Enhancing synchrony in chaotic oscillators by dynamic relaying

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In a chain of mutually coupled oscillators, the coupling threshold for synchronization between the outermost identical oscillators decreases when a type of impurity (in terms of parameter mismatch) is introduced in the inner oscillator(s). The outer oscillators interact indirectly via dynamic relaying, mediated by the inner oscillator(s). We confirm this enhancing of critical coupling in the chaotic regimes of the Lorenz system, in the Rössler system in the absence of coupling delay, and in the Mackey-Glass system with delay coupling. The enhancing effect is experimentally verified in the electronic circuit of Rössler oscillators.

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Chaotic trajectories in coupled nonlinear dynamical systems are known to synchronize when the strength of the coupling exceeds a critical value [1–3]. Such complete synchronization (CS) occurs only when the oscillators are identical. When the oscillators are mismatched, CS becomes unstable due to attractor bubbling or bursting instabilities [4], but lag synchronization (LS) may arise at a lower value of the critical coupling [5,6]. An unexpected consequence of this lag synchrony is that it can promote complete synchrony in a chain of diffusively coupled chaotic oscillators when there are isolated *impurities*, namely, mismatched oscillators. The simplest case arises when there are three oscillators coupled, as shown in Fig. 1(a), with oscillators 1 and 3 being identical to each other but mismatched with oscillator 2. The central oscillator with either positive or negative mismatch induces a common time lag or lead with the outer oscillators, leading to a LS scenario at a lower critical coupling. An examination of the master stability function (MSF) [7] in Fig. 1(b) for two outer oscillators in a set of three chaotic Lorenz oscillators shows that CS occurs at a lower critical coupling for mismatch in some regions of the parameter space of the central oscillator. This enhancing effect is seen at the right side of the black (red) hill where the coupled dynamics is chaotic. On the left side, the coupled dynamics is periodic, where a diminishing effect is observed instead. The MSF is calculated between the (1, 3) pair of identical oscillators, which are not directly coupled but interact through the exchange of signals mediated by oscillator 2. The effect is found to occur both in instantaneous and delay coupled systems.

In the presence of long conduction delays, zero-lag synchronization (ZLS) was reported [8,9] in two distantly located populations of neurons in cerebral cortical areas when mediated by a third population. The ZLS was verified in laboratory experiments on a laser [10] and electronic circuit [11]. In these experiments, one common strategy was to detune (positive or negative) the frequency of the central oscillator to optimize the ZLS. The robustness of the ZLS to frequency detuning in the central oscillator [12,13] and also to parameter mismatch in the outer oscillators [13] was investigated. An isochronal synchronization [14] was also reported in three instantaneously coupled laser sources. The central oscillator played a leader/laggard role [15,16] in the presence of long coupling delay. However, the role of parameter mismatch (or frequency detuning) on the critical coupling of ZLS has not been investigated. In this Brief Report, we address a natural question: does the parameter mismatch in the central oscillator play any role on the ZLS or isochronal synchrony? Investigating a chain of three delay coupled Mackey-Glass systems with coupling delay, we find that two identical outer oscillators evolve to a CS state at a lower critical coupling for parameter mismatch in the central oscillator: this is defined as the enhancing of synchrony in the outer oscillators.

The coupled Mackey-Glass system is given by

$$\dot{x}_i = -ax_i + \frac{m_i x_i(t-\tau_0)}{1+x_i^{10}(t-\tau_0)} + \varepsilon[x_{i-1}(t-\tau_1) + x_{i+1}(t-\tau_1) - 2x_i], \quad i = 1,2,3, \tag{1}$$

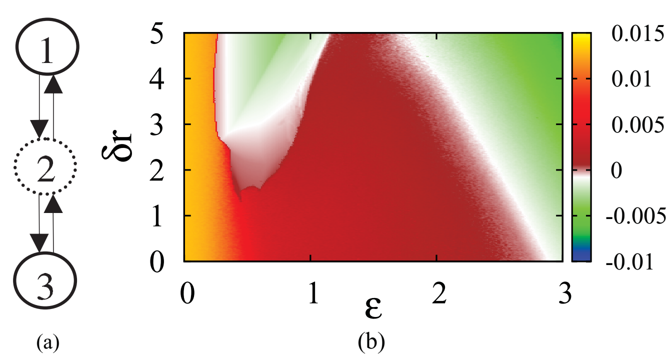

FIG. 1. (Color online) MSF (λ_{max}) as a function of coupling and mismatch. Each circle in (a) denotes a Lorenz system: $\dot{x}_i = \sigma(y_i - x_i)$, $\dot{y}_i = r_i x_i - y_i - x_i z_i$, $\dot{z}_i = -bz_i + x_i y_i$ (i = 1,2,3) : b = 8/3, σ = 10, $r_1 = r_3 = 28$, $r_2 = 28 + \delta r$, δr is a mismatch, and the coupling strength is ε. Arrows denote mutual coupling via the y variable. λ_{max} plot (b) has two CS regions [gray (green-white)] delineated by a critical coupling boundary on both sides of a black (red) hill when λ_{max} crosses a positive [black (red)] to a negative value [gray (green-white)]. The slope of the critical coupling boundary is negative at right, while positive at left.

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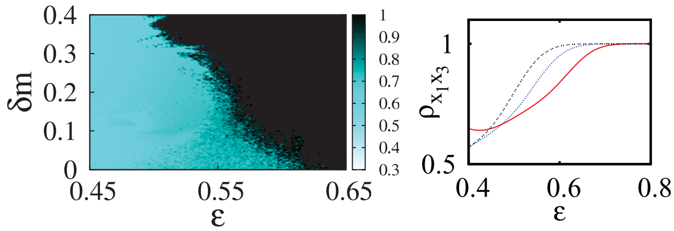

FIG. 2. (Color online) Mackey-Glass system: $\rho_{x_1x_3}$ plot of the $(x_1,\, x_3)$ pair of time series in the $\varepsilon - \delta m$ plane (a); CS in black region. Critical coupling boundary delineated by black and gray (blue) regions shows a negative slope. $\rho_{x_1x_3}$ of the $(x_1,\, x_3)$ pair in (b) with ε for $\delta m = 0$ [black (red)], 0.2 [gray (blue)], and 0.35 (dotted lines). $a = 1$, $m_1 = m_3 = 3$, $\tau_0 = 2$.

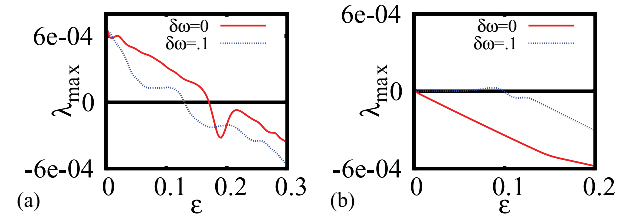

FIG. 4. (Color online) MSF ($\lambda_{\max}$) of outer Rössler oscillators with ε. In the chaotic regime, (a) $\varepsilon = \varepsilon_c = 0.18$ for CS in outer oscillators for the identical case [black (red) line], $\varepsilon_c = 0.13$ for the mismatch ($\delta\omega = 0.1$) in the central oscillator [gray (blue) line]. In the limit cycle case, (b) b = 0.2, c = 2.5 (other parameters same), and the ε_c for CS in outer oscillators is larger for the mismatched case [gray (blue) line] than the identical case [black (red) line].

with zero flux $[x_0(t-\tau_1) = x_2(t), x_4(t-\tau-1) = x_3(t)]$. Here, a_i, m_i are constants and τ_0 is the intrinsic time delay of the system. A mismatch, $\delta m (m_2 = m_{1,3} \pm \delta m)$, is introduced in the central oscillator. We consider a large coupling delay $\tau_1 = 4$ with a coupling strength ε to enact the long range ZLS [8–11]. Numerically computed cross-correlation $\rho_{x_1x_3}$ [17] of the $(x_1,\, x_3)$ pair of time series of the outer oscillators is plotted in the ε-δm plane in Fig. 2(a). In the case of CS, $\rho_{x_1x_3} = 1$. Figure 2(a) shows a negative slope of the boundary line that separates the CS regime (black) from the nonsynchronous regime [gray (blue)]. The negative slope of the boundary confirms the enhancing effect. The boundary is possibly fractal, and a further exploration of this feature is presently underway. We reconfirm the enhancing effect by plotting $\rho_{x_1x_3}$ with ε in Fig. 2(b) for two specific choices of mismatch. The critical coupling decreases with mismatch in the central oscillator. The $(x_1,\, x_3)$ pair of time series in Fig. 3(a) shows poor correlation ($\rho_{x_1,x_3} \approx 0.70$) in Fig. 3(d) at zero lag ($\tau = 0$) for $\varepsilon = 0.53$ when all oscillators are identical. Figure 3(b) shows CS ($\rho_{x_1x_3} = 1$) between the $(x_1,\, x_3)$ pair of time series for the same $\varepsilon = 0.53$ at zero lag in Fig. 3(e) when a mismatch ($\delta m = 0.3$) is introduced in the central oscillator. The outer oscillators maintain a time lag with the central one as shown in Fig. 3(c). However, the lag ($\tau = 4.22$) is slightly larger than the coupling delay ($\tau_1 = 4.0$) as shown in the maxima of $\rho_{x_1x_3}$ in Fig. 3(f). By taking coupling delay $\tau_1 = 0$, we check that the additional lag appears due to the parameter mismatch. This indicates that a LS scenario leads

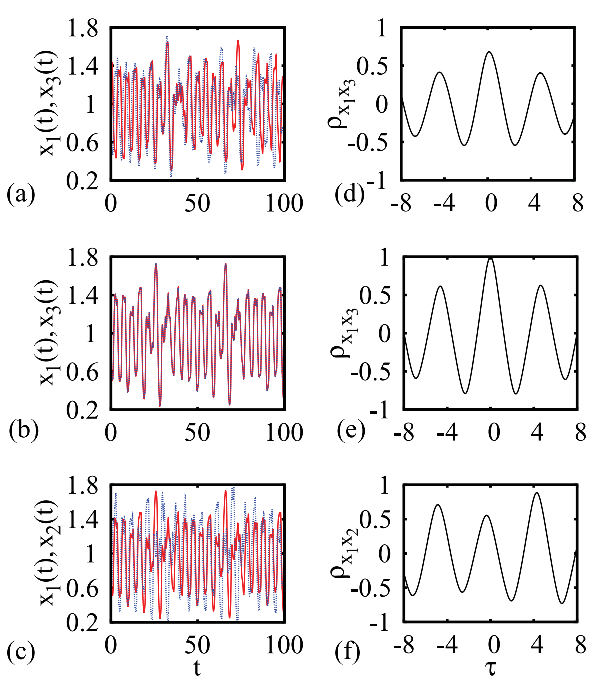

FIG. 3. (Color online) Mackey-Glass System: $(x_1,\, x_3)$ pair of time series shown in black (red) and gray (blue) lines for the identical case (a), mismatch case (b), of the $(x_1,\, x_2)$ pair shown in black (red) and gray (blue) lines for the mismatch case (c). Corresponding $\rho_{x_1x_3}$ of the $(x_1,\, x_3)$ pair of time series in (d) and (e), of $(x_1,\, x_2)$ in (f). $\varepsilon = 0.53$, a = 1.1, $m_1 = m_3 = 3$, $\delta m = 0.3$, $\tau_0 = 2.0$, $\tau_1 = 4.0$.

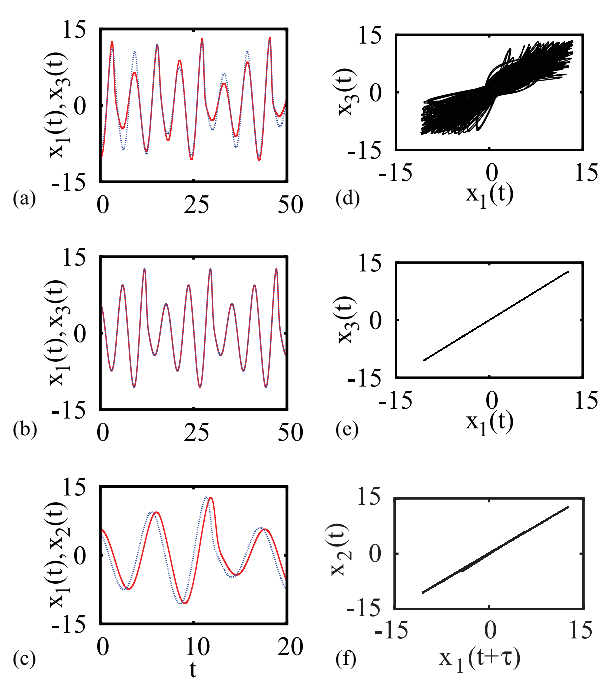

FIG. 5. (Color online) Module of three Rössler oscillators. Pair of $(x_1,\, x_3)$ times series shown in black (red) and gray (blue) lines for $\varepsilon = 0.13$ in the identical case (a), the mismatch case (b), of $(x_1,\, x_2)$ time series shown in black (red) and gray (blue) lines in the LS state (c). Plot of x_1 vs $x_3(t)$ in the identical case (d), the mismatch case (e), of x_1 vs $x_2(t+\tau)$ in LS (f) when $\tau = 0.5$.

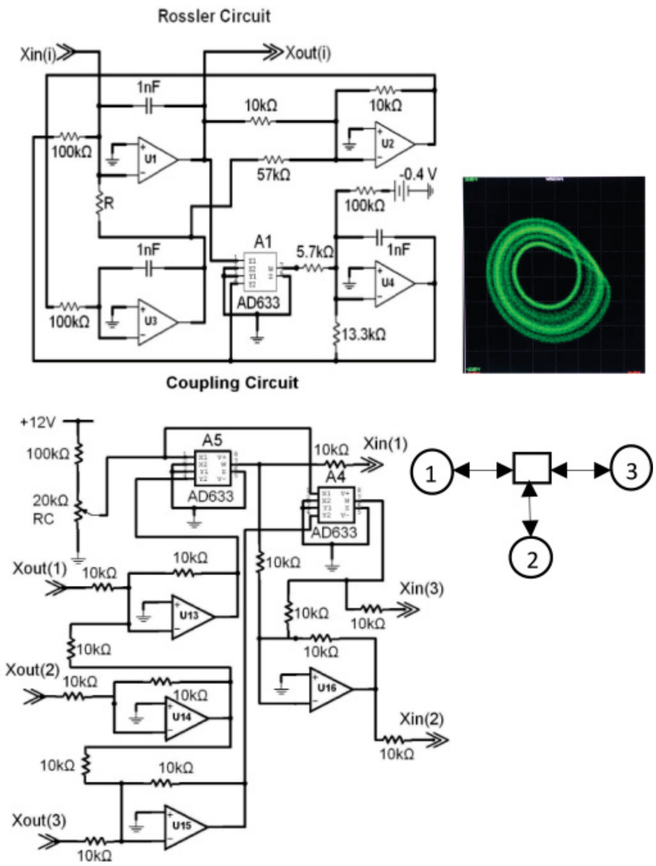

FIG. 6. (Color online) Rössler oscillator and coupling circuit. Upper circuit is a single Rössler oscillator; attractor of an isolated oscillator at right (oscilloscope picture). Lower circuit derives the diffusive coupling scheme (square box) in three oscillators (circle). R in oscillator (2) is varied to induce mismatch. Coupling strength increases with R_C as a proportional constant voltage.

to an enhancement of the CS in the outermost oscillators due to a parameter mismatch in the central oscillator.

To further elucidate that the LS effect plays a key role in lowering the critical coupling of CS, we consider a chain of three instantaneously coupled Rössler oscillators,

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon(x_{i+1} + x_{i-1} - 2x_i), \\ \dot{y}_i &= \omega_i x_i + a y_i, \quad \dot{z}_i = b + z_i(x_i - c), \quad i = 1,2,3, \end{aligned} \tag{2}$$

with zero flux ($x_0 = x_1, x_4 = x_3$). We choose a = 0.2, b = 0.4, c = 7.5, $\omega_1 = \omega_3 = 1$, $\omega_2 = \omega_1 + \delta\omega$, and $\delta\omega = 0.1$ for a chaotic regime.

Numerical results in Fig. 4(a) show that $\lambda_{\max}$ crosses from a positive to a negative value at a lower critical coupling [gray (blue)] when a mismatch is introduced in the central oscillator. The (1, 3) pair of oscillators is not in CS in Figs. 5(a) and 5(d) for $\varepsilon = 0.13$ when three oscillators are identical. In fact, they need a larger coupling ($\varepsilon > \varepsilon_c = 0.18$) to develop a CS state. Instead, when a mismatch ($\delta\omega = 0.1$) is introduced in the central oscillator, the (1, 3) pair of oscillators emerges into a CS state for the lower critical coupling, $\varepsilon_c = 0.13$, as shown in Figs. 5(b) and 5(e). The outer oscillators then emerge into a LS state with the central oscillator, as confirmed by the (x_1, x_2) pair of time series in Fig. 5(c). A lag ($\tau = 0.5$) is created in the

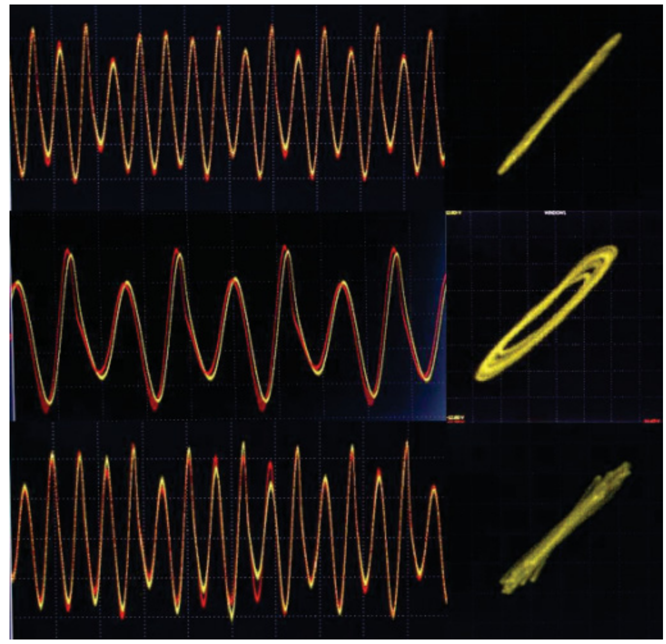

FIG. 7. (Color online) Oscilloscope pictures. Left panels: pair of time series of outer oscillators in almost CS in upper row, of the central and one of the outer oscillators in LS in middle row, and of outer oscillators in lower row when all are identical. Right panels: phase portraits of similar state variables of two outer oscillators; each right panel corresponds to its immediate left panel.

LS state, as shown in the x_1 vs $x_2(t + \tau)$ plot in Fig. 5(f). The onset of CS in the outer identical oscillators is thus enhanced by the mismatched central oscillator via dynamic relaying. The lag synchrony lowers the critical coupling [5,6] and eventually enhances CS in the outer oscillators in the chaotic regimes of the example systems. On the contrary, a diminishing effect is seen in Fig. 4(b) when the Rössler oscillators are in the limit cycle regime. This also holds in two-dimensional (2D) limit cycle systems, in which the effect is yet to be fully understood.

We experimentally support the enhancing effect using the electronic analog of Rössler oscillators (circle) coupled diffusively (square box) as shown in a block diagram (Fig. 6). A single Rössler oscillator circuit is only shown with details of the coupling circuit. The resistance R of the outer oscillators (1, 3) is selected as 100 kΩ to make them closely identical, while it is chosen as 91 kΩ for the middle oscillator(2) to introduce a mismatch. The resistance R_C is then varied to find a critical coupling (1.8 kΩ) when (1, 3) oscillators are almost in CS, as seen in the upper panels of Fig. 7. This leads the outer oscillators to a LS state with the central one (middle panels).

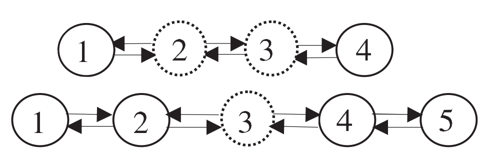

FIG. 8. 1D arrays of bidirectionally coupled N oscillator: identical (solid circle) and mismatched (dotted circle) oscillators. Upper row: identical outer oscillators (1, 4) mediated by two identically mismatched oscillators. Lower row: four identical oscillators mediated by a mismatched oscillator (3).

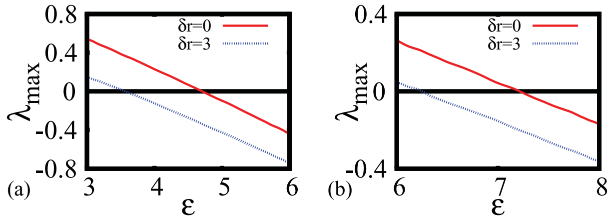

FIG. 9. (Color online) MSF (λ_{max}) of outer oscillators in a chain of N Lorenz oscillators in Fig. 8, (a) N = 4, (b) N = 5. Critical coupling of CS in outer oscillators is lower [gray (blue) line] for mismatched central oscillator than all identical case [black (red) line] for both (a) and (b). $\sigma = 10$, r = 28, b = 8/3, δr = 3.0.

Next, keeping R_C unchanged, R in the central oscillator is changed to 100 kΩ for making the three oscillators almost identical when they are all desynchronized (lower panels). A larger coupling ($R_C = 2.5$ kΩ) is found necessary to induce CS in three identical oscillators. Experimental results are found in good agreement with the numerical results in Fig. 5.

The enhancing effect is found in 1D arrays of N oscillators (N > 3) (Fig. 8) as well. In an array of four oscillators, the two outermost identical oscillators are mediated by two oscillators identically mismatched. In a second array of five oscillators, four mutually coupled identical oscillators are mediated by a mismatched oscillator. The oscillators at symmetric positions on both sides of the central oscillator emerge into CS at a lower critical coupling. In both cases, λ_{max} of the outer two oscillators crosses to a negative value at a lower critical coupling for a mismatch introduced in the intermediate oscillator(s) in Fig. 9.

Disorder enhanced synchrony was reported earlier [18] in an array of coupled Josephson junctions, as is noise induced enhancing of phase synchronization [19] or coherence resonance [20]. However, the mechanism of enhancing synchrony reported here is different.

To summarize, an enhancing of synchrony is reported here in a chain of identical oscillators mediated by mismatched oscillator(s). A common time lag is created between the identical outer oscillators and the mismatched central oscillator(s) leading to a LS scenario at a lower critical coupling. This time lag played a role of dynamic relaying of the outer oscillators to establish an indirect coupling between them and thereby enhances CS in the outer oscillators. We presented several example systems to verify the LS scenario causing the enhancing effect both in the presence and in the absence of coupling delay. We provided experimental evidence using an electronic circuit of Rössler oscillators. An enhancing of synchrony was reported earlier [21] in two oscillators by an induced coupling delay when the coupled system switches from a chaotic to a periodic state. In the present instance, the coupled oscillators remain chaotic before and after the coupling. This enhancing effect is found for a negative mismatch too where the central oscillator leads the outer ones instead of lagging. The effect is also found true for unidirectional coupling when the central oscillator drives the identical outer oscillators. A consequence of this observation is that a mismatched central oscillator can drive many identical oscillators in a starlike configuration into enhanced synchrony. Further, in a ring of coupled oscillators, the enhancing is seen in oscillators in symmetric positions to a mismatched oscillator. The effect appears to be a general feature of nonlinear dynamical systems since a parameter regime of a LS scenario [6] can always be found in chaotic systems. We are currently investigating details of the LS scenario in both the limit cycle and chaotic regimes of coupled oscillators for a further understanding of the enhancing effect.

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