) \neq M(t',l)$ for the the majority of l . In Sec. III we derive some more rigorous results.

II. THE STATISTICAL PHYSICS OF MSM

A. MSM with general distribution

Let us consider a general distribution for ϵ ,

$$\rho_0(q, \epsilon) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dh \exp[-h\epsilon + q\phi(h)], \quad (14)$$

where q is a parameter indicating some effective length. We choose the distribution $\rho(\epsilon)$ with some shift,

$$\rho(\epsilon) = \rho_0\left(\frac{V}{k}, \epsilon - \lambda\right), \quad (15)$$

to ensure the constraint given by Eq. (6):

$$\int_{-\infty}^{\infty} d\epsilon \rho(\epsilon) \equiv e^{\beta\lambda + \frac{V}{k}\phi(\beta)} = 1. \quad (16)$$

Thus we take

$$\lambda = -\frac{V\phi(\beta)}{\beta k}, \quad (17)$$

where k is our parameter describing the maximal hierarchy level. Hence, we have the following expression for the correlation function:

$$\langle u_t u_{t'} \rangle \sim e^{(V - \ln|t - t'|/\ln b)\phi(2\beta) + 2\phi(\beta)\ln|t - t'|/\ln b}. \quad (18)$$

B. The statistical physics versus the dynamics

Let us define a partition function

$$z(i_0, e^V) = \sum_{i=i_0}^{i_0+e^V} x_i \prod_{l=1}^k M(i, l), \quad (19)$$

where $M(i, l)$ is chosen from the distribution given by Eq. (15). Considering Eq. (1) as a dynamic process for a large period of time M , we define the probability distribution

$$P(z) = \frac{e^V}{M} \sum_{n=1}^{M/e^V} \delta_{z, z(1+e^V(n-1), e^V n)}. \quad (20)$$

We can define a statistical physics as well by considering z as a partition function for the 1D model with quenched disorder.

The average free energy, denoted as $\langle \ln Z \rangle$, is

$$\langle \ln Z \rangle \equiv \langle \ln z(i_0, e^V) \rangle. \quad (21)$$

Let us consider a related model with standard distribution for ϵ given by $\rho_0(\epsilon)$, without the constraint of Eq. (4).

We define

$$z_0(i_0, e^V) = \sum_{i=i_0}^{i_0+e^V} x_i \prod_{l=1}^k M(i, l), \quad (22)$$

where M_i^l are defined through the distribution $\rho_0(\epsilon)$ and the corresponding free energy is

$$\langle \ln Z_0 \rangle \equiv \langle \ln z_0(i_0, e^V) \rangle. \quad (23)$$

Equations (22) and (23) define a statistical physics model with e^V configurations and special quenched disorder in 1D space. Later we will focus on $\langle \ln Z \rangle$.

It is easy to check that

$$\langle \ln Z \rangle = \langle \ln Z_0 \rangle - V\phi(\beta). \quad (24)$$

Equation (24) is an exact relation, correct for any value of β .

It is easier to solve the model for $\langle \ln Z_0 \rangle$. To calculate $\langle \ln Z_0 \rangle$, we will map the model onto the REM and use the standard methods of REM. One can easily identify the most interesting transition in REM, from the high temperature phase to the spin-glass (SG) phase, by just looking at the point in the high temperature phase where the entropy disappears. We will calculate the partition function's moments $\langle Z_0^n \rangle$ and identify them with the $\langle Z_{\text{REM}}^n \rangle$.

C. The moments in the MSM model

First of all, we calculate

$$\langle (Z_0)^2 \rangle = e^V e^{V\phi(2\beta)}. \quad (25)$$

The cross terms vanishes due to integration by x_i . Let us consider now

$$\langle (Z_0)^{2n} \rangle = \sum_{t_1} \cdots \sum_{t_{2n}} \langle u_{t_1} \cdots u_{t_{2n}} \rangle. \quad (26)$$

While calculating the n -fold sum, we consider two principal contributions.

The first case is when all t are close to each other and Eq. (12) is valid. There are $N_1 \sim e^V$ such terms, and the sum gives

$$\ln\langle(Z_0)^{2n}\rangle = V + V\phi(2n\beta) + O(1). \quad (27)$$

The second case corresponds to the integration from the regions where $t_{i+1} - t_i$ are of the same order, and therefore the condition given by Eq. (13) is satisfied. There are N_2 such terms,

$$\frac{\ln N_2 - nV}{V} \ll 1. \quad (28)$$

As the vast majority of M^l are different, the average gives

$$\ln\langle(Z_0)^{2n}\rangle = nV + nV\phi(2\beta) + O(1). \quad (29)$$

D. The corresponding REM

Consider now e^V energy levels E_i and define the partition function

$$Z_{\text{REM}} = \sum_{i=1}^{e^V} x_i e^{-\beta E_i}, \quad (30)$$

where $-E_i$ have independent distributions by given Eq. (14) with $q = V$ and x_i have normal distribution with variance 1. The moments of the partition function for this model can be calculated exactly by following [3] and [12].

These moments are identical to the expressions given by Eqs. (27) and (29). We assume that two models with identical integer moments have an identical free energy as well. The free energy of the REM model by Eq. (30) can be calculated rigorously following [12].

At high temperatures, we have the Fisher zeros (FZ) phase with

$$\langle \ln Z_0 \rangle = \frac{1}{2} \ln\langle Z_0 \rangle^2 = V \frac{\phi(2\beta) + 1}{2}. \quad (31)$$

The transition point is at the point where the entropy disappears:

$$\beta_c \phi'(2\beta_c) = \frac{\phi(2\beta_c) + 1}{2}. \quad (32)$$

Below this temperature the system is in the SG phase with the free energy

$$V\beta\phi'(2\beta_c). \quad (33)$$

Thus, we have found two phases. The phase given by Eq. (31) corresponds to the Fisher zeros phase, while that given by Eq. (33) is the SG phase.

E. Asymmetric distribution

So far we have considered the case of a symmetric distribution of x_i . Let us now consider the asymmetric case described through the parameter γ , where

$$\langle x_i \rangle = e^{-\gamma V}. \quad (34)$$

Now it is possible for the existence of a paramagnetic (PM) phase with the free energy

$$\langle \ln Z_0 \rangle = \ln\langle Z_0 \rangle = [-\gamma + 1 + \phi(\beta)]V. \quad (35)$$

F. Large event

Let us assume that at the starting moment of time there is a large event described through the parameter A , while for the other times Eq. (2) is valid. We consider the following partition function:

$$z_0(i_0, e^V) = -e^{AV} + \sum_{i=i_0+1}^{i_0+e^V} x_i \prod_{l=1}^k M(i, l). \quad (36)$$

Now we can have the fourth, ferromagnetic (FM), phase with

$$\langle \ln |Z_0| \rangle = \ln\langle Z_0 \rangle = AV. \quad (37)$$

Actually, we can consider an infinite series of time when after e^V there is a member $r_i = -e^{AV}$, while for other moments of time we calculate r_i according to Eqs. (1) and (3).

G. Transition points

We should choose the proper phase by comparing the expressions given by Eqs. (31), (33), (35), and (37) and then selecting the one that gives the maximum. For example, the system transforms from the FZ phase to the PM phase at

$$e^{(1-\gamma+\phi(\beta))V} > J^2 e^{\frac{V}{2}(1+\phi(2\beta))} \quad (38)$$

where J^2 is the variance of x_i given by Eq. (2). In Figs. 1 and 2 we compare the numerics with our analytical results for the free energy.

H. The scale dependence of the free energy

Consider again the average distribution of Z , except that, instead of considering the sum over e^V terms as in Eq. (20), consider the sum over $l \equiv e^{\alpha V}$ terms,

$$P(z) = \frac{l}{M} \sum_{n=1}^{M/l} \delta_{z, z(\alpha, 1+l(n-1), ln)}, \quad (39)$$

where ϵ have the distribution by Eq. (14). At high temperatures we have the FZ phase with

$$\langle \ln Z_0 \rangle = V \frac{\phi(2\beta) + \alpha}{2}, \quad (40)$$

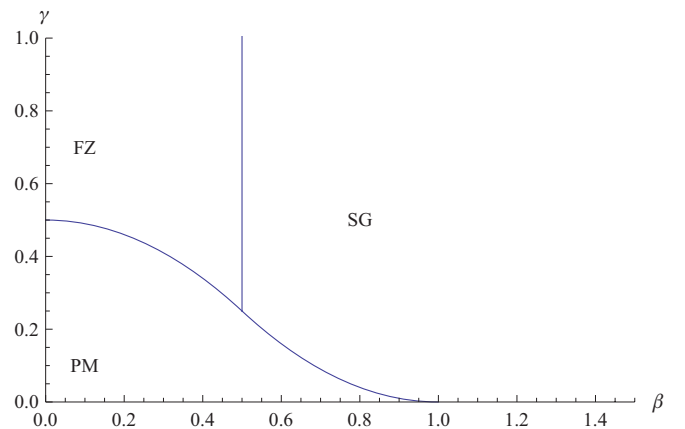


FIG. 1. (Color online) The phase structure of the model with asymmetric distribution of weights. The case of normal distribution $\phi(\beta) = \beta^2$.

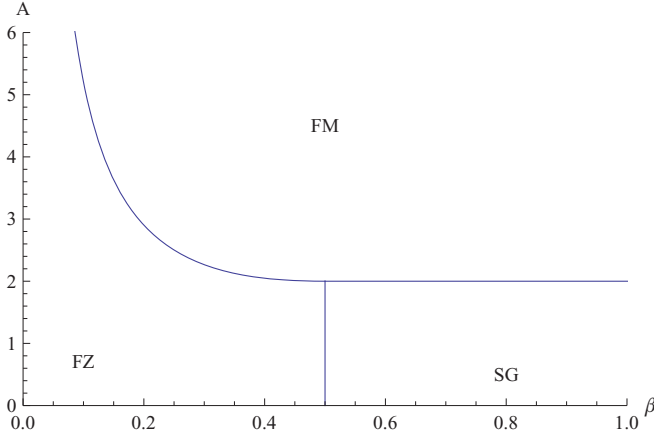


FIG. 2. (Color online) The phase structure of the model with a large event. The case of normal distribution $\phi(\beta) = \beta^2$.

and for the critical point,

$$\beta_c \phi'(2\beta_c) = \frac{\phi(2\beta_c) + \alpha}{2}. \quad (41)$$

As $\alpha < 1$, β_c decreases with the decrease of α .

III. THE CASE OF CONTINUOUS BRANCHING

A. The calculation of moments

All the formulas in the previous sections have been derived for the case of general values of b . Consider the case

$$a = 1, \quad k = \frac{V}{\delta v}, \quad b = 1 + \delta v, \quad \delta v \rightarrow 0 \quad (42)$$

and the random variables are distributed according to $\rho_0(\delta v, \epsilon)$.

The i th level of hierarchy is unchanged during the period of time t with a probability

$$\exp[-te^{v-V}], \quad v = i\delta v. \quad (43)$$

Multiplying the probabilities in Eq. (43) for different levels of hierarchy, we obtain

$$\begin{aligned} \langle u_{t_1} u_{t_2} \rangle &= \prod_{i=1}^k \{ [1 - \exp(-te^v)] e^{\delta v 2\phi(\beta)} + \exp(-te^v) e^{\delta v \phi(2\beta)} \} \\ &= \exp \left[\sum_i \ln((1 - \exp(-te^v)) e^{\delta v 2\phi(\beta)} \right. \\ &\quad \left. + \exp(-te^v) e^{\delta v \phi(2\beta)}) \right]. \end{aligned} \quad (44)$$

where $t = t_2 - t_1$. Replacing the product by an integral and introducing variables $x_i = t_i/e^V$, we derive

$$\begin{aligned} \int_0^{e^V} dt_1 dt_2 \langle u_{t_1} u_{t_2} \rangle &= e^{V(2(1+\phi(\beta)))} \int_0^1 dx_1 dx_2 \\ &\quad \exp \left[\int_0^V dv [\Phi_2(e^v, x_1, x_2)] \right] \\ \Phi_2(e^v, x_1, x_2) &= e^{-|x_2 - x_1|e^v} (\phi(2\beta) - 2\phi(\beta)). \end{aligned} \quad (45)$$

We get an asymptotic expression with the e^{-V} accuracy in the limit $V \rightarrow \infty$:

$$\frac{\int_0^{e^V} dt_1 dt_2 \langle u_{t_1} u_{t_2} \rangle}{e^{2V+2V\phi(\beta)}} = \int_0^1 dx_1 dx_2 e^{\int_0^\infty dv \Phi_2(e^v, x_1, x_2)}. \quad (46)$$

Similarly, we derive the expression for the multiple correlations:

$$\frac{\int_0^{e^V} dt_1 \dots t_n \langle u_{t_1} \dots u_{t_n} \rangle}{e^{nV+n\phi(\beta)}} = \int_0^1 dx_1 \dots dx_n e^{\int_0^\infty dv \Phi_n(e^v, x_1, \dots, x_n)}. \quad (47)$$

The latter expression is $O(1)$, as has been assumed before in Eq. (29).

For the 3-point correlation function we obtain

$$\begin{aligned} \Phi_3(y, x_1, x_2, x_3) &= 3 + 3\phi(\beta) \\ &\quad + e^{-(x_{12}+x_{23})y} (\phi(3\beta) - 3\phi(\beta)) \\ &\quad + e^{-x_{12}y} (1 - e^{-x_{23}y}) (\phi(2\beta) - 2\phi(\beta)) \\ &\quad + (1 - e^{-x_{12}y}) e^{-x_{23}y} (\phi(2\beta) - 2\phi(\beta)) \end{aligned} \quad (48)$$

where we denote $x_{12} = |x_1 - x_2|$, $x_{23} = |x_2 - x_3|$. For n -point correlation function we need to consider 2^{n-1} terms in the expression of Φ_n .

We can identify these terms with different paths on a tree with branching number 2, the jumps to the right give a coefficient $F(x, 1)$ and $F(x, -1)$ for the left jump:

$$F(x, 1) = \exp[-xe^v], \quad F(x, -1) = (1 - \exp[-xe^v]) \quad (49)$$

The path is fractured into clusters, when we have l subsequent right jumps. We define

$$f_l = \phi(l\beta) \quad (50)$$

We should consider all the paths, the identify the n clusters of the given path with the length l_m for the m -th cluster. Then we calculate

$$\begin{aligned} \Phi_n(e^v, x_1, \dots, x_n) &= \\ &\sum_{\text{paths}} \left[\prod_{i=0}^{n-1} F(x_{i,i+1}, \alpha) \right] \left(\sum_m f_{l_m} - n\phi(\beta) \right) \end{aligned} \quad (51)$$

B. Multiscaling

If we consider the model for an $l = e^{\alpha V}$ period of time and a normal distribution, we have in the high temperature phase

$$F \equiv \frac{\langle \ln Z_0 \rangle}{\alpha V} = \frac{1}{2} + \frac{\beta^2}{\alpha} \quad (52)$$

and in the SG phase

$$F \equiv \frac{\langle \ln Z_0 \rangle}{\alpha V} = \beta \sqrt{2/\alpha}. \quad (53)$$

The transition point is at

$$\beta_c = \sqrt{\alpha/2}. \quad (54)$$

In Fig. 3 we compare the numerics with our analytical results for the free energy.

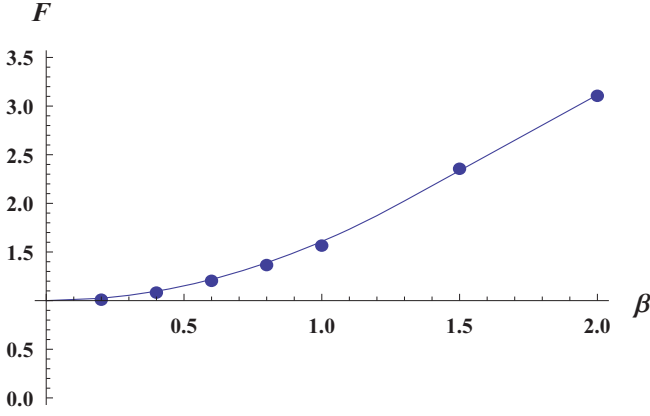


FIG. 3. (Color online) Free energy F vs an inverse temperature β for the continuous branching model given by Eq. (42) and a normal distribution. The analytical results from Eqs. (52) and (53), $L = 100\,000$, are given by continuous line.

Let us calculate the multifractal properties of the MSM. We need to calculate the moments of the partition function,

$$\int_0^l dt_1 \dots dt_n \langle u_{t_1} \dots u_{t_n} e^{-n\phi(\beta)V} \rangle \equiv e^{\ln \frac{l}{L} \xi(n,\beta)} A(L)^n C_n, \quad (55)$$

where $\xi(n,\beta)$ defines the multi-scaling, $A(L)$ is some large numbers while $C_n \sim O(1)$.

While calculating the moments, we slightly modify the formulas of the previous sections. Instead of Eq. (47) now we obtain

$$\int dt_1 \dots dt_n u(t_1) \dots u(t_n) = \int dx_1 \dots dx_n \exp \left[nV(1 + \phi(\beta)) + \int_0^\infty dv \Phi_n \left(\frac{l}{L} e^v, x_1 \dots x_n \right) \right]. \quad (56)$$

We consider the case $1 \ll l \ll L$. Then, using the equation

$$\Phi_n(0, x_1 \dots x_n) = n\phi(\beta) - \phi(n\beta), \quad (57)$$

we derive, integrating by parts:

$$\begin{aligned} & \int_1^\infty \frac{dy}{y} \Phi_n \left(y \frac{l}{L}, x_1 \dots x_n \right) \\ &= (\phi(n\beta) - n\phi(\beta)) \ln \frac{l}{L} - \int_0^\infty dy \ln y \Phi_n'(y, x_1 \dots x_n). \end{aligned} \quad (58)$$

Thus we get a multiscaling with

$$\xi(n,\beta) = n + n\phi(\beta) - \phi(n\beta). \quad (59)$$

Considering the moments of

$$z = \sum_{t=1}^l u_t, \quad (60)$$

where for u_t we use the distribution given by Eq. (15), we obtain

$$\frac{\langle z^n \rangle}{L^n n!} = e^{\xi(n,\beta) \ln \frac{l}{L}} \int_0^1 dx_1 \dots dx_n e^{-\int_0^\infty dy \ln y \Phi_n'(y, x_1 \dots x_n)} \quad (61)$$

and here $x_1 \dots x_n$ are time ordered.

C. The moments for the model with random Boltzmann weights

Let us calculate now the moments $\langle z^n \rangle$ for

$$z = \sum_{t=1}^l x_t u_t. \quad (62)$$

Using Eq. (8) from [19], we derive

$$\begin{aligned} \frac{\langle z^{2n} \rangle}{(LJ)^n} &= \frac{2^n \Gamma(\frac{1+2n}{2})}{\sqrt{\pi}} \left(\frac{l}{L} \right)^{\xi(n,2\beta)} n! \int_0^1 dx_1 \dots dx_n \\ &\times \exp \left[- \int_0^\infty dy \ln y \Phi_n'(y, 2\beta, x_1 \dots x_n) \right], \end{aligned} \quad (63)$$

where there is a time ordering $t_1 < t_2 < \dots < t_n$.

D. The multiscaling properties of different phases

There are no simple order parameters to distinguish the FZ and SG phases. If we enlarge the free energy expression to the complex temperatures $\beta = \beta_1 + i\beta_2$, then in the FZ phase there is a finite density $\bar{\rho}$ of partition function zeros, defined through the formula [22]

$$\bar{\rho}(\beta_1, \beta_2) = \frac{1}{2\pi} \frac{\langle \ln z(\beta_1, \beta_2) \rangle}{V}, \quad (64)$$

while in the SG phase this density is zero.

The SG and FZ phases have different schemes of replica symmetry (breaking) [23]. There is a slow relaxation in the SG phase. Unfortunately, there are no results about the dynamics of the FZ phase to compare.

It is more interesting to distinguish the two phases looking at the multiscaling properties. We investigated well the multiscaling properties of the FZ phase. Let us investigate now the SG phase. Gardner and Derrida [3] give results for the moments of partition functions and the probability distribution.

Consider again the model with $z = \exp[\alpha V]$ configurations. For the case

$$\beta > \beta_c \sqrt{\alpha}, \quad \beta_c \sqrt{\alpha} > n\beta \quad (65)$$

we have, rescaling the result of [3],

$$\ln \langle z^n(\alpha) \rangle = n\beta V \sqrt{\alpha}. \quad (66)$$

In case of the multiscaling the right hand side is proportional to α .

Thus there is a lack of any scaling in the SG phase, and we can distinguish the SG and FZ phases by checking the multiscaling property. We can distinguish the FZ and SG phases also by looking at the tails of the distributions.

For the FZ phase a simple rescaling of the results of [11] gives for the large z

$$P(z) \sim \frac{1}{z^{1 + \frac{\alpha\beta_c^2}{2\beta^2}}}, \quad (67)$$

while in the SG phase we used the rescaled result by [3],

$$P(z) \sim \frac{1}{z^{1 + \frac{\sqrt{\alpha}\beta_c}{\beta}}}, \quad (68)$$

where $\beta_c/2$ is the transition point at $\alpha = 1$. Equation (68) is the result for the REM. For the logarithmic REM to get a more

accurate expression we should multiply the right hand side of Eq. (68) by $\ln Z$ [24].

IV. THE DYNAMIC MODEL

Let us return to the dynamic model given by Eq. (1). Mapping

$$e^{-\gamma V} = \mu \delta t, \quad J^2 = \sigma^2 dt, \quad (69)$$

we identify this version of the model at $\beta = 0$ as a finite time version of driven Brownian motion [25]. In case of simple random walks with $\beta = 0$, there is a single phase. In the model given by Eq. (1) with $\beta \neq 0$ we have an intrinsic large parameter $L \equiv e^V$, describing the effective number of configurations in the model. In this way the statistical physics enters into the dynamical problem. Contrary to the driven Brownian motion and the Heston model [26], we have four different phases in the MSM model. This is also the situation with other models of multiscaling random walks [16].

To identify the choice among the two phases (FZ versus PM) we consider

$$C = \frac{\sigma}{|\mu|\sqrt{L}}. \quad (70)$$

When $C \ll 1$, the system is in the PM phase. Otherwise, at $C \gg 1$ we have the FZ phase.

V. CONCLUSION

We considered the dynamic Markov-switching multifractal models and connected them with a new class of solvable statistical physics models of quenched disorder in one dimension. In

these models there is both translational invariance and general distribution of disorder. We investigated the statistical physics properties of the model and found the phase structure. We found the exact phase structure of the model. At different phases there should be different distributions $P(z)$. In the case of a symmetric random walk there are two phases in the considered model. At small parameters β , the model is in the phase with nonzero density of Fisher's zeros. At high β , the system is in the spin-glass phase, a pathologic phase with a slow relaxation dynamics. For an asymmetric distribution of x_i there is a possibility for the third, paramagnetic, phase. A slight modification of the model allows the existence of the fourth, ferromagnetic, phase. It is possible to distinguish different phases measuring the multiscaling properties of the model. The multiscaling is broken in the SG phase. We also introduced a continuous hierarchy branching version of the MSM, gave expressions for the moments of the partition function, and calculated the multiscaling indices.

For applications it is important to calculate the fractional moments of the partition function. Perhaps we can use expressions for integer moments and use some approximate methods of extrapolation. Another interesting open problem is to investigate the dynamics of the model, looking for a new phase transition point in the dynamics, as is the case of the spherical spin-glass model [27].

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