

Universal statistics of vortex lines

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We study the vortex lines that are a feature of many random or disordered three-dimensional systems. These show universal statistical properties on long length scales, and geometrical phase transitions analogous to percolation transitions but in distinct universality classes. The field theories for these problems have not previously been identified, so that while many numerical studies have been performed, a framework for interpreting the results has been lacking. We provide such a framework with mappings to simple supersymmetric models. Our main focus is on vortices in short-range-correlated complex fields, which show a geometrical phase transition that we argue is described by the $CP^{k|k}$ model (essentially the CP^{n-1} model in the replica limit $n \rightarrow 1$). This can be seen by mapping a lattice version of the problem to a lattice gauge theory. A related field theory with a noncompact gauge field, the ‘NCCP $^{k|k}$ model’, is a supersymmetric extension of the standard dual theory for the XY transition, and we show that XY duality gives another way to understand the appearance of field theories of this type. The supersymmetric descriptions yield results relevant, for example, to vortices in the XY model and in superfluids, to optical vortices, and to certain models of cosmic strings. A distinct but related field theory, the $RP^{2|2l}$ model (or the RP^{n-1} model in the limit $n \rightarrow 1$) describes the unoriented vortices that occur, for instance, in nematic liquid crystals. Finally, we show that in two dimensions, a lattice gauge theory analogous to that discussed in three dimensions gives a simple way to see the known relation between two-dimensional percolation and the $CP^{k|k}$ σ model with a θ term.

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I. INTRODUCTION

Ensembles of random geometric objects lead to some of the most subtle and interesting questions in critical phenomena. A geometric viewpoint is often useful in analyzing phase transitions: for example, one can view an Ising transition as a proliferation of domain walls, or an XY transition as a proliferation of vortices, leading to important dual descriptions [1–5]. But in addition to this, some of the most intriguing issues concern the critical behavior of the geometrical objects themselves. In this paper we consider the universal statistical properties of *vortex lines* in random media in three dimensions, a subject that has received considerable numerical attention [6–10], but which has so far lacked a field-theoretic formulation.

Geometrical critical behavior can occur even in the absence of a conventional thermodynamic phase transition. The best-known example of this is percolation, in which nonlocal geometrical correlation functions (such as the probability that two sites lie in the same cluster) yield universal exponents, despite the fact that correlation functions of local observables remain trivial at the critical point [11]. In fact, it is precisely because of the short-range correlations of the local degrees of freedom that the percolation universality class is ubiquitous: generic random media are short-range correlated.

Two-dimensional percolation can be thought of as a fluctuating soup of loops [12]. On the lattice, these are the cluster boundaries; in the continuum, they are the zero lines of a random real height field $v(x)$. When $v(x)$ is biased toward either positive or negative values, the loops have a finite typical size, but these cases are separated by a critical point at which this length scale diverges and the loop ensemble becomes conformally invariant.

Vortex lines in three dimensions constitute an analogous problem. The most generic version of it concerns the statistics

of the zero lines of a short-range-correlated *complex* field $w(x)$. Again, these vortices form a fluctuating soup of oriented loops, with the orientation of a loop given by the sense in which the phase of $w(x)$ rotates around the vortex. Although the correlations of $w(x)$ are trivial by construction, there is an interesting geometrical phase transition as a function of bias—the ratio of the average value of $w(x)$ to the width of its distribution [6,10]. As will be explained in detail in the following, there is an *extended* phase at small or zero bias [6–9], in which infinite vortex lines proliferate (and are found numerically to be Brownian), while at large bias there is a *localized* phase in which long vortices are exponentially suppressed. In between lies a critical point, where loops have a nontrivial fractal structure (and a fractal dimension close to $5/2$). Similar transitions can also occur in the absence of applied bias, for example, in the vicinity of the critical point of the XY model, or for line defects in other order parameters: in particular, one can consider the unoriented \mathbb{Z}_2 vortices in nematics [10,13].

Universal aspects of vortex statistics have been examined numerically in a wide range of contexts, including cosmic strings, beginning with the simulations of Vachaspati and Vilenkin [7,8,10,13–15], line defects in random light fields, known as optical vortices [9,16], vortices in the XY [17,18] and Abelian Higgs models [19–22], in turbulent superfluids [23], and in a Potts-like model introduced to describe polymers [6]. In addition, similar scaling behavior has been observed in simulations of lattice models of completely packed loops [24,25].

Despite this previous work on vortices, numerous questions remain unanswered because of the lack of a continuum formulation in which the relevant correlation functions can be expressed. In particular, it has not been clear how to classify these problems or what the criteria are for two instances to be in

the same universality class. It has been uncertain, for example, whether critical exponents for the geometrical vortex transition coincide with those for conventional three-dimensional (3D) percolation, or whether universal behavior for oriented and unoriented vortices coincide (even in the extended phase) or how the geometrical transition for vortices in the XY model relates to the thermodynamic one. Even basic facts such as the appearance of Brownian statistics in the extended phase have not been derived, though previously this has been explained heuristically [8] by analogy with polymers in the melt, long known to be Brownian [26].

Our effort to address these questions starts from a set of duality transformations. We begin by showing that the partition function for tricolour percolation, a Potts-like lattice model used for the numerical investigation of vortex statistics [6,8,10], can be mapped exactly to that of a supersymmetric lattice gauge theory via a high-temperature (graphical) expansion of the latter. By considering the continuum limit of this lattice gauge theory and, separately, by discussing mappings from vortex problems that are formulated directly in the continuum, we arrive at field theories describing the geometrical properties of vortices.

For oriented vortices, these field theories involve a complex supervector $\psi = (z^1, \dots, z^{k+1}, \chi^1, \dots, \chi^k)$, with $k+1$ bosonic and k fermionic components, coupled to a $U(1)$ gauge field A . (Supersymmetry [27,28] is a standard tool in problems involving loops or polymers, allowing representation of geometrical correlation functions.) In order to describe the transition for vortices in short-range-correlated complex fields $w(x)$, the gauge field A must be taken to be compact, leading to a model in the universality class of the $CP^{k|k}$ σ model [29]. Heuristically, this can be thought of as the replica limit $n \rightarrow 1$ of the CP^{n-1} σ model [30,31], which at $n = 2$ is equivalent to the classical Heisenberg [or $O(3)$] model. The ordering transition in the σ model describes the geometrical transition in the vortex ensemble. The $CP^{k|k}$ field theories have been related to lattice models of completely packed loops in both two dimensions (by Read and Saleur [29] and Candu, Jacobsen, Read, and Saleur [32]) and three dimensions [33,34]. (Strictly speaking we are abusing terminology, since there is a *separate* $CP^{k|k}$ model for each integer $k \geq 0$, the model with $k = 0$ being trivial. The correspondence with vortices holds for arbitrary k , but the range of geometrical correlation functions that can be expressed in the theory grows with k .)

The $CP^{k|k}$ description provides the appropriate framework for questions about vortex statistics. Immediate consequences include the explanation for Brownian statistics in the extended phase, which corresponds to the ordered phase in the σ model language, and qualitative features of the critical point (for example, the fact that the fractal dimension is close to $d_f = 5/2$ is easily understood as a symptom of confinement in the gauge theory). Most importantly, it clarifies the relations between the different universality classes, allowing us to classify these problems and explaining why various numerically studied models are in the same universality class and others are not. For instance, it explains the coincidence of exponents measured for the bias-induced transition for vortices [6] and for the completely packed 3D loop models [24,35], which also permit a mapping to the $CP^{k|k}$ model [33].

In the absence of bias in the distribution for $w(x)$, the gauge field in the supersymmetric theory is noncompact, yielding the noncompact $CP^{k|k}$ or $NCCP^{k|k}$ model, to borrow the terminology used for noncompact CP^{n-1} models. The appearance of this theory can be simply understood by reference to a well-known duality that maps the XY model to an Abelian Higgs theory with a complex scalar field z coupled to a gauge field [3–5,36–39]. Under this mapping, XY vortices become world lines of the particles created by z^* [to use a $(2+1)$ -dimensional language]. However, this standard dual description does not allow us to probe the long-distance behavior of vortices, because it does not allow us to write down the required geometric correlation functions. The $NCCP^{k|k}$ model is a supersymmetric extension of the Abelian Higgs model with a richer operator content (and possibly a richer phase diagram).

Unoriented vortices (e.g., in a short-range-correlated random nematic order parameter) are described by a different field theory, the $RP^{2|2l}$ model. This is essentially the replica limit of the RP^{n-1} σ model. A soft-spin version of the CP^{n-1} model in the replica limit allows us to clarify the relation between these two geometric problems, as well as the relation to 3D percolation, and also shows that the upper critical dimension for $CP^{k|k}$ is 6, meaning that exponents can be calculated in a $6 - \epsilon$ expansion [40–43]. The $CP^{1|1}$ model also describes Anderson localization in symmetry class C, which is closely related to classical loop models [24,35,44–46], so we expect 6 to be an upper critical dimension for this problem too. A similar conclusion has been reached previously by Senthil [43], who considered the soft-spin formulation of $CP^{1|1}$.

As mentioned above, another important conceptual connection is with 2D percolation. One of the continuum descriptions of this problem is as a $CP^{k|k}$ model with a θ term, as was shown via a mapping to a supersymmetric spin chain [29,32]. To connect this with our 3D results, we give a treatment of 2D percolation as a lattice gauge theory (with an unconventional Boltzmann weight for the gauge field) analogous to our treatment of tricolor percolation in three dimensions. This leads in a simple way to the continuum description. A similar lattice regularization of CP^{n-1} has been discussed by Affleck [47].

The structure of the paper is as follows. We begin in Sec. II with a heuristic overview of the key ideas. In Sec. III we give duality mappings for oriented vortices in short-range-correlated random systems on the lattice (Sec. III A) and in the continuum (Sec. III B). For the lattice derivation, we use a specific model, tricolor percolation [6,8], an elegant realization of the vortex problem that may be mapped to a lattice gauge theory. Section III C gives the basic consequences of the field theory for vortex statistics.

Section IV considers variations of the vortex problem. First we briefly describe the soft-spin version of the CP^{n-1} model in the limit $n \rightarrow 1$ and the $6 - \epsilon$ expansion (Sec. IV A). We then consider a perturbation leading to crossover to conventional percolation (Sec. IV B). Here we also discuss the issue of vortex intersections, which is important in many realistic vortex problems. Section IV C considers unoriented vortices, occurring when the fundamental group of the order parameter manifold is Z_2 . Section IV D considers the stability of the universal behavior of vortices in a short-range-correlated

$w(x)$ to the introduction of long-range correlations. We then consider vortices near the (thermodynamic) critical point of the XY model, their description in terms of the $\text{NCCP}^{k|k}$ model (Sec. IV E), and the phase structure of this field theory. This concludes our discussion of vortices. In Sec. V we consider loop models in two dimensions and their relation to lattice gauge theories with θ terms. We close with a summary in Sec. VI.

II. HEURISTIC CONSIDERATIONS

We begin with a hand-waving argument as to the form of the field theory for vortices in a short-range-correlated random complex function $w(x)$. This is intended to provide an overview and to supplement the formal derivations of Sec. III.

To reiterate, we expect the following phase structure. When the probability distribution for $w(x)$ is $U(1)$ symmetric, or only weakly biased, vortices have a fractal dimension of 2 and a finite fraction of the vortex density is in infinite vortex lines. This is the extended phase. A nonzero (translationally invariant) mean value suppresses vortices; when the mean is sufficiently large compared with the typical size of fluctuations, we enter the localized phase and infinite vortices disappear [6,15]. At the continuous transition between these phases, vortices are random fractals.

In general, a simple way to relate ensembles of loops to field theory is to view the loops as world lines of quantum particles (in this case in $2 + 1$ dimensions) and to ask what sort of particles these must be and what kind of interactions they must have. The partition function for the field theory is a sum over histories of the quantum system: expressed in the right basis, this is a weighted sum over world-line configurations. The Lagrangian must be chosen so that this sum reproduces the weighted sum over loop configurations that we started with.

Of course, to be useful the correspondence must go beyond the partition function: we should be able to represent geometrical correlation functions [such as the probability that two points x and y lie on the same loop, denoted $G(x - y)$] as correlation functions of local operators in the field theory. The field theory we seek must allow us to do this despite the fact that $G(x - y)$ is nonlocal in the original description, in the sense that we cannot necessarily tell, by looking only in the vicinity of x and y , whether the two points lie on the same loop.

Since vortices have an orientation determined by the sense of rotation of $\arg w(x)$, it would be natural to assume that we will need a single species of charged boson (with vortices that go forward and backward in the imaginary time direction corresponding to world lines of particles and antiparticles, respectively). As usual, these bosons would be represented by a single complex scalar field $z(x)$. However, this construction would not allow us to write correlators like $G(x - y)$. To be able to do that, we must give the world lines (and particles) an additional color label $\alpha = 1, \dots, n$, so that z expands to a vector z^α . This yields operators such as $z^1(x)z^{2*}(x)$, which, when inserted into a correlation function, forces a meeting at x between an outgoing world-line strand of color 2 and an incoming strand of color 1. Therefore, neglecting for now the possibility of strands escaping to infinity, the two-point function $\langle z^1 z^{2*}(x) z^2 z^{1*}(y) \rangle$ forces x and y to be joined by a

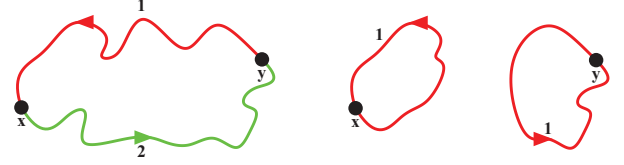


FIG. 1. (Color online) On the left: the two-point function $\langle z^1 z^{2*}(x) z^2 z^{1*}(y) \rangle$ forces x and y to be joined by a loop. With only a single-color index, we could not separate such configurations from those with the topology shown on the right.

loop, one arm of which is of color 1 and the other of color 2 (see Fig. 1). This is just what we need to construct $G(x - y)$. By contrast, if we had only a single species of particle at our disposal, we could not separate out configurations with a loop joining x and y from those with x and y lying on separate loops.

However, while the colors solve this problem, they introduce another one. The sum over color indices for each loop modifies the weight for a given configuration of loops by the unwanted factor $n^{\text{no. loops}}$. We are no longer describing the problem of vortices, but a completely different problem where loops have a fugacity equal to n .

One way to get around this is to take a replicalike limit $n \rightarrow 1$ at the end of any calculation [48]. A more concrete alternative [27–29] is to exchange z for a supervector

$$\psi = (z^1, \dots, z^{k+1}, \chi^1, \dots, \chi^k), \quad (1)$$

with $k + 1$ bosonic and k fermionic components. Fermionic loops come with a minus sign. Thus, summing over the possible colors for a loop yields a fugacity of $k + 1 - k = 1$ as desired, independently of k .

We now ask what the Lagrangian for ψ should be. It must be invariant under supersymmetry transformations preserving $\psi^\dagger \psi$. At a technical level, this ensures that the partition function is independent of the number of fermions k . In terms of loops, the invariance is a consequence of the fact that the color index we have introduced is just a label: the weight of a configuration does not change if we alter the color of one of the loops (except by minus signs associated with fermionic colors). It is tempting also to assume that since $w(x)$ is short-range correlated, interactions between vortices will be short range and therefore that we should write down a field theory with local interactions. The natural candidate is

$$\mathcal{L}_{\text{incorrect}} = |\nabla \psi|^2 + \mu |\psi|^2 + \lambda |\psi|^4. \quad (2)$$

However, this is not what we want. The phase transition (as a function of μ) at which ψ condenses does indeed represent a proliferation of ψ world lines. However, this phase transition is thermodynamically nontrivial (by setting $k = 0$ we see it is in the XY universality class) and so cannot represent the vortex phase transition we are discussing, where all local degrees of freedom are short-range correlated.

As a hint toward a better candidate, notice that correlation functions of the form

$$\langle [\psi(x)]^{N_{\text{in}}} [\psi^*(x)]^{N_{\text{out}}} (\dots) \rangle, \quad (3)$$

where (\dots) contains insertions away from x (and we omit color indices), have no meaning in the vortex problem unless

$N_{\text{in}} = N_{\text{out}}$: for obvious topological reasons, the number of vortex strands entering the vicinity of x must equal the number leaving. We are thus free to set these correlation functions to zero, which can be done by introducing a gauge symmetry:

$$\mathcal{L} = |(\nabla - iA)\psi|^2 + \mu|\psi|^2 + \lambda|\psi|^4 + \dots \quad (4)$$

The significance of this change depends on what Lagrangian for the gauge field is hidden in the ellipsis in Eq. (4). In the trivial limit of zero-gauge coupling, when nongauge fluctuations of A are completely suppressed, the thermodynamics of Eq. (2) will be unchanged and we will not have made any progress. However, a nontrivial action for A will mediate interactions between particles that depend on their charge. This is what we want: it reflects the fact that the weight of a given configuration of vortices depends on their relative orientations, as is clear from thinking about the winding of arg $w(x)$.

Our previous assumption of local interactions for ψ was in fact incorrect. The weight for a vortex configuration \mathcal{C} comes from integrating over all compatible configurations of w and this yields a long-range interaction between vortices. The role of the integral over A is to simulate this part of the statistical weight. This is the fundamental idea of XY duality [3–5,36–39,49]; in the context of the present discussion, see especially Refs. [4,38]. The coupling of loops to a gauge field has also been exploited to derive exact results for self-avoiding loops in two dimensions [50,51].

Specifically, in the presence of the gauge field each configuration \mathcal{C} of oriented loops is weighted by the expectation value

$$W(\mathcal{C}) = \left\langle \exp i \int_{\mathcal{C}} A \cdot dx \right\rangle_A \quad (5)$$

of a Wilson loop calculated using the functional integral over A only, and it is the desired form of this weight that determines the required action for A . It is particularly easy to see what this should be in the limit of large bias. Suppose that $\langle w \rangle$ is large and positive, and consider $W(\mathcal{C})$ for a single large loop. Then arg $w(x)$ will be close to zero for most x , but the vortex must be the boundary of a sheet where arg $w(x) = \pi$. This sheet will cost an ‘energy’ scaling as the minimal area \mathcal{A} enclosed by \mathcal{C} , so that $\ln W(\mathcal{C}) \sim -\mathcal{A}$.

Such an area law for the Wilson loop implies that A should be a compact gauge field [38,52,53]: we must include singular Dirac monopole configurations in the functional integral over A . While we used large bias for this discussion, monopoles are in fact present at any nonzero bias. (Since they are irrelevant in the extended phase, the zero-bias point is not distinguished from the rest of the extended phase as far as the universal statistics of vortices are concerned.)

The field theory we have arrived at, with a supervector coupled to a compact gauge field A as in Eq. (4), will appear naturally in our derivations. However, it is often more convenient to use alternative formulations that describe the same universal physics. We now discuss two such formulations. We temporarily simplify the discussion by exchanging ψ for the n -component bosonic vector z , with the replica limit $n \rightarrow 1$ in mind.

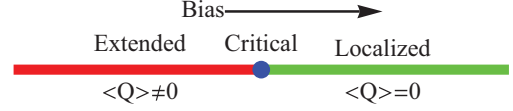


FIG. 2. (Color online) Phase diagram for vortices in short-range-correlated $w(x)$.

For the first, we switch to a σ model, with the constraint $z^\dagger z = 1$ rather than a potential for z , and the Lagrangian:

$$\mathcal{L}_{\text{CP}^{n-1}} = \frac{1}{g^2} (|\nabla z|^2 - |z^\dagger \nabla z|^2), \quad z^\dagger z = 1. \quad (6)$$

The peculiar-looking kinetic term is designed to be insensitive to the phase of z , so that we retain the gauge symmetry $z \rightarrow e^{i\phi} z$. Because of this gauge symmetry, the local degrees of freedom take values in CP^{n-1} , the manifold of n -component complex unit vectors z subject to the identification $z \sim e^{i\phi} z$ [30,31]. The σ model describes the ordering transition of spins living on this space. At long distances it is equivalent to the compact gauge theory, in which the effect of the gauge field is to confine the charged particles z in neutral composite particles, or alternately to render the overall phase of z a redundant degree of freedom.

The order parameter for the transition in this σ model is a gauge-invariant traceless $n \times n$ matrix

$$Q = zz^\dagger - z^\dagger z/n, \quad \text{tr} Q = 0. \quad (7)$$

In the limit $n \rightarrow 1$, or in the supersymmetric formulation described below, the transition is that between localized and extended vortex phases, and the extended phase is that in which $\langle Q \rangle \neq 0$. The phase diagram is shown in Fig. 2.

When $n = 2$, the order-disorder transition is simply that of the $O(3)$ model, to which the CP^1 model is equivalent. This equivalence is seen by using the Pauli matrices to form a three-component Heisenberg spin $S^i = \text{Tr} \sigma^i Q = z^\dagger \sigma^i z$: when written in terms of S , the σ model Lagrangian (6) becomes that of the $O(3)$ σ model. The fact that $n = 1$ (the CP^0 model) must be regarded as a limiting case is seen in the fact that $Q = 0$ in this limit.

A familiar idea in the context of the $O(3)$ model is that a soft-spin formulation in which S^2 is unconstrained is often more useful than a σ model in which $S^2 = 1$. The second formulation we will need is an analogous soft-spin theory for Q ,

$$\mathcal{L}_{\text{soft spin}} = \text{tr}(\nabla Q)^2 + \mu \text{tr} Q^2 + \nu \text{tr} Q^3 + \dots \quad (8)$$

(the cubic term vanishes in the special case $n = 2$). This formulation, in the $n \rightarrow 1$ limit, will be useful to us in Sec. IV in considering the effect of perturbations to the vortex problem. An immediate inference from it is that the upper critical dimension for this universality class (and for the related problem of the Anderson transition in class C [43]) is 6. Exponents for vortices can thus be computed in a $6 - \epsilon$ expansion similar to that for percolation [40–42].

The supersymmetric replacement for the replica limit of the σ model (6) is obtained simply by replacing z with ψ ,

$$\mathcal{L}_{\text{CP}^k} = \frac{1}{g^2} (|\nabla \psi|^2 - |\psi^\dagger \nabla \psi|^2), \quad \psi^\dagger \psi = 1. \quad (9)$$

In two dimensions, these supersymmetric σ models and their connection with loop models have been studied in

Refs. [29,32]. The gauge-invariant operator Q becomes a supermatrix [54],

$$Q = \psi \psi^\dagger - (\psi^\dagger \psi), \quad \text{str} Q = 0. \quad (10)$$

The transition into the ordered (extended) phase as a function of the coupling g is equivalent to the spontaneous breaking of supersymmetry and the appearance of an expectation value $\langle Q \rangle$.

This simple phase diagram can be extended in various ways. We can, for example, modify the problem so as to induce a crossover to the universality class of conventional percolation or to that for nematic vortices (Sec. IV). The gauge theory (4) also prompts us to ask about the geometrical critical behavior described by the version with a *noncompact* gauge field. Adopting the standard terminology for the case without fermions [55], we will call this the noncompact $\text{CP}^{k|k}$ model, or $\text{NCCP}^{k|k}$. Since compactness (the presence of Dirac monopoles) is a consequence of bias, the $\text{NCCP}^{k|k}$ model describes the case without an applied bias. It is a supersymmetric extension of the well-known dual theory for the XY transition [3–5,36–39] and is relevant to the statistics of vortices in the XY model at zero magnetic field [17,18]. Here we regard $w(x)$ as the order parameter for an XY model and we are no longer necessarily in the domain of short-range correlations. We will consider this noncompact theory in Sec. IV E. An added complication, in comparison to the compact case, is that it may show two separate phase transitions, one thermodynamic (associated with the onset of the Higgs mechanism) and one geometrical (associated with the breaking of supersymmetry). This is suggested by numerical results for XY vortices [17,18]. Surprisingly, in this scenario the geometrical transition may again be described by a *compact* $\text{CP}^{k|k}$ model.

III. VORTICES IN A SHORT-RANGE-CORRELATED RANDOM ENVIRONMENT

We now turn to derivations of the $\text{CP}^{k|k}$ model for vortices, beginning (Sec. III A) with a lattice model in which geometrical observables can be explicitly related to correlation functions of ψ . We then present a derivation directly in the continuum in order to emphasize the universality of the results and to see the relation with more standard treatments of duality (Sec. III B). Section III C then gives simple consequences of the field-theory description.

A. Tricolor percolation

Tricolor percolation is an elegant discretization of the vortex problem [6,8], which has become a standard setup for numerical simulations, partly because some more obvious alternatives to this model suffer from aesthetic deficiencies. For example, while it would be natural to consider random phases $\theta \equiv \arg w$ on the sites of a cubic lattice, with vortex lines piercing the plaquettes, this requires an additional rule for deciding when a plaquette is pierced by a vortex line. Also, in such a model nothing prevents vortex lines from intersecting each other, requiring a further decision about how to resolve the intersections. For a generic function $w(x)$ in the

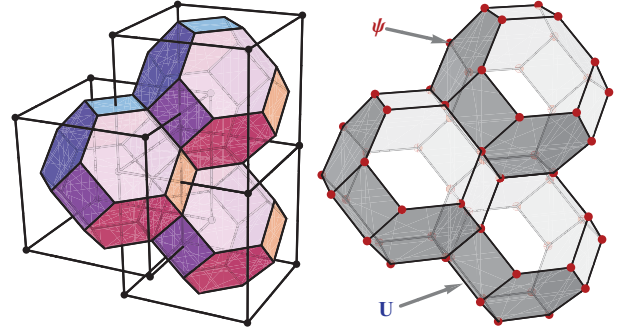


FIG. 3. (Color online) Lattice for tricolor percolation. Shown on the left is the relation to the bcc lattice. On the right the locations of the dual degrees of freedom (ψ on sites and U on links) are indicated.

continuum, by contrast, zero lines intersect with probability zero. (Intersections will be discussed in detail in Sec. IV B.)

Tricolor percolation avoids these issues by a smart choice of lattice and by discretizing not only real space but also the target space for the degrees of freedom. This idea is familiar from 2D percolation, where a single universality class encompasses both continuum percolation, which deals with zero lines of a continuous random function $v(x)$, and lattice site percolation, where the function takes only the values $v = \pm 1$ and the intersection of cluster boundaries is avoided by using sites on the triangular rather than the square lattice. For a random complex function, the minimal discretization is to let w run over the cube roots of unity, $w = 1, e^{2\pi i/3},$ and $e^{4\pi i/3}$ [7]. The configurations are therefore configurations in percolation with three colors [6] and vortices (or tricords) are lines where all three colors meet (Fig. 4).

The construction described below was introduced by Scherrer and Frieman [8] as an improvement on the model of Vachaspati and Vilenkin [7] and continues to be used for numerics in the context of cosmic strings [10]. It was introduced independently and simulated extensively by Bradley, Debierre, and Strenski [6], who realized that the model displays an interesting geometric phase transition at nonzero bias. Their interest came from a different direction: they were looking for an efficient means of generating configurations of self-avoiding polymers and were inspired by earlier work on smart walks [56].

1. Model

Figure 3 depicts a tiling of space with cells of a certain shape (truncated octahedra). The centers of these cells lie on a bcc lattice and they are the Wigner-Seitz cells for this lattice. The loops will exist on the edges of the cells, which form a second lattice known as the tetrakaidecahedral lattice. Its key features are that each link is surrounded by three cells and that its sites are four-coordinated.

Tricolor percolation configurations are generated by randomly and independently coloring the cells either red, green, or blue, with probabilities $p_R, p_G,$ and p_B . Assigning each color to a cube root of unity

$$\{\mathbf{R}, \mathbf{G}, \mathbf{B}\} \longrightarrow \{1, \omega, \omega^2\}, \quad \omega = e^{2\pi i/3}, \quad (11)$$

gives the relation with a discrete random field $w(x)$; the threefold anisotropy in its distribution is not expected to

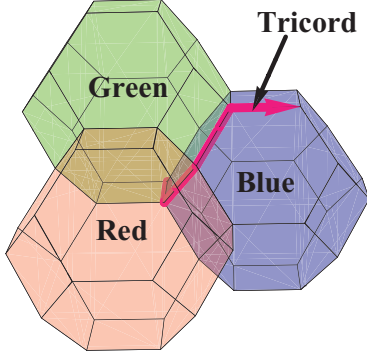


FIG. 4. (Color online) Section of tricord in tricolor percolation.

change the universality class of the transition (Sec. III B). Vortices are the lines along which all three colors meet. The design of the lattice ensures that these curves are nonbranching, self- and mutually-avoiding, and have an orientation defined by the order in which the three colors encircle the link.

Close to the symmetric point $p_R = p_G = p_B = 1/3$, tricords are in the extended phase [6,10]. Moving sufficiently far from this point, we enter the localized phase. Numerical studies of the transition have been performed only on the line $p_R \neq p_G = p_B$, which intersects the critical line at the two points $p_R = 0.417(1)$ and $0.255(5)$ [6], but the derivation below leads us to believe that the critical behavior will be the same at all points on the critical line, as expected by the authors of Ref. [6]. Note that the critical line lies *inside* the region in which all three colors percolate, so the transitions do not coincide with percolation transitions of the colors (the critical probability for site percolation on the bcc lattice is $0.2460(3)$ [6]).

2. Symmetric point

We begin with the symmetric point, where the mapping to lattice gauge theory is simplest. The gauge theory we will need is unconventional in that its Boltzmann weight is not naturally written as the exponential of something simple. This situation is standard in the study of loop ensembles: lattice magnets with Boltzmann weights of nonexponential form are useful because they give straightforward graphical expansions, which can be interpreted as loop models [57]. For a pedagogical introduction to these ideas, see Ref. [58]. The motivation here is similar: A graphical expansion of the kind standard in lattice gauge theory [1] will yield tricolor percolation configurations.

Let i, j, \dots label sites of the tetrakaidekahedral lattice and let $\langle ij \rangle$ denote a link directed from i to j . We introduce gauge fields U_{ij} on the links, which are unimodular numbers with $U_{ij} = U_{ji}^*$, and unit supervectors [27,28]

$$\psi_i = (z_i^1, \dots, z_i^{k+1}, \chi_i^1, \dots, \chi_i^k), \quad \psi_i^\dagger \psi_i = 1 \quad (12)$$

at sites. The trace Tr will denote an integral over all degrees of freedom, with the normalization $\text{Tr}1 = 1$. The normalization of ψ then implies that $\text{Tr} \psi_i^\alpha \psi_i^{\beta*} = \delta^{\alpha\beta}$.

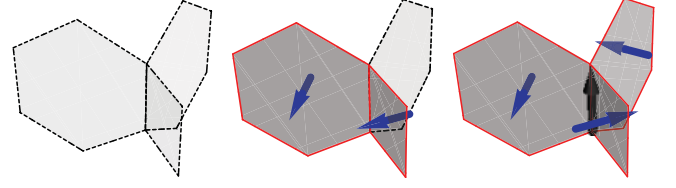


FIG. 5. (Color online) Possibilities for the graphical expansion at a link, as described in the text. In the mapping to tricolor percolation, the shaded surfaces become domain walls between different colors.

Letting F denote a face (where two cells meet) and l a link, the partition function for the lattice gauge theory is

$$Z = \text{Tr} \prod_F \left(1 + \prod_{l \in F} U_l + \prod_{l \in F} U_l^* \right) \times \prod_{\langle ij \rangle} (1 + U_{ij}^3 \psi_i^\dagger \psi_j + \text{c.c.}). \quad (13)$$

The links in the product $\prod_{l \in F}$ are oriented consistently around the face F —changing the orientation corresponds to exchanging $\prod U$ and $\prod U^*$.

To see the relation to tricolor percolation, we expand the two products in Eq. (13) (one over faces F and one over links $\langle ij \rangle$) and represent the terms graphically. The diagram for a given term is built up as follows. For each face F , we must choose either 1, $\prod U$, or $\prod U^*$. If we choose 1 we add nothing to the diagram, while if we choose $\prod U$ or $\prod U^*$ we draw in the face F together with an orientation (equal or opposite to that of the links in the product, respectively). We represent this orientation by a normal vector as in Fig. 5. For each link we must choose either 1, in which case we draw nothing, or $U_{ij}^3 \psi_i^\dagger \psi_j$, in which case we draw in the link with an arrow from j to i , or $U_{ji}^3 \psi_j^\dagger \psi_i$, in which case we orient the arrow the other way.

Only a subset of the diagrams generated this way survive after integrating over U , namely those having an equal number of U_l and U_l^* on each link l . For each link there are three possibilities (shown in Fig. 5). Either (i) none of the three faces bordered by the link is included in the diagram and neither is the link; (ii) two of the faces are included, they are consistently oriented so as to form part of a sheet of oriented surface, and the link is not included; or (iii) the link is included and so are all three of the faces, with orientations determined by that of the link and the right-hand rule. We are then left with sheets of oriented surface that close on themselves or meet along directed lines \mathcal{L} . On a finite lattice, these lines are closed loops. They are self- and mutually-avoiding and neither branch nor terminate.

Such configurations are easily mapped to tricolor percolation by regarding the sheets as boundaries between domains of different color. Fixing the color of one cell, we can color all the others by the rule that the color changes cyclically,

$$\mathbf{R} \rightarrow \mathbf{G} \rightarrow \mathbf{B} \rightarrow \mathbf{R}, \quad (14)$$

upon crossing a sheet in the same direction as its normal. The lines \mathcal{L} where three sheets meet are then the tricords. A convenient convention is to consider a finite bcc lattice and define the color *outside* the lattice to be (say) red. Thus a cell on

the boundary is red if its exterior faces (those on the boundary) are not covered by surface in the graphical expansion.

Let \mathcal{T} denote a tricolor percolation configuration with loops \mathcal{L} . At this point, having integrated over U but not ψ , we have

$$Z = \sum_{\mathcal{T}} \left\{ \text{Tr} \prod_{\mathcal{L} \in \mathcal{T}} (\psi_1^\dagger \psi_2) \cdots (\psi_L^\dagger \psi_1) \right\}, \quad (15)$$

where $1, \dots, L$ label the sites on a given loop \mathcal{L} . To evaluate the integral over the supervectors, note that each loop gives

$$\text{Tr}(\psi_1^{\alpha*} (\psi_2^\alpha \psi_2^{*\beta}) (\psi_3^\beta \psi_3^{*\gamma}) \cdots (\psi_L^\gamma \psi_L^{*\omega}) \psi_1^\omega). \quad (16)$$

The formula $\text{Tr} \psi_i^\alpha \psi_i^{\beta*} = \delta^{\alpha\beta}$ requires all color indices to be equal, say, to α , to give a nonzero result. For the fields ψ_1 and ψ_1^* , however, a reordering is necessary in order to use this formula: This gives a minus sign when α corresponds to a fermionic component of ψ . Thus the sum on α yields a fugacity per loop of $k+1-k=1$ and Z becomes as a sum over equally weighted tricolor percolation configurations $Z = \sum_{\mathcal{T}} 1 = 3^{\text{no. cells}}$.

In addition, we can use ψ to construct geometrical correlation functions. Let the operators O_i depend only on ψ . Then the graphical expansion goes through as before and we have

$$\begin{aligned} &\langle O_1, \dots, O_N \rangle \\ &= \frac{1}{Z} \sum_{\mathcal{T}} \text{Tr} O_1, \dots, O_N \prod_{\mathcal{L} \in \mathcal{T}} (\psi_1^\dagger \psi_2) \cdots (\psi_L^\dagger \psi_1). \end{aligned}$$

This formula allows us to translate correlators of ψ into the probabilities of various geometrically defined events. The other input required is a formula for the single-site traces,

$$\begin{aligned} &\text{Tr}(\psi^{\alpha_1} \cdots \psi^{\alpha_q} \psi^{*\beta_1} \cdots \psi^{*\beta_q}) \\ &= \frac{1}{q!} \langle \psi^{\alpha_1} \cdots \psi^{\alpha_q} \psi^{*\beta_1} \cdots \psi^{*\beta_q} \rangle_G, \end{aligned} \quad (17)$$

where $\langle \cdots \rangle_G$ is a Gaussian average evaluated using Wick's theorem and $\langle \psi^\alpha \psi^{*\beta} \rangle_G = \delta^{\alpha\beta}$. A simple example is the two-point function $\langle Q_i^{12} Q_j^{21} \rangle$ of the gauge-invariant operator $Q_i^{12} = z_i^1 z_i^{*2}$. The operator insertions restrict the sum over \mathcal{T} to configurations in which i and j lie on the same tricord and so gives the probability of this event. Note that in order to construct this correlation function we require $k \geq 1$. We will return to correlation functions in Sec. III C.

The naive continuum limit of the lattice gauge theory (13) involves a superfield ψ coupled to a gauge field A . The compactness of the microscopic gauge degrees of freedom U_l translates in the continuum into the presence of Dirac monopoles in A [52,53]. However, because ψ couples to U^3 rather than to U in Eq. (13), these monopoles are of three times the minimal charge allowed by the Dirac quantization condition. [We will take the minimal value allowed by the Dirac quantization condition as our unit of monopole charge, hence the factor of 3 in Eq. (18) below.]

Crudely, monopoles arise because there is a 6π ambiguity in the magnetic flux B_F passing through each face F (here F denotes a face together with a choice of orientation) and thus also in the divergence of this quantity. The flux B_F is a vector field on the links of the dual lattice, which pierce the faces F ,

and is defined modulo 6π by

$$\exp\left(\frac{i B_F}{3}\right) = \prod_{l \in F} U_l. \quad (18)$$

The divergence $(\nabla \cdot B)_B$ is defined at each site of the dual lattice or, equivalently, each tricolor percolation cell \mathcal{B} by the sum of B_F over the (outwardly oriented) faces of the cell.

Recall that an ambiguity of this kind can be resolved in two ways, which correspond to thinking about B either as a vector field whose divergence is nonzero at monopole defects or as a vector field with strictly zero divergence but with Dirac strings. If we resolve the ambiguity by restricting B_F to the domain $[-3\pi, 3\pi)$, it can have nonzero divergence at the locations of monopoles, whose charge ρ is a multiple of 3. Letting \mathcal{B} denote a site of the dual lattice or, equivalently, a tricolor percolation cell,

$$(\nabla \cdot B)_B = 2\pi \rho_B, \quad \rho_B \in 3\mathbb{Z}. \quad (19)$$

Alternatively, we can resolve the ambiguity (call the new version B') in such a way that

$$(\nabla \cdot B')_B = 0. \quad (20)$$

Here B' differs from B by the inclusion of Dirac strings, which carry 6π flux away from the monopoles along strings of adjacent plaquettes and on which B' lies outside the region $[-3\pi, 3\pi)$.

In a continuum formulation, the functional integral over A must include configurations with pointlike monopoles (of charge $\rho \in 3\mathbb{Z}$), together with singular Dirac strings, which do not incur any weight in the action [53,59]. A fugacity will also be associated with each monopole. Anticipating Sec. III B, these charge-3 monopoles are a result of threefold anisotropy in the argument of our random field $\arg w$ [38]. We postpone further discussion until Sec. III B since they will not play an important role. The symmetric point lies in the phase where ψ is condensed and A is massive by the Higgs mechanism; the appropriate description here is in terms of Goldstone modes associated with the breaking of supersymmetry, and the monopoles are irrelevant.¹ However, monopoles of charge one appear on moving away from the symmetric point and will play an important role at the transition.

3. Alternative view of the graphic expansion

For an alternative view on the graphical expansion and to connect with the discussion in Sec. II, we can expand only the second product (that over links) in the partition function (13). The diagrams now consist only of oriented links (to each of which is attached a factor $U^3 \psi^\dagger \psi$). Next, integrating over ψ kills all configurations except those in which the links form closed loops. However, it does not prevent these loops from

¹Charge-3 monopoles would play an important role in a transition into the localized phase, at zero bias, induced by Potts-like interactions between the colors of the cells [these can be included by modifying the Boltzmann weight for the gauge field of Eq. (13)].

intersecting. If \mathcal{C} denotes such a configuration of loops, then

$$Z = \sum_{\mathcal{C}} \text{Tr}_U \prod_F \left(1 + \prod_{l \in F} U_l + \prod_{l \in F} U_l^* \right) \prod_{\text{loops } \mathcal{L}} \left(\prod_{l \in \mathcal{L}} U_l^3 \right).$$

In the world-line picture, we see that each loop configuration is weighted by the expectation value of a Wilson loop $W(\mathcal{C})$,

$$Z \propto \sum_{\mathcal{C}} W(\mathcal{C}), \quad W(\mathcal{C}) = \left\langle \prod_{\mathcal{L} \in \mathcal{C}} \prod_{l \in \mathcal{L}} U_l^3 \right\rangle_U, \quad (21)$$

evaluated using the U -dependent part of the Boltzmann weight:

$$\langle \cdots \rangle_U = \frac{1}{Z_U} \text{Tr}(\cdots) \prod_F \left(1 + \prod_{l \in F} U_l + \prod_{l \in F} U_l^* \right). \quad (22)$$

The expectation value $W(\mathcal{C})$ assigns the correct entropic weight to each tricord configuration. This is a nontrivial weight given by summing over the possible domain wall configurations compatible with a given set of tricords. In particular the weight is zero for loop configurations that are not allowable tricord configurations, including those with intersections of loops.

4. Away from the symmetric point

At first sight, extending the correspondence between tricolor percolation and lattice gauge theory away from the symmetric point presents a problem, since the basic objects in the graphical expansion of the lattice gauge theory are the domain walls between colors rather than the colors themselves. However, this is not in fact the case. To begin with we consider changing the probabilities for a single cell: This allows the appearance of a Dirac monopole of unit charge at the center of that cell (Fig. 6).

Again we consider a system with a boundary and stipulate that the color outside the boundary is **R**. The color of an interior cell \mathcal{B} is then determined by the signed number of domain walls we must cross (modulo 3) in order to reach a point outside the boundary, starting from \mathcal{B} .

Pick any path \mathcal{P} from the interior of \mathcal{B} to the boundary. Now introduce a variable η that will run over the cube roots of

unity, and modify the Boltzmann weight for the lattice gauge theory on the faces crossed by \mathcal{P} by the substitution

$$\left(1 + \prod_{l \in F} U_l + \prod_{l \in F} U_l^* \right) \rightarrow \left(1 + \eta \prod_{l \in F} U_l + \eta^* \prod_{l \in F} U_l^* \right).$$

The factor η goes with the term in which the plaquette's normal is parallel to the path \mathcal{P} and the factor η^* with the term where the normal and \mathcal{P} are antiparallel.

For a given term in the graphical expansion, we can read off the color of \mathcal{B} from the power of $\eta \bmod 3$: if all the η^* and η cancel, \mathcal{B} is red, and if there is an η^* or an η left over, \mathcal{B} is green or blue, respectively. Thus, to change the probabilities away from $1/3$ for this cell, we make η a dynamical variable and include in the partition function a sum over η with the weight

$$\sum_{\eta=1, \omega, \omega^2} (p_R + p_G \eta + p_B \eta^*) (\cdots), \quad (23)$$

where the ellipsis stands for the other factors in the partition function. The graphical expansion goes through as before, with each tricolor percolation configuration acquiring a power of η determined by the color of \mathcal{B} . Performing the sum over η , only one of the three terms p_R , $p_G \eta$, or $p_B \eta^*$ gives a nonzero contribution, yielding the appropriate probability.

We now specialize to the line $p_R \geq p_G = p_B$ (we will briefly discuss the general case in Sec. III B). Writing $\eta = \omega^{\rho_B}$, where $\rho_B \in \{0, \pm 1\}$ will become the monopole charge in the cell, the above weight is

$$\sum_{\rho_B=0, \pm 1} h^{|\rho_B|} (\cdots), \quad h = \frac{3}{2} \left(p_R - \frac{1}{3} \right). \quad (24)$$

At the symmetric point, ρ_B is forced to be zero and the modification disappears. Moving away from the symmetric point allows ρ_B to fluctuate.

To see the effect of this on the gauge theory, consider the Boltzmann weight in terms of gauge field configurations, rather than in terms of the graphical expansion. The Boltzmann weight for the plaquettes on \mathcal{P} is now maximized not by a magnetic flux $B_F = 0 \pmod{6\pi}$, but by a flux $B_F = 2\pi\rho_B \pmod{6\pi}$ oriented antiparallel to \mathcal{P} . Thus \mathcal{P} is a Dirac string for a monopole of charge ρ_B located at \mathcal{B} , and h is a fugacity for monopoles.

We may immediately extend the above argument to a uniform (rather than local) change in the probabilities. Each cell acquires a monopole charge ρ_B and we write

$$Z = \sum_{\{\rho_B\}} h^{N_\rho} \text{Tr} \prod_F (1 + e^{i(B_F - D_F)/3} + \text{c.c.}) \times \prod_{\langle ij \rangle} (1 + U_{ij}^3 \psi_i^\dagger \psi_j + U_{ji}^3 \psi_j^\dagger \psi_i). \quad (25)$$

Here N_ρ is the total number of monopoles $N_\rho = \sum_B |\rho_B|$; B_F is the magnetic flux through the face F , defined in Eq. (18); and D_F is the flux through F due to the Dirac strings \mathcal{P} emanating from the monopoles. Since the geometry of these strings is arbitrary, D_F is arbitrary except that it must be a multiple of 2π on every face and (viewed as a vector field on the dual lattice) must have the right divergence $(\nabla \cdot D)_B = -2\pi\rho_B$. We fix it by any suitable convention, for example by setting all the Dirac strings to be parallel to a given axis.

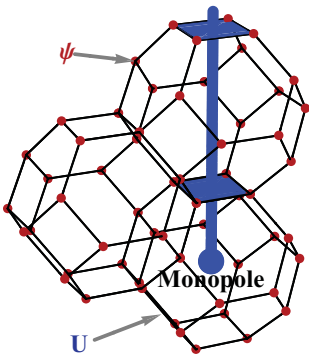


FIG. 6. (Color online) Away from the symmetric point, monopoles of unit charge appear. In the presence of monopoles, the Boltzmann weight for the gauge field is modified on the plaquettes crossed by Dirac strings.

A peculiarity of the above mapping is that monopoles appear in two distinct ways: as a result of the microscopic compactness of U and as a result of a modification to the Boltzmann weight for the gauge field. In the continuum derivation (Sec. III B) all monopoles appear in the same way.

5. Interpretation

The lattice model (25) describes a supersymmetric matter field, whose world lines are the tricords, coupled to a gauge field. The partition function includes a sum over monopole configurations, with monopoles weighted by a fugacity that tends to zero at the symmetric point. If we write a continuum Lagrangian with a gauge field

$$\mathcal{L} = |(\nabla - iA)\psi|^2 + \mu|\psi|^2 + \lambda|\psi|^4 + \kappa B^2, \quad (26)$$

we must take into account that in the kinetic-energy term B^2 for the gauge field we should not include contributions from Dirac strings and that the monopole fugacity is hidden in the measure for the path integral [53,59] (Sec. III B). However, as mentioned in Sec. II, the universal behavior is equally well captured by a σ model formulation. The Lagrangian for this σ model is given in Eq. (9).

The microscopic form of the action (25) does not of course give us the values of the parameters (such as μ or g) in the coarse-grained action. However, it is easy to identify the phases of tricolor percolation with the phases of the above theory.

The extended phase is the ordered phase of the σ model at small g , or the Higgs phase (at negative renormalized μ) in the language of the gauge theory. The gauge symmetry (Higgs mechanism) removes one Goldstone mode, but $2k$ bosonic and $2k$ fermionic Goldstone modes remain. They are governed by free-field theory and this leads to Brownian behavior for the vortex lines (see Sec. III C for further details). Note that that while the symmetric point may seem special in that charge-1 monopoles are forbidden there, it is no different from the rest of the extended phase as far as universal properties are concerned.

The localized phase is the disordered phase of the σ model. In the gauge theory, both ψ and A are massive, the latter because of the proliferation of monopoles.

Note that when $k = 0$ the two phases collapse into one, and the Goldstone modes that distinguish the extended from the localized phase disappear. This reflects the fact that when $k = 0$ we can no longer construct the geometrical correlation functions needed to probe the transition.

B. Vortices in the continuum

Focusing on tricolor percolation allowed us to introduce the dual degrees of freedom in an explicit way, but it had the disadvantage that the more general applicability of the $\text{CP}^{k|k}$ model was obscured by microscopic details specific to that model, such as threefold anisotropy. In order to make the universality of the result plausible, we now give an alternative derivation in the continuum, building on standard duality arguments for the XY model [3–5,36–39].

Let $\theta(x)$ denote the phase of the random complex function

$$\theta(x) = \arg w(x). \quad (27)$$

This field is compact, $\theta(x) \equiv \theta(x) + 2\pi$, and ill-defined at the vortices. We expect universal features of their statistics

to be independent of the precise details of its probability distribution so long as correlations in $\theta(x)$ are short range (just as universal quantities in percolation are robust to changes in the probability distribution of the microscopic variables). Thus we choose the most convenient distribution, taking $\theta(x)$ to be governed by the Lagrangian \mathcal{L}_{XY} of the classical XY model,

$$\mathcal{L}_{XY} = \frac{K}{2}(\nabla\theta)^2 - H \cos\theta. \quad (28)$$

When $H = 0$, this has a phase transition at a critical coupling K_c , with short-range correlations in the disordered regime $K < K_c$. The magnetic field H allows us to introduce bias, i.e., to tune the mean value of the random function $w(x)$: at $K < K_c$ and sufficiently small $|H|$ vortices are in the extended phase, and we enter the localized phase by increasing $|H|$. This gives a critical line separating extended from localized vortices. Note that in this formulation the core energy of a vortex is hidden in the short-distance regularization.

As is well known, the phase transition of the 3D XY model at zero magnetic field is dual to a Higgs transition for a Lagrangian with a noncompact gauge field coupled to a single complex scalar z [3–5,36–39],

$$\mathcal{L}_{\text{dual}} = |(\nabla - iA)z|^2 + \kappa(\nabla \times A)^2 + \mu|z|^2 + \lambda|z|^4,$$

where the role of the gauge field A is to encode the long-range interactions between vortices that come from integrating over all $\theta(x)$ compatible with a given vortex configuration. The disordered phase of the XY model, in which vortices proliferate, corresponds to the Higgs phase of the gauge theory (at negative renormalized μ), where the field z has condensed and A is massive by the Higgs mechanism. The ordered phase of the XY model corresponds to the disordered phase of the noncompact dual theory in which z is massive and A massless.

This dual theory differs from the one we require in two ways. First, we must consider the theory at nonzero H . The operators $e^{\pm i\theta}$ correspond in the dual language to the insertion of magnetic monopoles, so the perturbation $H \cos\theta$ leads to compactness of the gauge field, a point also stressed by Kleinert [38]. Usually this perturbation would lead simply to a massive theory. However, in order to capture geometrical correlation functions, we must extend the theory, using supersymmetry to introduce extra degrees of freedom. This allows us to obtain a nontrivial field theory (the $\text{CP}^{k|k}$ model) at finite H .

Following the general approach described in Ref. [39], the functional integral over $\theta(x)$ can be split into a sum over vortex configurations \mathcal{C} , and for each \mathcal{C} an integral over vortex field configurations. To do this we associate a representative field configuration $\phi(x)$ with each vortex configuration (by any suitable convention) and write $\theta(x)$ in terms of $\phi(x)$ and a vortex-free field $\tilde{\theta}(x)$:

$$\theta(x) = \phi(x) + \tilde{\theta}(x). \quad (29)$$

Here $\tilde{\theta}(x)$ should be treated as a single-valued field in $[-\infty, \infty)$: while Eq. (29) defines $\tilde{\theta}(x)$ only modulo 2π , this ambiguity can be removed using the line integral of $\nabla[\theta - \phi]$.

The path integral over θ now becomes $\int_{\mathcal{C}} \int \mathcal{D}\tilde{\theta}$, where $\int_{\mathcal{C}}$ is shorthand for the sum over vortex configurations. To make this precise we would have to fix a regularization of the theory. This will not be necessary for our present purposes; however, it

is important to note that \int_C includes a local weight determined by physics on the scale of the UV cutoff. This will include an action cost per unit length of vortex and local interactions between vortex strands. Note that one way to think of the σ model action (28) is as an extreme case in which fluctuations in the amplitude of $w \sim e^{i\theta}$ are restricted to vortex tubes with a thickness on the scale of the short-distance cutoff.

Inserting the expression (29) for θ into the action and making a Hubbard-Stratonovich transformation with a three-component vector field ξ [39], one obtains the Lagrangian

$$\mathcal{L}'_{XY} = (2K)^{-1}\xi^2 - i\xi \cdot (\nabla\tilde{\theta} + \nabla\phi) - H \cos\theta. \quad (30)$$

When $H = 0$, the integral over $\tilde{\theta}$ forces ξ to be divergence-free. A nonzero H relaxes this constraint, allowing pointlike monopole defects where $\nabla \cdot \xi \neq 0$. This can be seen by a standard trick [53]: expanding the Boltzmann weight in H gives

$$\exp\left(H \int d^3x \cos\theta(x)\right) = \int_\rho \exp\left(i \int d^3x \rho(x)\theta(x)\right), \quad (31)$$

where $\rho(x)$ is a density for a variable number N of monopoles, of strengths $\rho_i = \pm 1$, at locations r_i ,

$$\rho(x) = \sum_{i=1}^N \rho_i \delta(x - r_i),$$

and \int_ρ is shorthand for an integral over monopole configurations, weighted by a fugacity $H/2$ per monopole and with a factor of $1/N!$ for indistinguishability:

$$\int_\rho \text{ denotes } \sum_{N=0}^{\infty} \frac{(H/2)^N}{N!} \prod_{i=1}^N \left(\sum_{\rho_i=\pm 1} \int d^3r_i \right). \quad (32)$$

Integrating over $\tilde{\theta}$, the XY partition function is

$$Z_{XY} = \int_C \int_\rho \mathcal{D}\xi \delta(\nabla \cdot \xi - \rho) \times \exp \int d^3x \left(-\frac{1}{2K} \xi^2 + i\xi \cdot \nabla\phi + i\rho\phi \right). \quad (33)$$

Since the pointlike charges in ρ are quantized, the 2π ambiguity in ϕ does not affect the Boltzmann weight.

We resolve ξ into a divergence-free part $\nabla \times A$ and a collection of Dirac strings D that carry the flux away from the monopoles along singular flux lines, by writing

$$\xi = \frac{1}{2\pi} (\nabla \times A - D) \quad \text{for} \quad \nabla \cdot D = -2\pi\rho. \quad (34)$$

(The geometry of the Dirac strings is chosen by some convention and does not fluctuate.)

As a consequence of Eq. (34), the functional integral over A includes Dirac monopole configurations with singular strings D . The form of the kinetic-energy term for the gauge field, involving ξ^2 rather than $(\nabla \times A)^2$, means that these strings do not cost any energy. Actions of this form for compact gauge fields are discussed in Refs. [53,59].

We now simplify the action in Eq. (33). Since $\nabla\phi$ is not the derivative of a continuous, single-valued function (in terms of

such a derivative, $\nabla\phi = -ie^{-i\phi}\nabla e^{i\phi}$), the last two terms do not cancel on integrating by parts. Instead they yield

$$i \int d^3x (\xi \cdot \nabla\phi + \rho\phi) = \frac{i}{2\pi} \int d^3x A \cdot (\nabla \times \nabla\phi) + i \int_{\text{surfaces of discontinuity}} D \cdot d^2S. \quad (35)$$

The second integral counts intersections of the Dirac strings with the surfaces on which ϕ jumps by 2π . Since this is an integer multiplied by 2π , the term does not contribute to the Boltzmann weight. However, the first term provides a coupling of the gauge field to a singular current

$$J = \frac{1}{2\pi} \nabla \times \nabla\phi, \quad (36)$$

which runs along the vortices.

At this point we have a theory of fluctuating vortex lines (flux lines of J) coupled to a compact gauge field, with the partition function

$$Z_{XY} = \int_C \int_\rho \int \mathcal{D}A \exp \left(- \int d^3x \left(\frac{1}{2K} \xi^2 + iA \cdot J \right) \right).$$

Note that because of the quantization of the monopole charge and of the vortex flux, the Dirac strings do not couple to J . The dependence on H is in the integral over monopole configurations (32); when $H = 0$, monopoles are absent and A is noncompact.

The integral over vortex configurations, so far denoted only symbolically by \int_C , is over configurations of oriented loops, with local interactions between segments of loop. The key point of XY duality is that we can equally well view these loops as world lines of charged particles [39]. In order to be able to construct geometrical correlation functions, we must introduce additional species of bosonic and fermionic particles, using a supervector ψ rather than a single complex scalar z [28]. A concrete example of this correspondence between loops and supervectors was displayed in Sec. III A. The $A \cdot J$ term in the action implies that the gauge field couples to ψ in the usual way. The interactions between vortex segments in \int_C translate into local interactions for ψ :

$$\mathcal{L}_{\text{dual}} = |(\nabla - iA)\psi|^2 + \mu|\psi|^2 + \lambda|\psi|^4 + \dots \quad (37)$$

The line of transitions out of the extended phase at $K < K_c$ and $|H| > 0$ is governed by the (compact) $\text{CP}^{k|k}$ model. When $H = 0$ we obtain instead the $\text{NCCP}^{k|k}$ model, a supersymmetric extension of the conventional dual theory for the XY transition. We will consider the phase diagram of this noncompact theory in Sec. IV E.

The discussion of tricolor percolation in Sec. III A focused on the line $p_G = p_B$ in the two-dimensional parameter space of the model. As a caricature of tricolor percolation away from this line, we can modify the potential of Eq. (28) to $H \cos\theta + H_3 \cos 3(\theta - \Theta)$. Expanding in H_3 leads to the charge-3 monopoles we expect from Sec. III A (the connection between monopoles and anisotropy has been noted in Ref. [38]). For general Θ these have a complex fugacity $H_3 e^{\pm 3i\Theta}/2$, just as Eq. (23) leads to complex monopole fugacities for general

p_R , p_G , and p_B .² Changing the bare values of H_3 or Θ is not expected to change critical behavior in the extended phase or at the $\text{CP}^{k|k}$ critical point, where the low-energy degrees of freedom are the neutral ones in the supermatrix Q . In particular, we expect to see the same behavior for tricolor percolation everywhere on the critical line.³

C. Statistics of vortex lines

Having identified the field theory for vortex lines in short-range correlated $w(x)$, we now use it to answer some natural questions about their statistics.

1. Geometrical correlation functions

The geometrical correlation functions of interest to us, the probabilities of various geometrically defined events, can be expressed as correlation functions of ψ . At a formal level the relationships follow from Eq. (17), as indicated in Sec. III A2, but they can easily be anticipated by thinking of vortices as particle world lines.

Let $G(x - y)$ be the probability that a single vortex visits both x and y . (Strictly, since we are working in the continuum and taking the vortices to be of infinitesimal thickness, we should define a small ϵ and say “visits regions of size ϵ around x and y .”) At the critical point and in the localized phase all vortices are finite, and the contributions to $G(x - y)$ come from configurations in which a finite loop visits the two points. In the extended phase there are also contributions of a different type, with x and y lying on a single infinite strand. We first discuss the cases where these infinite strands are absent.

The operators needed to construct geometrical correlators are combinations of the gauge-invariant bilinear operators $Q^{\alpha\beta} = \psi^\alpha \psi^{*\beta} - (\psi^\dagger \psi) \delta^{\alpha\beta}$. In fact, we will need only $Q^{\alpha\beta}$ for $\alpha, \beta \leq k + 1$ and $\alpha \neq \beta$, i.e., $z^\alpha z^{*\beta}$ for $\alpha \neq \beta$. As discussed in Sec. II, such an insertion restricts the loop expansion of the partition function to configurations in which the point x has an incoming strand of color α and an outgoing strand of color β —or equivalently, configurations in which there is a loop passing through x that changes color from α to β at x . Thus we see that the correlation function $\langle Q^{\alpha\beta}(x) Q^{\beta\alpha}(y) \rangle$ forces a single loop to visit both x and y :

$$G(x - y) \propto \langle Q^{12}(x) Q^{21}(y) \rangle. \quad (38)$$

Similarly, the probability that m loops connect the vicinity of x to the vicinity of y scales like $\langle Q^{12}(x)^m Q^{21}(y)^m \rangle$. As a further example, the probability that the three sites x , y , and z all lie on the same vortex can be written $F(x, y, z) \propto \langle Q^{12}(x) Q^{23}(y) Q^{31}(z) \rangle$.

2. Vortices in the critical region

At the critical point, the two-point function decays as $G(x) \sim 1/|x|^{1+\eta}$, where η is the anomalous dimension of

Q . Moving slightly away from criticality, a finite correlation length appears, scaling as $\xi \sim |h - h_c|^{-\nu}$ if h parametrizes the bias: in the localized (disordered) phase ξ is a measure of the typical size of a large loop [e.g., $G(x)$ decays with the Ornstein-Zernicke form $|x|^{-1} e^{-|x|/\xi}$] and in the extended (ordered) phase, anticipating the following section, it sets the scale beyond which correlations are Brownian. In this phase Q acquires a nonzero expectation value $\langle Q \rangle \sim (h_c - h)^\beta$, which gives the probability that a given point lies within a fixed small distance of an infinite vortex or that a given link in tricolor percolation lies on an infinite tricord.

A scaling argument [60,61] relates η to the fractal dimension d_f of the critical vortices and to the exponent τ governing the number density $n(l)$ of vortex loops of length l . With $n(l) \sim l^{-\tau}$,

$$d_f = \frac{5 - \eta}{2}, \quad \tau = \frac{11 - \eta}{5 - \eta}. \quad (39)$$

Hyperscaling in three dimensions gives $\beta = \nu(1 + \eta)/2$.

Precise numerical values are available from simulations of a lattice loop model [24] that is also in the $\text{CP}^{k|k}$ universality class [33]:

$$d_f = 2.534(9), \quad \tau = 2.184(3), \quad \nu = 0.9985(15). \quad (40)$$

This yields $\eta = -0.068(18)$.

Clearly, this critical point is not accurately described by mean-field theory, which would require $\nu = 1/2$. However, it is interesting to consider what the mean-field prediction for d_f is: surprisingly, it is not the trivial value $d_f^{\text{MF}} = 2$, but the fairly accurate value $d_f^{\text{MF}} = 5/2$. The reason one might have expected $d_f^{\text{MF}} = 2$ is that mean-field theory corresponds to a free-field description of the critical point and free-field theory is usually associated with Brownian walks (as in the extended phase, where free-field theory for the Goldstone modes leads to Brownian behavior). Here, however, that is not the case because the appropriate free theory is formulated in terms not of ψ but of the composite field Q : in a replica approach,

$$\mathcal{L}_{\text{free}} = \text{tr}(\nabla Q)^2. \quad (41)$$

This leads to $\eta = 0$ and a fractal dimension $d_f^{\text{MF}} = 5/2$. Since η is small, as is typical for critical points in three dimensions, this approximation is quite accurate. The fact that d_f is close to $5/2$ rather than to 2 is a clear symptom of confinement and the fact that vortices are world lines not of Q but of its square root ψ . [In the context of quantum magnets described by $\text{CP}^1/\text{NCCP}^1$ models [55], a large value of η has been used as a diagnostic for deconfinement, i.e., the noncompactness of the $U(1)$ gauge field.] The above assumes a soft-spin formulation for Q , which we discuss further in Sec. IV.

3. Extended phase: Goldstone modes and Brownian loops

In three dimensions, a random walker need never return to its starting point, but can escape to infinity. The same is true of vortices in the extended phase. Of course, in a finite system there is no such thing as escape to infinity: what takes its place depends on the boundary conditions. The simplest and most natural choice is to allow vortices to terminate on the boundary. The possibility of escaping to infinity then means that a vortex strand passing through the origin has a finite probability of

²It is a matter of convention whether the charge-1 or the charge-3 monopoles have the nonpositive fugacity.

³While a complex fugacity for monopoles may seem a drastic perturbation, note that this fugacity will naturally renormalize toward positive values. Crudely, after a coarse-graining step we cannot distinguish a charge-3 monopole from three charge-1 monopoles, so the renormalized fugacity $\sim aH_3 e^{3i\Theta} + bH^3 + \dots$ with $a, b > 0$.

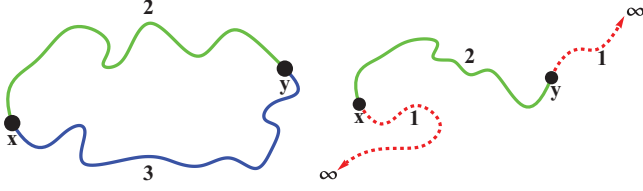


FIG. 7. (Color online) Dominant contributions to the correlation functions $\langle Q^{23}(x)Q^{32}(y) \rangle \sim |x - y|^{-2}$ and $\langle Q^{12}(x)Q^{21}(y) \rangle \sim |x - y|^{-1}$ in the extended phase as described in the text. The orientations of the loops are not indicated.

reaching the boundary, even in the thermodynamic limit. If the linear size of the system is L , then these system-spanning lines have an arc length of order L^2 , since their fractal dimension is 2. In contrast, if we were to impose periodic boundary conditions, these lines would join up into loops with typical length L^3 [25,33,62].

In order to discuss geometric correlation functions in the extended phase we must separate out the infinite strands. We take a finite system in which vortex lines can end on the boundary and require that any line that does so has a specific (bosonic) color, say, $\alpha = 1$. With this protocol the fugacity per vortex is still one, regardless of whether the vortex forms a closed loop or ends on the boundary. In the $\text{CP}^{k|k}$ language, these boundary conditions correspond to applying a symmetry-breaking field for the $\text{CP}^{k|k}$ order parameter on the boundary.⁴ The extended phase is the ordered phase of the field theory and this boundary condition fixes the direction of the order parameter to be $\psi \propto (1, 0, \dots, 0)$. We may fix the gauge freedom by taking the first component to be real and positive. Fluctuations involve k (complex) bosonic Goldstone modes, $\phi^2, \dots, \phi^{k+1}$, as well as k fermionic ones, which are just the original fermions χ : $\psi \sim (\sqrt{1 - |\phi|^2 - |\chi|^2}, \phi, \chi)$. To leading order in these modes (for $\alpha, \beta = 2, \dots, k + 1$),

$$Q^{11} \sim 1, \quad Q^{\alpha 1} \sim \phi^\alpha, \quad Q^{\alpha\beta} \sim \phi^\alpha \phi^{*\beta}.$$

Since $\langle \phi^\alpha(x) \phi^{*\beta}(y) \rangle \propto \delta^{\alpha\beta} |x - y|^{-1}$ we therefore have

$$\langle Q^{23}(x)Q^{32}(y) \rangle \sim \frac{1}{|x - y|^2}, \quad \langle Q^{12}(x)Q^{21}(y) \rangle \sim \frac{1}{|x - y|}.$$

The first of these gives the probability that x and y lie on the same *finite* loop since strands of colors 2 and 3 are prohibited from escaping to infinity (Fig. 7). In contrast, the second of them is dominated by configurations in which a single *infinite* vortex passes through both points, that is, they are connected by a single finite strand rather than by two strands as for points sharing a finite vortex.

In general, one finds that the probability $G_M(x)$ that two regions separated by a distance $|x|$ are connected by M (finite) strands of vortex scales like $|x|^{-M}$:

$$G_M(x) \sim |x|^{-M}.$$

⁴For tricolor percolation this is done by modifying the Boltzmann weight for the boundary links, replacing $U_{ij}^3 \psi_i^\dagger \psi_j$ with $U_{ij}^3 z_i^{*1} z_j^1$. In the scaling limit this yields a boundary magnetic field having the effect described.

(The correlator G discussed in the context of the localized phase and critical point corresponds to G_2 ; in this regime $G_M = 0$ for odd M .) These simple exponents indicate that the vortices behave on large scales like free random walks—each factor of $1/|x|$ is just the probability that a random walker, starting in one of the regions, happens to visit the other. It follows that the fractal dimension is 2 and the number density of loops of length l behaves as $n(l) \sim l^{-5/2}$.

As another example, we easily check that the probability that three distant sites lie on the same vortex behaves as

$$F(x_1, x_2, x_3) \sim \frac{r_{12} + r_{23} + r_{31}}{r_{12}r_{23}r_{31}}, \quad r_{ij} = |x_i - x_j|,$$

being dominated by contributions from infinite vortices, while the probability that they lie on the same finite vortex scales as $(r_{12}r_{23}r_{31})^{-1}$.

This result—Brownian statistics for vortices in the extended phase—is expected on the basis of numerical simulations [7,8,10]. Surprisingly, a clear theoretical derivation has not previously been provided, although a heuristic explanation [8] has been given by analogy with polymers in the melt [26].

IV. VARIATIONS AND PERTURBATIONS

So far we have discussed universal behavior for vortices in short-range-correlated environments. In this section we consider the stability of this universality class to perturbations, including long-range correlations in the complex field and a perturbation leading to a crossover to conventional percolation, and also the universal behavior of other kinds of vortex problem: unoriented vortices in nematics and vortices in the vicinity of the XY critical point. We also briefly discuss the $6 - \epsilon$ expansion for vortices.

A useful language for considering perturbations of the $\text{CP}^{k|k}$ model is the soft-spin Lagrangian for the gauge-invariant composite field Q [Eq. (8)]. Since some perturbations cannot be written in the supersymmetric theory, and also for simplicity, we use the replica formulation in which Q is an $n \times n$ traceless Hermitian matrix with the implied limit $n \rightarrow 1$.

Microscopically, this matrix is given by $Q = zz^\dagger - 1/n$, with $z^\dagger z = 1$, and thus satisfies the nonlinear constraint $(Q + 1/n)^2 = (Q + 1/n)$ in addition to the linear one $\text{tr} Q = 0$. In a soft-spin approach, we might imagine imposing a softened version of the former via a potential

$$\begin{aligned} &\text{tr}((Q + 1/n)^2 - (Q + 1/n))^2 \\ &= \text{const} + \frac{n^2 - 6n + 6}{n^2} \text{tr} Q^2 + \frac{2(2 - n)}{n} \text{tr} Q^3 + \text{tr} Q^4. \end{aligned}$$

Of course these bare coefficient values have no significance except to draw our attention to the fact that (except in the case $n = 2$, where there is an additional symmetry $Q \rightarrow -Q$) there is a symmetry-allowed cubic term [41,43,63]. For $n > 2$, this term implies that the mean-field prediction (expected to be valid in a large number of spatial dimensions) is a first-order transition. However, this conclusion does not apply to the replica limit $n \rightarrow 1$. Such a situation is familiar from percolation [64].

A. Upper critical dimension and $6 - \epsilon$ expansion

The cubic term implies that the upper critical dimension for CP^{n-1} in the limit $n \rightarrow 1$ (and $\text{CP}^{k|k}$) is expected to be 6. Certain correlation functions for Anderson localization in symmetry class C are also described by this field theory and can in fact be mapped to correlation functions in classical loop models described by $\text{CP}^{k|k}$ [24,29,33,44–46], so we expect 6 to be an upper critical dimension for that problem too [43].

We may also consider the $6 - \epsilon$ expansion, which was performed for theories with a single cubic invariant in Refs. [40–42]. Equation (8) at $n > 1$ is precisely the theory considered in Ref. [41], in the form given by writing Q in terms of $n^2 - 1$ real scalar fields t_i and the $\text{SU}(n)$ generators T_i via $Q = t_i T_i$. With the tensor d_{ijk} defined by $\{T_i, T_j\} = \frac{1}{n} \delta_{ij} + d_{ijk} T_k$,

$$\mathcal{L}_{\text{soft spin}} = \frac{1}{2}(\nabla t)^2 + \frac{\mu}{2}t^2 + \frac{\nu}{4}d_{ijk}t_i t_j t_k. \quad (42)$$

Coefficients in the ϵ expansion are given by contractions of the ds [40]. In the limit $n \rightarrow 1$, to order ϵ^2 ,

$$\begin{aligned} \nu &= \frac{1}{2} + \frac{5\epsilon}{76} + \frac{733\epsilon^2}{27436} + \dots, \\ \eta &= -\frac{\epsilon}{19} - \frac{166\epsilon^2}{6859} + \dots. \end{aligned} \quad (43)$$

In three dimensions, this gives $\nu \simeq 0.94$ and $\eta \simeq -0.38$, to be compared with Eq. (40). The expressions (43) are compatible with the $\mathcal{O}(\epsilon)$ results for class C localization in Ref. [43]. Note that the σ model formulation (9) leads to an alternative approximation scheme for exponents via the $2 + \epsilon$ expansion, which has been performed to high orders; see Ref. [44] and references therein.

The field theory in higher dimensions can again be related to loops, and it is natural to conjecture that in d dimensions these have the statistics of zero lines in a random $(d - 1)$ -component vector field (w_1, \dots, w_{d-1}) . (Such lines are oriented in any number of dimensions, with orientation given by the vector $\epsilon_{\mu_1, \dots, \mu_d} \partial_{\mu_2} w_1 \cdots \partial_{\mu_d} w_{d-1}$.)

B. Vortex gluing and percolation

In cases where intersections of vortices cannot be neglected (such as in the lattice XY model), a convention is required for defining geometrical correlation functions or, in other words, for determining which vortex strands are regarded as connected. We may either adopt a convention in which vortices remain topologically one dimensional, or ‘glue’ intersecting strands together to form branching, netlike clusters [17,18]; our choice may affect the observed critical behavior. We now address this issue. The conclusion is that with the former convention (when vortices are topologically one dimensional), the results of the supersymmetric dual theory continue to apply even in the presence of intersections, whereas vortex gluing induces a crossover to conventional percolation. This crossover can be seen in the soft-spin language. Another reason for clarifying the relation to percolation is that it has previously been believed that some or all of the exponents for tricolor percolation coincide with those for conventional

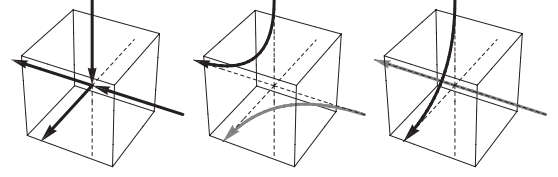


FIG. 8. Shown on the left are vortex strands intersecting at a site of the dual lattice (cube of the original lattice) in a cubic lattice XY model. According to the stochastic rule, we choose randomly from the ways of resolving the intersection (middle and right).

percolation [6,10,65]. The soft-spin formulation makes clear that this is not the case.

A microscopic model with intersections can be got by modifying the lattice for tricolor percolation (specifically, we can replace the bcc with the fcc lattice, so the cells are rhombic dodecahedra). Alternately, we can use the lattice XY model. Considering such models shows that the dual supersymmetric theory ($\text{CP}^{k|k}$ or $\text{NCCP}^{k|k}$) still applies if we resolve vortex intersections according to the stochastic rule used in some numerical simulations [17,18]: at an intersection with q incoming and q outgoing vortex strands, we choose randomly from the $q!$ ways of pairing the former with the latter (Fig. 8). This rule is a simple consequence of formulas like Eq. (17) for the traces at a single site.

To induce a crossover to percolation, we modify the rule by gluing intersecting vortices with a probability $c > 0$ [17,18].⁵ In the replica language, each strand acquires a color $\alpha = 1, \dots, n$, and to glue strands we must force their colors to coincide; the probability G_{perc} of two links lying in the same cluster will then be given by

$$G_{\text{perc}} \sim \langle Q^{\alpha\alpha} Q^{\alpha\alpha} \rangle, \quad (44)$$

being simply related to the probabilities of their being the same or different colors. We can check in the graphical expansion of a lattice model with intersections (we omit details) that the perturbation induced by the gluing is of the form $\delta\mathcal{L} \sim -c \sum_{\alpha} |z^{\alpha}|^4$. As expected, this operator creates a meeting of four world-line strands of the same color. The perturbation breaks rotational symmetry in replica space, yielding an additional mass term for the off-diagonal elements of Q in $\mathcal{L}_{\text{soft spin}}$. Subtracting a multiple of $\text{tr} Q^2$, which in the absence of the symmetry-breaking term would merely shift the critical point,

$$\delta\tilde{\mathcal{L}} \sim \sum_{\alpha \neq \beta} |Q^{\alpha\beta}|^2. \quad (45)$$

$\delta\tilde{\mathcal{L}}$ is expected, on the basis of its naive scaling dimension, to be relevant at the $\text{CP}^{k|k}$ critical point. (Both $\delta\tilde{\mathcal{L}}$ and the perturbation discussed in Sec. IV C can be written in terms of the traceless symmetric tensor $S_{\beta\delta}^{\alpha\gamma}$ built out of $Q^{\alpha\beta} Q^{\gamma\delta}$, modulo a $\text{tr} Q^2$ term.) Symmetry considerations then lead us to anticipate a crossover to behavior in the universality class of the n -state Potts model in the limit $n \rightarrow 1$, which is a well-known description of percolation [58]. Note that the Goldstone

⁵In terms of connectivity, setting $c = 1$ is equivalent to the maximal definition of Ref. [17].

modes in the extended phase acquire a mass, consistent with the fact that in conventional percolation the percolating phase is massive.

The above picture implies that the bias-induced vortex transition is in a distinct universality class from that of percolation. Previously this has been obscured by the fact that some exponents are similar [6,10,65]. In particular, the fractal dimension of critical tricords was found [6] to be $d_f = 2.54(1)$ [see also Eq. (40)], close to that of critical percolation clusters, for which a more recent study [66] gives $d_f^{\text{perc}} = 2.5226(1)$. In light of the above, this similarity is not surprising and just reflects the small anomalous dimension of Q at each of the respective critical points. (The result $d_f^{\text{perc}} \sim 5/2$ can also be seen without invoking field theory via a Flory-like argument due to de Gennes [67].) The correlation length exponents for the two transitions are clearly different; compare $\nu^{\text{perc}} = 0.8733(5)$ [66] with Eq. (40).

C. Unoriented vortices

So far we have discussed oriented vortices in random complex fields, which are associated with the fundamental group $\pi_1(S^1) = \mathbb{Z}$. It is natural to expect unoriented vortices, which appear when the relevant fundamental group is equal to \mathbb{Z}_2 , to correspond to a different field theory since the distinction between positively and negatively charged fields (ψ and ψ^*) in the $\text{CP}^{k|k}$ model is due to the distinction between vortices and antivortices. To obtain the field theory for \mathbb{Z}_2 vortices from the $\text{CP}^{k|k}$ model, we must introduce a perturbation that reduces the symmetry (from superunitary to orthosymplectic); the introduction of crossings into 2D loop models for noncrossing loops leads to the same symmetry breaking [68].

Unoriented vortices occur, for example, in nematic order parameters, where the order parameter manifold is RP^{N-1} with $N > 2$. This is the space of N -component unit vectors \mathbf{v} with the identification $\mathbf{v} \sim -\mathbf{v}$. Vortices of this kind, for $N = 3$, have been observed in nematic liquid crystals [69,70] and their fractal geometry has been studied numerically in the context of cosmic strings [10,13].

As for oriented vortices, we may consider the phase diagram of vortices in short-range correlated RP^{N-1} fields as a function of bias favoring some point in the order parameter manifold. At small bias, there is an extended phase similar to that for oriented vortices [10,13]. We argue below that this phase is again Brownian (thus we disagree with earlier suggestions that the fractal dimension is less than 2 in this phase [10]). However, this Brownian behavior results from the Goldstone phase of a field theory different from the $\text{CP}^{k|k}$ model, so the bias-induced critical point is in a different universality class from that for oriented vortices. The initial investigation of this transition by Strobl and Hindmarsh [10] is probably not sufficiently precise to confirm this and further numerical work would be interesting.

To determine the field theory, we access the unoriented vortex problem by applying a perturbation to that of oriented vortices. Take an RP^2 order parameter $\mathbf{v} = (v_1, v_2, v_3)$ with $\mathbf{v}^2 = 1$ and $\mathbf{v} \sim -\mathbf{v}$. The subspace $v_3 = 0$ is $\text{RP}^1 = S^1$; thus in the limit where fluctuations in v_3 are completely suppressed, we return to the oriented vortex problem. Turning fluctuations in v_3 back on allows the orientation of a vortex to fluctuate along its length. Specifically, the leading effect of

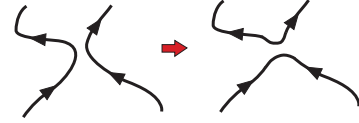


FIG. 9. (Color online) Rewiring of vortices induced by perturbing the oriented vortex problem.

reintroducing fluctuations in v_3 is to allow configurations of the type shown in Fig. 9, where two vortices passing close to each other are rewired in a manner incompatible with the orientations of the strands.

This perturbation is most easily discussed in the replica theory. In terms of world lines, the right-hand side of Fig. 9 corresponds to the annihilation of two particles of the same color β and the creation of two particles of color α , so the perturbation leads to the following $\text{SU}(n)$ -symmetry-breaking term:

$$\delta\mathcal{L} = - \sum_{\alpha, \beta=1}^n (z^{*\alpha} z^{*\alpha})(z^\beta z^\beta) = -|z^T z|^2. \quad (46)$$

In the soft-spin theory for Q , which we decompose into real and imaginary parts $Q = Q_R + iQ_I$, this yields

$$\delta\tilde{\mathcal{L}} \sim \sum_{\alpha, \beta=1}^n (Q_I^{\alpha\beta})^2. \quad (47)$$

We see that allowing fluctuations of the vortex orientation yields a mass term for the imaginary part of Q . The simplest assumption is that this leads to a crossover to a similar theory but with real rather than complex Q . The resulting traceless symmetric matrix is the order parameter for RP^{n-1} as opposed to CP^{n-1} . (This RP^{n-1} field should not be confused with the RP^{N-1} field that hosts the vortices.) This can easily be checked in the case $n = 2$, where the CP^{n-1} model is the $\text{O}(3)$ model and the perturbation is a mass term for one of the components of the $\text{O}(3)$ spin, leading to a crossover to the XY or, equivalently, RP^1 universality class.

So we expect that unoriented vortices in a (short-range-correlated) random nematic field are described by the replica limit of the RP^{n-1} σ model. Although for brevity we have given only a perturbative argument, additional evidence comes from the existence of lattice loop models with RP^{n-1} degrees of freedom [33]. It is also possible to construct a tricolor-percolation-like model that can be formulated as a lattice gauge theory with the same symmetry.

Note that in the extended phase it makes no difference whether we think of the field theory for unoriented vortices as the replica limit of an RP^{n-1} σ model, with target space $\text{RP}^{n-1} = S^{n-1}/\mathbb{Z}_2$, or an $\text{O}(n)$ σ model, with target space S^{n-1} . The topological difference between these manifolds is not seen by the Goldstone modes. However, the difference does become important at the critical point: The $\text{O}(1)$ σ model has a thermodynamic (Ising) transition, whereas the vortex problem has a geometric transition that is invisible in thermodynamic quantities. This is reflected in the fact that the manifold RP^{n-1} becomes a point when $n = 1$.

In SUSY language, the replica limit of the RP^{n-1} model becomes a σ model for a real supervector field $\Phi =$

$(S^1, \dots, S^{2l+1}, \eta^1, \dots, \eta^{2l})$, with $\Phi \Lambda \Phi = 1$, where

$$\Lambda = \begin{pmatrix} 1_{2l+1} & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & -1_l & 0 \end{pmatrix}$$

(the subscripts give the dimensions of the unit matrices), similar to the 2D supersymmetrized $O(n)$ model described in Ref. [68] in the context of lattice loop models with unoriented loops, but with the additional identification $\Phi \sim -\Phi$. Note that we can also cast the perturbation (46) in supersymmetric language. We must take the $CP^{k|k}$ model with $k = 2l$. Then Eq. (46) should be replaced with

$$\delta \mathcal{L} = -(\psi^T \Lambda \psi)(\psi^\dagger \Lambda \psi^*).$$

It can be checked in a lattice model (such as the modification of tricolor percolation described in Sec. IV B) that this is the desired perturbation and that it gives all loops the correct fugacity of one.

D. Stability to long-range correlations in $w(x)$

We have discussed short-range-correlated $w(x)$ at length. The question arises as to how short ranged the correlations must be in order not to alter the universal behavior in the extended phase or at the critical point.

Weak correlations in $w(x)$ can be addressed in a standard way using the extended Harris criterion [58,71]. Correlations in $|w|$ are more relevant than correlations in $\arg w$ and we will consider only the former. Letting these correlations decay as $1/|x - y|^A$ at large distances, one finds that the extended phase is extremely robust, being unaffected by sufficiently weak correlations for any $A > 0$. (The statistics of vortices in correlated complex fields have been studied numerically in Ref. [14] and the results appear to be compatible with this claim.) The critical theory, in contrast, is stable so long as

$$A > 2/\nu \simeq 2.003(3). \quad (48)$$

These criteria may be derived by viewing correlations in $w(x)$ as due to quenched, correlated disorder in the parameters of the probability distribution for w [71] (for example, quenched disorder in the parameters p_R , p_G , and p_B of tricolor percolation) and averaging over this disorder to give an effective Lagrangian with nonlocal couplings between the operators of the $CP^{k|k}$ model, whose relevance or irrelevance can then be determined. This average does not require use of the replica trick, since the partition function takes a trivial value that is independent of the disorder realization.

In addition, correlations decaying *slower* than Eq. (48) may still count as short range if they oscillate with distance [72]. This phenomenon occurs for random superpositions of plane waves $\exp(ik \cdot x)$ with fixed $|k|$. According to Berry's random wave model [73], such superpositions

$$w(x) = \int_{k^2=E} d^{d-1}k a(k) e^{ik \cdot x}, \quad (49)$$

with Gaussian δ -correlated $a(k)$, give a good description of the statistics of eigenfunctions of the Laplacian,

$$-\nabla^2 w = E w, \quad (50)$$

in chaotic d -dimensional domains. In two dimensions the version of this problem with real w has been studied extensively in the context of quantum chaos [74] and the nodal lines $w(x) = 0$ have been found to have the statistics of percolation cluster boundaries. Bogomolny and Schmit [72] reconciled this with the square-root decay of the correlation function at large distances $\langle w(x)w(y) \rangle \sim \cos(E^{1/2}|x - y| - \pi/4)/\sqrt{|x - y|}$, which would usually be slow enough to make percolation results inapplicable [71], by taking oscillations into account. A simple generalization of this analysis shows that in three dimensions, for complex w , the collapse transition of the lines $w(x) = h$ induced by increasing $|h|$ is expected to be in the $CP^{k|k}$ universality class despite the slow decay of the correlator $\langle w^*(x)w(y) \rangle \propto \sin(E^{1/2}|x - y|)/|x - y|$.

E. Vortices near the XY critical point

It has long been recognized that vortices play an important role in the phase transition of the 3D XY model. XY duality makes this more concrete, giving a dual formulation in which the transition is represented by the condensation of a vortex field. A natural question is thus whether the thermodynamic phase transition coincides with the geometric phase transition for the vortices, or whether the two transitions are separate. Simulations of lattice XY models [17,18,75] suggest that the geometric transition occurs slightly inside the XY ordered phase, but very close to the thermodynamic transition [letting K be the coupling for an XY model on the cubic lattice, the most accurate determination of the geometrical critical point in Ref. [17] gives $(K_c^{\text{geom}} - K_c^{\text{therm}})/K_c^{\text{therm}} \sim 4 \times 10^{-4}$]. Vortices must of course be defined using the stochastic rule (Sec. IV B).

At first sight, the $NCCP^{k|k}$ formulation of this problem

$$\mathcal{L}_{NCCP^{k|k}} = |(\nabla - iA)\psi|^2 + \kappa(\nabla \times A)^2 + \mu|\psi|^2 + \lambda|\psi|^4$$

implies that the transitions should coincide, occurring where the field ψ condenses. However, this assumption is not justified, because in the theory with a noncompact gauge field we must distinguish between the condensation of ψ and the condensation of the bilinear Q .

The geometric transition into the extended phase is signaled by the appearance of an expectation value for Q , i.e., by the spontaneous breaking of supersymmetry: $\langle Q \rangle \neq 0$. By contrast, the thermodynamic transition into the XY disordered phase is signaled by the onset of the Higgs mechanism, due to the condensation of ψ , and the generation of a mass for the gauge field. Following standard convention, we denote this $\langle \psi \rangle \neq 0$ (though, of course, this expectation value is zero by gauge invariance). In the simplest scenario, these events would occur simultaneously. However, the numerical results⁶ [17,18] lead us to consider the possibility of a phase with $\langle \psi \rangle = 0$ but $\langle Q \rangle \neq 0$. Heuristically, this is a pair condensate [77] of vortices

⁶While Refs. [17,18] appear to show two separate transitions, scaling collapse for the geometrical observables in Ref. [18] required the use of the standard XY correlation length exponent [76], which is surprising unless either it or the splitting of the transitions can be attributed to finite-size effects.

in which the XY order is not disrupted because vortices and antivortices are locally paired.

If the geometric transition does occur within the XY ordered phase in, for example, lattice XY models, a natural guess is that it is in the same universality class as the bias-induced transition. Note that the nonzero value of the XY order parameter yields a bias. In this scenario, long-distance behavior precisely at the geometrical phase transition would be described by a Lagrangian of the form

$$\mathcal{L} = \kappa(\nabla \times A)^2 + \mathcal{L}_{\text{critical CP}^{k|k}}[Q], \quad (51)$$

where the massless noncompact gauge field A (describing Goldstone fluctuations in the XY order) is decoupled, at long wavelengths, from the neutral degrees of freedom in Q . These are then described by a critical compact $\text{CP}^{k|k}$ model. The symmetry-allowed couplings between A and Q are irrelevant at this critical point. This scenario is also compatible with the extended Harris criterion (note that in the Goldstone phase correlations in the modulus of the complex field decay much faster than correlations in its argument).

Previous attempts have been made to relate the fractal dimension of vortices in critical XY or Abelian Higgs models to local correlators in field theory using the duality between these two theories [21,75,78]. However, the authors did not make use of SUSY or replica, which are necessary in order to form nontrivial geometrical correlators. As a result, the scaling relations put forth were not correct, as has been previously noted [22,79]. Note that vortices in Abelian Higgs models, with the Lagrangian $\mathcal{L}_{\text{SC}} = |(\nabla - ia)\Phi|^2 + \tilde{\kappa}(\nabla \times a)^2 + V(\Phi)$, can be treated in a similar manner to vortices in ungauged fields, leading to a dual SUSY theory without a gauge field, Eq. (2).⁷

V. LOOPS IN TWO DIMENSIONS

We now turn to a different topic, namely, 2D percolation. It is interesting to see how the relation to a lattice gauge theory is modified for two-dimensional deterministic walks in a random environment in the universality class of percolation hulls (cluster boundaries). Essentially the idea will be to take the black clusters in percolation (Fig. 10) to be sheets of random surface in the graphical expansion of a lattice gauge theory. This can be compared with the treatment of tricolour percolation in Sec. III A, where the random surfaces generated by a lattice gauge theory were interpreted as domain walls.

The end result in two dimensions is again a $\text{CP}^{k|k}$ model, but this time with a topological θ term. This is a known continuum representation of percolation [29,32], derived previously by mapping bond percolation to a supersymmetric spin chain. The present approach makes the appearance of the θ term transparent. There is also closely related work mapping (non-SUSY) lattice CP^{n-1} models with a θ term to solid-on-solid models on the honeycomb lattice [47]. That work considers a graphical expansion in terms of loops very similar to the one described below. However, since it is not concerned with percolation, it uses a more conventional Boltzmann weight for

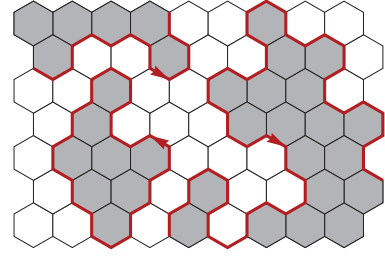


FIG. 10. (Color online) Percolation cluster boundaries. We will view the black clusters as surfaces in the graphical expansion of a lattice gauge theory.

the gauge field, which does not yield the simple expansion in terms of percolation configurations that we desire.

A. Lattice gauge theory for 2D loops

The heuristic argument of Sec. II can be adapted to the 2D case, but it is just as easy to give a concrete lattice construction. For definiteness, let us consider site percolation on the triangular lattice. The unit cells of the triangular lattice are faces of the honeycomb lattice, which are colored black or white with probabilities p and $1 - p$; the critical point is $p = 1/2$. We orient the cluster boundaries (hulls) so as to have the black hexagons to the left; see Fig. 10.

As for tricolour percolation (Sec. III A), we introduce supervectors $\psi_i = (z_i^1, \dots, z_i^{n+k}, \chi_i^1, \dots, \chi_i^k)$ on the sites i of the honeycomb lattice (now with arbitrary numbers of bosonic and fermionic components and $\psi^\dagger \psi = n$) and gauge degrees of freedom U_{ij} on the links. We also introduce a fugacity x per unit length of loop, with $x = 1$ for uncorrelated percolation. Letting F denote a hexagon, the partition function we need is

$$Z = \text{Tr} \prod_F \left[(1 - p) + p \prod_{(ij) \in F} U_{ij} \right] \times \prod_{(ij)} [1 + x U_{ij} \psi_i^\dagger \psi_j + \text{c.c.}].$$

In the products over the links in a hexagon, the links are oriented counterclockwise. To be definite, we consider a finite lattice (made up of complete hexagons) with the topology of the disk.

Now consider two different approaches to the graphical expansion of this partition function. In the first, we begin by expanding the first product, representing the terms via the rule that a hexagon is colored white if we take the first term and black if we take the second. Each term then corresponds to a percolation configuration \mathcal{P} and comes with a product of U s along the links of the cluster boundaries. Fixing \mathcal{P} , we then expand the second product (over links). Only one term survives after tracing over U , namely, that which cancels the U s on the on the cluster boundaries, and this term comes with ψ world lines, which encircle the black clusters counterclockwise. The trace over ψ gives the loops a fugacity:

$$Z = \sum_{\mathcal{P}} p^B (1 - p)^W n^{\text{no. loops}} x^{\text{total loop length}}.$$

Here B and W are the numbers of black and white hexagons in the configuration \mathcal{P} . Note that our boundary conditions mean

⁷Here a is a dynamical field. The phase structure of the XY model in a fixed external gauge field has been considered in Ref. [80].

that the region outside the lattice is considered to be white for the purposes of drawing the cluster boundaries. Percolation is the case $n = 1$; by varying n at $p = 1/2$ we can consider the well-known $O(n)$ loop models for various n [47,57,58].

For the second approach, we expand only the second product in Z , representing the terms by drawing in oriented links $\langle ji \rangle$ of the lattice for each $U_{ij} \psi_i^\dagger \psi_j$ we take. Tracing over ψ , only configurations of mutually avoiding closed oriented loops survive (see also Ref. [47]). Then for each loop configuration \mathcal{C} we trace over U to get a Wilson loop:

$$Z \propto \sum_{\mathcal{C}} n^{|\mathcal{C}|} W(\mathcal{C}), \quad W(\mathcal{C}) = \left\langle \prod_{\text{links} \in \mathcal{C}} U \right\rangle_U. \quad (52)$$

The Wilson loop $W(\mathcal{C})$, which is evaluated using the Boltzmann weight $\prod_F [(1-p) + p \prod_{(ij) \in F} U_{ij}]$ for the gauge field, has simple properties. It is zero if \mathcal{C} is not equivalent to a percolation configuration: loops must alternate in their orientations (along any line) so as to allow the hexagons to be consistently colored. Then $W(\mathcal{C}) \propto (1-p)^W p^B$. There is exponential suppression of black areas (regions encircled counterclockwise) when $p < 1/2$ and of white areas (regions encircled clockwise) when $p > 1/2$. When $p = 1/2$, this exponential suppression disappears.

These properties suggest that the continuum action for the gauge field A will have a topological term $i(\theta/2\pi) \int d^2x E$, where $E = \epsilon_{\mu\nu} \partial_\mu A_\nu$, and that $\theta = \pi$ corresponds to $p = 1/2$. Particles coupled to a gauge field at this value of θ are known as half asymptotic: In a $(1+1)$ -dimensional quantum language, they can be separated at no energy cost so long as they alternate (particle and antiparticle) in space [47,81]. An illuminating discussion of half-asymptotic particles is given by Shankar and Murthy in Ref. [81], who also mention a world-line interpretation pertinent to our considerations here.

This topological action in fact arises from the naive continuum limit of the Boltzmann weight for our lattice gauge theory. The product of the U 's around a hexagon gives the flux through that hexagon,

$$\prod_{(ij) \in F} U_{ij} \sim \exp \left(i \int_F d^2x E \right). \quad (53)$$

Using this, a formal expansion in E turns the part of the Boltzmann weight dependent only on E into

$$\exp \int d^2x [i(\theta/2\pi)E - p(1-p)E^2/2 + \dots], \quad (54)$$

with $\theta = 2p\pi$. Since E is not small, we should be suspicious of these values of the couplings. However, the fact that $\theta = \pi$ when $p = 1/2$ is robust because it reflects an additional symmetry that is present at $p = 1/2$, where the loop ensemble is invariant under switching black and white hexagons. Neglecting boundary effects, this exchange equates to reversing the orientation of each loop or, equivalently, to complex conjugation of ψ and ψ^\dagger (note that the necessary convention is $\chi^{**} = -\chi$ for the fermions [54]). This reversal of the charge of ψ is equivalent to a change in the sign of θ in the continuum description below. Thus the value of θ that corresponds to $p = 1/2$ must be physically equivalent to $-\theta$, up to boundary effects; this is the case for $\theta = \pi$.

Taking account of the symmetries and dropping higher-order terms, the continuum action is

$$\mathcal{L} = i(\theta/2\pi)E + |(\nabla - iA)\psi|^2 + \mu|\psi|^2 + \lambda|\psi|^4, \quad (55)$$

with $(\theta - \pi) \sim (p - 1/2)$.

Let us briefly discuss the phase structure of this theory when $n = 1$ and $p = 1/2$, under the assumption that $\lambda > 0$. In order to vary μ , we introduce an Ising coupling of strength J between the colors of the hexagons, setting $x = e^{-2J}$ in the partition function for the lattice gauge theory. For strong Ising coupling $J > J_c$, domain walls are suppressed and ψ is a massive field. Decreasing J , we pass through the Ising transition, at which ψ condenses; since this is a thermodynamic transition as well as a geometrical one, it is nontrivial even when we remove the fermions by setting $k = 0$, leaving a single complex scalar z [47]. [It is interesting that the Ising transition can be described either in terms of a single boson z coupled to the $\theta = \pi$ gauge field or, as we can deduce from Eq. (55) by using bosonic duality and then the bosonization rules, as a single Dirac fermion coupled to a $\theta = \pi$ gauge field; the Ising transition in the latter theory has been discussed in Ref. [81].] Finally, for weak Ising coupling $J < J_c$, we flow to the percolation critical point. In this regime domain walls proliferate and ψ is condensed. Since ψ is condensed, it is natural here to use a σ model with a fixed length for $\psi^\dagger \psi$, as in Refs. [29,32].

VI. CONCLUSION

Lineline topological defects are ubiquitous in three-dimensional systems and continue to be of theoretical interest; for example, an exciting recent development is the realization that in certain quantum systems vortices can sustain topologically protected zero-energy states [82]. Here we have addressed the universal fractal geometry of vortices in disordered systems, a topic that has been studied numerically in diverse contexts but has lacked a field-theoretic description. By mapping a lattice model for vortices in a random complex field to a lattice gauge theory, and also via a continuum treatment based on XY duality, we related geometrical correlation functions for vortices to correlation functions in the $CP^{k|k}$ model and explored simple consequences of this correspondence. For unoriented line defects, such as vortices in nematic fields, we argued that the appropriate field theory is the $RP^{2l|2l}$ model. These models play an important role in the classification of universality classes of geometrical critical phenomena in three dimensions.

We also argued that similar mappings can be fruitfully applied to 2D random curve ensembles, such as contour lines in a random height field, for which a lattice regularization is provided by site percolation and which are described by $CP^{k|k}$ with a θ term. Future work will apply similar ideas to other random curve problems in two dimensions.

Although we have focused on vortices, we know from the results of Ref. [33] that $CP^{k|k}$ and $RP^{k|k}$ models also apply to a class of three-dimensional loop models that undergo thermodynamically trivial geometrical phase transitions, and we expect these field theories to be generic descriptions for line defects that are topologically one dimensional (and cannot branch or terminate at a dangling end) or for deterministic walks in a random environment such as trajectories in a Lorentz

lattice gas. Note that while at first sight the loop models seem remote from vortices, configurations in one of them (that on Cardy's L lattice [35]) can be specified in terms of percolation configurations on two interpenetrating lattices, and thus in terms of two height fields (albeit on different lattices) taking values ± 1 . Heuristically, then, there is a relation to zero lines of a complex (i.e., two-component real) function here too.

It is interesting that the distinction between compact and noncompact CP^{n-1} models, much discussed recently in the context of deconfined criticality [55], appears even in the replica limit $n \rightarrow 1$. It remains to be seen whether this actually gives a new universality class for vortices, or whether the geometrical transition described by $\text{NCCP}^{k|k}$ is inevitably separate from the thermodynamic (inverted XY) transition and reduces to the universality class of the compact $\text{CP}^{k|k}$

model (Sec. IV E). On the other hand, we do believe that unoriented vortices (or loops in an appropriate loop model) will show a localized-extended transition in a universality class distinct from that of oriented vortices; it remains to perform detailed simulations for this universality class and to confirm that distinct exponents are indeed obtained.

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