Nonlinear Schrödinger equation: Generalized Darboux transformation and rogue wave solutions

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In this paper, we construct a generalized Darboux transformation for the nonlinear Schrödinger equation. The associated N-fold Darboux transformation is given in terms of both a summation formula and determinants. As applications, we obtain compact representations for the Nth-order rogue wave solutions of the focusing nonlinear Schrödinger equation and Hirota equation. In particular, the dynamics of the general third-order rogue wave is discussed and shown to exhibit interesting structures.

I. INTRODUCTION

The Darboux transformation, originating from the work of Darboux in 1882 on the Sturm-Liouville equation, is a powerful method for constructing solutions for integrable systems. The theory is presented in several monographs and review papers (see [1–3]). In the literature, various approaches have been proposed to find a Darboux transformation for a given equation, for instance, the operator factorization method [4], the gauge transformation method [3,5,6], and the loop group transformation [7].

It is remarked that the Darboux transformation is very efficient for construction of soliton solutions. Indeed, through iterations, one is often led to compact representations in terms of special determinants such as the Wronskian or Grammian for N-soliton solutions. Such N-soliton solutions are appealing both from the theoretical viewpoint and from the practical application viewpoint.

In addition to the soliton solutions, rational solutions are interesting and the Darboux transformation may be adopted for this purpose. For the celebrated Korteweg–de Vries (KdV) equation, Matveev [8] introduces the so-called generalized Darboux transformation and the positon solutions are calculated. Recently, rogue waves, appearing in oceans, have been studied and applied extensively in other fields such as Bose-Einstein condensates, optics, and superfluids (see [9] and references therein). The very first model for rogue waves was the focusing nonlinear Schrödinger (NLS) equation,

$$iq_t + \tfrac{1}{2}q_{xx} + |q|^2 q = 0, \tag{1}$$

which has been an important integrable equation. The simplest rogue wave solutions were calculated by Akhmediev and coworkers, and the construction of higher order analogs is one of the challenges as remarked in Ref. [10]. In this regard, as pointed out by Dubard *et al.* [11], the solutions obtained by Eleonskii, Krichever, and Kulagin [12] represent a class of multi-rogue-wave solutions. It is remarked that the construction method proposed in Refs. [11,13] is very specific and technical, so it may not be easy to apply to other models.

The aim of this article is to propose a simple method for construction multi-rogue-wave solutions. The main tool is the generalized Darboux transformation. We re-examine Matveev's generalized Darboux transformation for the KdV equation and derive it in a way that can be easily extended to other models. Then we apply the idea to the focusing NLS equation and work out a formula for generation of multi-rogue-wave solutions.

This article is organized as following: In Sec. II, we propose a new way to derive the generalized Darboux transformation for the KdV equation. In Sec. III, we first apply the proposed method to the NLS equation and obtain the corresponding generalized Darboux transformation for it, then we reformulate the N-fold generalized Darboux transformation in terms of determinants. Also, we provide formulas for Nth-order rogue wave solutions for the NLS equation and Hirota equation. With the help of these formulas, we consider the dynamics of a general third-order rogue wave. The spatial-temporal pattern of the solution can form as a triangle or a pentagon. The final section contains some discussion.

II. GENERALIZED DARBOUX TRANSFORMATION FOR KdV

Let us first recall the well-known classical Darboux transformation for the KdV equation. Consider the Sturm-Liouville equation,

$$-\Psi_{xx} + u\Psi = \lambda\Psi, \tag{2}$$

and introduce the first-order operator

$$T[1] = \partial_x - \frac{\Psi_{1x}}{\Psi_1},$$

where Ψ_1 is the fixed solution of (2) with $\lambda = \lambda_1$. Then the Darboux transformation,

$$\Psi[1] = T[1]\Psi = \frac{\mathrm{Wr}(\Psi_1, \Psi)}{\Psi_1},$$

converts Eq. (2) into

$$-\Psi[1]_{xx} + u[1]\Psi[1] = \lambda\Psi[1], \tag{3}$$

where

$$u[1] = u - 2(\ln\Psi_1)_{xx}$$

and $\mathrm{Wr}(\Psi_1, \Psi) = \Psi_1\Psi_x - \Psi_{1,x}\Psi$ is the standard Wronskian determinant.

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The most interesting point here is that one can iterate the above Darboux transformation. Indeed, the N-times iterated or N-fold Darboux transformation yields the Crum theorem,

$$-\Psi_{xx}[N] + u[N]\Psi[N] = \lambda\Psi[N],$$
$$u[N] = u - 2(\ln \text{Wr}(\Psi_1, \ldots, \Psi_N))_{xx},$$

where

$$\Psi[N] = \frac{\text{Wr}(\Psi_1, \ldots, \Psi_N, \Psi)}{\text{Wr}(\Psi_1, \ldots, \Psi_N)},$$

and $\Psi_1, \ldots, \Psi_N$ are solutions of (2) at $\lambda = \lambda_1, \ldots, \lambda_N$, respectively.

It is obvious that $\Psi_1[1] = T[1]\Psi_1 = 0$, namely, Ψ_1 is mapped to a trivial solution. This fact implies that a seed solution may not be used more than once when considering the iterations for the Darboux transformation. However, as pointed out by Matveev and Salle [1], a generalized Darboux transformation does exist. Let us derive this result in a way which may be readily generalized. We start with the assumption that

$$\Psi_2 = \Psi_1(k_1 + \epsilon),$$

where $k_1 = f(\lambda_1)$ is a monotonic function and ϵ is a small parameter. Expanding Ψ_2 in a series in ϵ,

$$\Psi_2 = \Psi_1 + \Psi_1^{[1]}\epsilon + \Psi_1^{[2]}\epsilon^2 + \cdots,$$

where $\Psi_1^{[i]} = \frac{1}{i!}\frac{\partial^i \Psi_1(k)}{\partial k^i}|_{k=k_1}$. Since $\Psi_2[1] \equiv T_1[1]\Psi_2$ is a special solution for (3), so is $\frac{\Psi_2[1]}{\epsilon}$. Taking the limit $\epsilon \to 0$ for this solution, we find

$$\Psi_1[1] = \lim_{\epsilon \to 0}\frac{T[1]\Psi_1(k_1 + \epsilon)}{\epsilon} = T[1]\Psi_1^{[1]},$$

which is a nontrivial solution for (3) at $\lambda = \lambda_1$. This solution may be adopted to do the second-step Darboux transformation, that is,

$$T[2] = \partial_x - \frac{\Psi_{1,x}[1]}{\Psi_1[1]}, \quad u[2] = u - 2(\ln \text{Wr}(\Psi_1, \Psi_1^{[1]}))_{xx}.$$

Combining these two Darboux transformations, we obtain

$$-\Psi_{xx}[2] + u[2]\Psi[2] = \lambda\Psi[2], \quad \Psi[2] = \frac{\text{Wr}(\Psi_1, \Psi_1^{[1]}, \Psi)}{\text{Wr}(\Psi_1, \Psi_1^{[1]})}.$$

This process may be continued and results in the so-called generalized Darboux transformation for system (2). Indeed, let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be n different solutions for (2) at $\lambda_1, \lambda_2, \ldots, \lambda_n$, and consider the expansions

$$\Psi_i(k_i + \epsilon) = \Psi_i(k_i) + \Psi_i^{[1]}\epsilon + \cdots + \Psi_i^{[m_i]}\epsilon^{m_i} + \cdots,$$
$$k_i = f(\lambda_i) \quad (i = 1,2,\ldots,n);$$

then we have the following.

Proposition 1 [14].

$$u[N] = u - 2(\ln(W_1))_{xx}, \quad \Psi[N] = \frac{W_2}{W_1},$$

with

$$W_1 = \text{Wr}(\Psi_1, \ldots, \Psi_1^{[m_1]}, \Psi_2, \ldots, \Psi_2^{[m_2]}, \ldots, \Psi_n, \ldots, \Psi_2^{[m_n]}),$$
$$W_2 = \text{Wr}(\Psi_1, \ldots, \Psi_1^{[m_1]}, \Psi_2, \ldots, \Psi_2^{[m_2]}, \ldots, \Psi_n, \ldots, \Psi_2^{[m_n]}, \Psi),$$

solve

$$-\Psi_{xx}[N] + u[N]\Psi[N] = \lambda\Psi[n],$$

where $m_1 + m_2 + \cdots + m_n = N - n$, $m_i \geqslant 0$, $m_i \in \mathbb{Z}$.

The generalized Darboux transformation presented above may be used to generate both solitons and rational solutions for the KdV equation. Let us illustrate this with the following examples. It is well known that the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4}$$

takes (2) as its spatial part of the spectral problem and the corresponding temporal part reads

$$\Psi_t = -4\Psi_{xxx} + 6u\Psi_x + 3u_x\Psi.$$

In the case of N distinct spectral parameters, we will have the Wronskian representation for the N-soliton solution. To get rational solutions, one starts with the seed solution $u = c$, where c is a real constant, and $\Psi_1 = \sin[k_1(x + (4k_1^2 + 6c)t) + P(k_1)]$, $k_1 = \sqrt{\lambda_1 - c}$, and $P(k_1)$ is a polynomial of k_1. Now expanding the function Ψ_1 at $k_1 = 0$ and taking $P(k_1) = 0$ for convenience, we have

$$\Psi_1 = (x + 6ct)k_1 + \left[-\tfrac{1}{6}(x + 6ct)^3 + 4t\right]k_1^3$$
$$+ \left[\tfrac{1}{120}(x + 6ct)^5 - 2t(x + 6ct)^2\right]k_1^5 + \cdots,$$

therefore

$$\Psi_1^{[0]} = x + 6ct, \quad \Psi_1^{[1]} = -\tfrac{1}{6}(x + 6ct)^3 + 4t,$$
$$\Psi_1^{[2]} = \tfrac{1}{120}(x + 6ct)^5 - 2t(x + 6ct)^2.$$

Then the generalized Darboux transformation provides us the rational solution for the KdV equation (4),

$$u[3] = c + \frac{G}{H^2},$$

where

$$G = 12[279936c^5t^5(216c^5t^5 + 360c^4xt^4 + 270c^3x^2t^3$$
$$+ 120c^2x^3t^2 + 35cx^4t + 7x^5) + 38880c^4t^4(180t^2 + 7x^6)$$
$$+ 25920c^3xt^3(x^6 + 180t^2) + 1620c^2x^2t^2(x^6 + 720t^2)$$
$$+ 60ct(x^9 + 2160t^2x^3 + 4320t^3) + 43200t^3x + x^{10}$$
$$+ 5400x^4t^2],$$
$$H = 3888c^4t^4(12c^2t^2 + 12cxt + 5x^2) + 4320c^3t^3(x^3 + 3t)$$
$$+ 540c^2xt^2(x^3 + 12t) + 36cx^2t(x^3 + 30t) + x^6$$
$$- 720t^2 + 60x^3t.$$

More general solutions of the rational type may be obtained if we expand

$$\Psi_1 = \sin\left[k_1(x + (4k_1^2 + 6c)t) + P(k_1)\right]$$
$$= \Psi_1^{[0]}k_1 + \Psi_1^{[1]}k_1^3 + \Psi_1^{[2]}k_1^5 + \cdots + \Psi_1^{[N]}k_1^{2N+1} + \cdots.$$

In particular, the positon solutions for the KdV equation may be found in this way, and for a detailed analysis we refer the reader to the work of Matveev [15].

Concluding this section, we mention that, for those equations which possess solutions represented in terms of Wronskians, there are alternative manners in which to explain the limit solutions [1,16,17].

III. GENERALIZED DARBOUX TRANSFORMATION FOR THE NLS

In this section, we extend the idea discussed in Sec. II to the NLS equation and construct a generalized Darboux transformation for it. Furthermore, we show that this Darboux transformation enables one to obtain, apart from the soliton solutions, rational solutions including multi-rogue-wave solutions.

The focusing NLS equation (1), is the compatibility condition of the linear spectral problems,

$$\Psi_x = [i\zeta\sigma_1 + iQ]\Psi, \tag{5a}$$

$$\Psi_t = \left[i\zeta^2\sigma_1 + i\zeta Q + \tfrac{1}{2}\sigma_1(Q_x - iQ^2)\right]\Psi, \tag{5b}$$

where

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}.$$

A. Generalized Darboux transformations

The Darboux transformation in this case is defined as (see [2] and references therein)

$$\Psi[1] = T[1]\Psi, \quad q[1] = q + 2(\zeta_1^* - \zeta_1)(P[1])_{21},$$

where

$$T[1] = \zeta - \zeta_1^* + (\zeta_1^* - \zeta_1)P[1], \quad P[1] = \frac{\Psi_1\Psi_1^\dagger}{\Psi_1^\dagger\Psi_1}, \tag{6}$$

and Ψ_1 is a special solution of the linear system (5a) and (5b) at $\zeta = \zeta_1$; $(P[1])_{21}$ represents the entry of matrix $P[1]$ in the second row and first column, and a dagger denotes the matrix transpose and complex conjugation.

If N distinct seed solutions $\Psi_k, (k = 1, 2, \ldots, N)$ of (5a) and (5b) are given, the basic Darboux transformation may be iterated. To do the second step of transformation, we employ Ψ_2 which is mapped to $\Psi_2[1] = T[1]|_{\zeta=\zeta_2}\Psi_2$. Therefore,

$$\Psi[2] = T[2]\Psi[1], \quad q[2] = q[1] + 2(\zeta_2^* - \zeta_2)(P[2])_{21},$$

where

$$T[2] = \zeta - \zeta_2^* + (\zeta_2^* - \zeta_2)P[2], \quad P[2] = \frac{\Psi_2[1]\Psi_2[1]^\dagger}{\Psi_2[1]^\dagger\Psi_2[2]}.$$

In the general case, we may have the following.

Theorem 1. Let $\Psi_1, \Psi_2, \ldots, \Psi_N$ be N distinct solutions of the spectral problem (5a) and (5b) at $\zeta_1, \ldots, \zeta_N$, respectively; then the N-fold Darboux transformation for the NLS equation (1), is

$$\Psi[N] = T[N]T[N-1]\cdots T[1]\Psi,$$

$$q[N] = q[0] + 2\sum_{i=1}^{N}(\zeta_i^* - \zeta_i)(P[i])_{21}, \tag{7}$$

with

$$T[i] = \zeta - \zeta_i^* + (\zeta_i^* - \zeta_i)P[i],$$

$$P[i] = \frac{\Psi_i[i-1]\Psi_i[i-1]^\dagger}{\Psi_i[i-1]^\dagger\Psi_i[i-1]},$$

$$\Psi_i[i-1] = (T[i-1]T[i-2]\ldots T[1])|_{\zeta=\zeta_i}\Psi_i,$$

$$q[0] = q.$$

We remark that the N-fold Darboux transformation given by (7) is equivalent to the determinant representation presented in Ref. [1], as shown in Sec. III B.

Now we manage to find a generalized Darboux transformation. As in the last section, suppose that $\Psi_2 = \Psi_1(\zeta_1 + \delta)$ is a special solution for system, then after transformation we have $\Psi_2[1] = T_1[1]\Psi_2$. Expanding Ψ_2 at ζ_1, we have

$$\Psi_1(\zeta_1 + \delta) = \Psi_1 + \Psi_1^{[1]}\delta + \Psi_1^{[2]}\delta^2 + \cdots + \Psi_1^{[N]}\delta^N + \cdots, \tag{8}$$

where $\Psi_1^{[k]} = \frac{1}{k!}\frac{\partial^k}{\partial\zeta^k}\Psi_1(\zeta)|_{\zeta=\zeta_1}$.

Through the limit process

$$\lim_{\delta\to 0}\frac{[T_1[1]]|_{\zeta=\zeta_1+\delta}]\Psi_2}{\delta} = \lim_{\delta\to 0}\frac{[\delta + T_1[1]|_{\zeta=\zeta_1}]\Psi_2}{\delta}$$
$$= \Psi_1 + T_1[1]|_{\zeta=\zeta_1}\Psi_1^{[1]} \equiv \Psi_1[1], \tag{9}$$

we find a solution to the linear system (5a) and (5b) with $q[1]$ and $\zeta = \zeta_1$. This allows us to go to the next step of the Darboux transformation, namely,

$$T_1[2] = \zeta - \zeta_1^* + (\zeta_1^* - \zeta_1)P_1[2],$$

$$q[2] = q[1] + 2(\zeta_1^* - \zeta_1)(P_1[2])_{21}, \tag{10}$$

where

$$P_1[2] = \frac{\Psi_1[1]\Psi_1[1]^\dagger}{\Psi_1[1]^\dagger\Psi_1[1]}.$$

Similarly, the limit

$$\lim_{\delta\to 0}\frac{[\delta + T_1[2](\zeta_1)][\delta + T_1[1](\zeta_1)]\Psi_2}{\delta^2}$$
$$= \Psi_1 + [T_1[1](\zeta_1) + T_1[2](\zeta_1)]\Psi_1^{[1]} + T_1[2](\zeta_1)T_1[1](\zeta_1)\Psi_1^{[2]}$$
$$\equiv \Psi_1[2]$$

provides us a nontrivial solution for the linear spectral problem with $q = q[2]$ and $\zeta = \zeta_1$. Thus we may do the third-step iteration of the Darboux transformation, which is the following:

$$T_1[3] = \zeta - \zeta_1^* + (\zeta_1^* - \zeta_1)P_1[3], \quad P_1[3] = \frac{\Psi_1[2]\Psi_1[2]^\dagger}{\Psi_1[2]^\dagger\Psi_1[2]},$$

$$q[3] = q[2] + 2(\zeta_1^* - \zeta_1)(P_1[3])_{21}. \tag{11}$$

Continuing the above process and combining all the Darboux transformation, a generalized Darboux transformation is constructed. We summarize our findings as follows.

Theorem 2. Let $\Psi_1(\zeta_1), \Psi_2(\zeta_2), \ldots, \Psi_n(\zeta_n)$ be n distinct solutions of the linear spectral problem, (5a) and (5b), and

$$\Psi_i(\zeta_i + \delta) = \Psi_i + \Psi_i^{[1]}\delta + \Psi_i^{[2]}\delta^2 + \cdots + \Psi_i^{[m_i]}\delta^N + \cdots$$
$$(i = 1, 2, \ldots, n)$$

be their expansions, where

$$\Psi_i^{[j]} = \frac{1}{j!}\frac{\partial^j}{\partial\zeta^j}\Psi_i(\zeta)|_{\zeta=\zeta_i} \quad (j = 1, 2, \ldots).$$

Defining

$$T = \Gamma_n \Gamma_{n-1} \ldots \Gamma_1 \Gamma_0, \quad \Gamma_i = T_i[m_i] \ldots T_i[1] \, (i \geqslant 1), \quad \Gamma_0 = I, \tag{12}$$

where

$$T_i[j] = \zeta - \zeta_i^* + (\zeta_i^* - \zeta_i) P_i[j], \quad P_i[j] = \frac{\Psi_i[j-1]\Psi_i[j-1]^\dagger}{\Psi_i[j-1]^\dagger \Psi_i[j-1]}, \quad 1 \leqslant j \leqslant m_i$$

$$\Psi_i[0] = (\Gamma_{i-1} \ldots \Gamma_1 \Gamma_0)|_{\zeta = \zeta_i} \Psi_i,$$

$$\Psi_i[k] = \lim_{\delta \to 0} \frac{[\delta + T_i[k]_{\zeta = \zeta_i}] \cdots [\delta + T_i[2]_{\zeta = \zeta_i}][\delta + T_i[1]|_{\zeta = \zeta_i}]\Gamma_{i-1}(\zeta_i + \delta) \cdots \Gamma_1(\zeta_i + \delta)\Gamma_0 \Psi_i(\zeta_i + \delta)}{\delta^k}$$

$$= \Psi_i + \sum_{s=1}^{k} \sum_{\substack{m_i \geqslant h_1^{(i)} > \cdots > h_{k_i}^{(i)} \geqslant 1, \\ i \geqslant g_1 > \cdots > g_l \geqslant 1, \\ \text{if } g_1 = i, \text{ then } h_1^{(1)} \leqslant k}}^{\sum_{j=1}^{l} k_j + s = k} \left(T_{g_1}[h_1^{(1)}] \cdots T_{g_1}[h_{k_1}^{(1)}] \cdots T_{g_l}[h_1^{(l)}] \cdots T_{g_l}[h_{k_l}^{(l)}] \right)\Big|_{\zeta = \zeta_i} \Psi_i^{[s]}$$

$(1 \leqslant k < m_i)$, then the transformations

$$\Psi[N] = T\Psi, \quad q[N] = q + 2 \sum_{i=1}^{n} \sum_{j=1}^{m_i} (\zeta_i^* - \zeta_i)(P_i[m_j])_{21} \quad \left(N = n + \sum_{k=1}^{n} m_k \right) \tag{13}$$

constitute a generalized Darboux transformation for the NLS equation.

We remark here that the solution formulas, (7) and (13), represented in terms of summations, have certain merit. Indeed, for nonzero Ψ_k, $k = 1, 2, \ldots, N$, all the denominators of $P[i]$ and $P_i[j]$ are easily seen to be nonzero in these forms; therefore, both (7) and (13) supply nonsingular solutions. The former could lead to N-soliton solutions, while the latter may yield rogue wave solutions.

Let us consider an example to illustrate the application of the above formulas to the construction of higher rogue wave solutions. To this end, we start with the seed solution $q[0] = e^{it}$. The corresponding solution for the linear spectral problem at $\zeta = ih$ is

$$\Psi_1(f) = \begin{pmatrix} i(C_1 e^A - C_2 e^{-A})e^{-\frac{i}{2}t} \\ (C_2 e^A - C_1 e^{-A})e^{\frac{i}{2}t} \end{pmatrix}, \tag{14}$$

where

$$C_1 = \frac{(h - \sqrt{h^2-1})^{1/2}}{\sqrt{h^2-1}}, \quad C_2 = \frac{(h + \sqrt{h^2-1})^{1/2}}{\sqrt{h^2-1}}, \quad A = \sqrt{h^2-1}(x + iht).$$

Let $h = 1 + f^2$; expanding the vector function $\Psi_1(f)$ at $f = 0$, we have

$$\Psi_1(f) = \Psi_1(0) + \Psi_1^{[1]} f^2 + \cdots, \tag{15}$$

where

$$\Psi_1(0) = \begin{pmatrix} (-2t + 2ix - i)e^{-\frac{1}{2}it} \\ (2it + 2x + 1)e^{\frac{1}{2}it} \end{pmatrix},$$

$$\Psi_1^{[1]} = \begin{pmatrix} \left[\frac{i}{2}x - \frac{5}{2}t + \frac{i}{4} - 2tx^2 + \frac{2i}{3}x^3 + \frac{2}{3}t^3 - 2ixt^2 - ix^2 + 2tx + it^2\right]e^{-\frac{1}{2}it} \\ \left[\frac{1}{2}x + \frac{5i}{2}t - \frac{1}{4} - 2itx^2 + \frac{2}{3}x^3 - \frac{2i}{3}t^3 - 2xt^2 + x^2 + 2ixt - t^2\right]e^{\frac{1}{2}it} \end{pmatrix}.$$

It is clear that $\Psi_1(0)$ is a solution for (5a) and (5b) at $\zeta = i$. By means of formula (9), we obtain

$$\Psi_1[1] = \lim_{f \to 0} \frac{[if^2 + T_1[1]]\Psi_1(f)}{f^2} = T_1[1]\Psi_1^{[1]} + i\Psi_1(0), \quad T_1[1] = 2i\left(I - \frac{\Psi_1(0)\Psi_1(0)^\dagger}{\Psi_1(0)^\dagger \Psi_1(0)}\right).$$

Substituting the above data into (10) yields the second-order rogue wave solution,

$$q[2] = \left[1 + \frac{G_1 + itG_2}{H}\right]e^{it},$$

where

$$G_1 = 36 - 288x^2 - 192x^4 - 1152t^2x^2 - 864t^2 - 960t^4, \quad G_2 = 360 + 576x^2 - 192t^2 - 384x^4 - 768x^2t^2 - 384t^4,$$

$$H = 64t^6 + 192t^4x^2 + 432t^4 + 396t^2 + 192t^2x^4 - 288t^2x^2 + 9 + 108x^2 + 64x^6 + 48x^4,$$

which is nothing but the solution first constructed by Akhmediev *et al.* [10]. The higher order rogue wave solutions may be calculated similarly, thus we have a general approach to produce these solutions.

B. Determinant forms and higher order rogue waves

In generic cases, iterated Darboux transformations may be given compactly by means of determinants and this is appealing mathematically. For the original Darboux transformation, (6), the result is well known [1]:

Theorem 3. Denoting $\Psi_i = (\psi_i, \phi_i)^T$ $(i = 1, 2, \ldots N)$, then the N-fold Darboux transformation between fields, (7), can be reformulated as

$$q[N] = q[0] - 2\frac{\Delta_2}{\Delta_1}, \tag{16}$$

where

$$\Delta_1 = \begin{vmatrix} \lambda_1^{N-1}\psi_1 & \cdots & \lambda_N^{N-1}\psi_N & -\lambda_1^{*(N-1)}\phi_1^* & \cdots & -\lambda_N^{*(N-1)}\phi_N^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_1 & \cdots & \psi_N & -\phi_1^* & \cdots & -\phi_N^* \\ \lambda_1^{N-1}\phi_1 & \cdots & \lambda_N^{N-1}\phi_N & \lambda_1^{*(N-1)}\psi_1^* & \cdots & \lambda_N^{*(N-1)}\psi_N^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ \phi_1 & \cdots & \phi_N & \psi_1^* & \cdots & \psi_N^* \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \lambda_1^{N}\phi_1 & \cdots & \lambda_N^{N}\phi_N & \lambda_1^{*N}\psi_1^* & \cdots & \lambda_N^{*N}\psi_N^* \\ \lambda_1^{N-2}\psi_1 & \cdots & \lambda_N^{N-2}\psi_N & -\lambda_1^{*(N-2)}\phi_1^* & \cdots & -\lambda_N^{*(N-2)}\phi_N^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ \psi_1 & \cdots & \psi_N & -\phi_1^* & \cdots & -\phi_N^* \\ \lambda_1^{N-1}\phi_1 & \cdots & \lambda_N^{N-1}\phi_N & \lambda_1^{*(N-1)}\psi_1^* & \cdots & \lambda_N^{*(N-1)}\psi_N^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ \phi_1 & \cdots & \phi_N & \psi_1^* & \cdots & \psi_N^* \end{vmatrix}.$$

To find the determinant representations of our generalized Darboux transformation, we observe that the limit process presented in Sec. III A may be taken directly; namely, (10) may be obtained by the following consideration:

$$q[2] = q[1] + \lim_{\zeta_2 \to \zeta_1} 2(\zeta_2^* - \zeta_2)(P[2])_{21}.$$

Thus, as for the KdV case worked out in Ref. [1], we may perform the limit on the determinant form, (16), and get the following.

Theorem 4. Assuming that N distinct solutions $\Psi_i = (\psi_i, \phi_i)^T$ $(i = 1, 2, \ldots, n)$ are given for the spectral problem, (5a) and (5b), at $\zeta = \zeta_1, \ldots, \zeta = \zeta_n$ and expanding

$$(\zeta_i + \delta)^j \psi_i(\zeta_i + \delta) = \zeta_i^j \psi_i + \psi_i[j,1]\delta + \psi_i[j,2]\delta^2 + \cdots + \psi_i[j,m_i]\delta^{m_i} + \cdots,$$

$$(\zeta_i + \delta)^j \phi_i(\zeta_i + \delta) = \zeta_i^j \phi_i + \phi_i[j,1]\delta + \phi_i[j,2]\delta^2 + \cdots + \phi_i[j,m_i]\delta^{m_i} + \cdots,$$

with

$$\psi_i[j,m] = \frac{1}{m!}\frac{\partial^m}{\partial\zeta^m}[\zeta^j\psi_i(\zeta)]|_{\zeta=\zeta_i}, \quad \phi_i[j,m] = \frac{1}{m!}\frac{\partial^m}{\partial\zeta^m}[\zeta^j\phi_i(\zeta)]|_{\zeta=\zeta_i}$$

$(j = 0,1,\ldots,N, m = 1,2,3,\ldots)$, then we have

$$q[N] = q - 2\frac{D_2}{D_1}, \quad D_2 = \det([H_1 \ldots H_n]), \quad D_1 = \det([G_1 \ldots G_n]), \tag{17}$$

where $N = n + \sum_{k=1}^{n} m_k$ and

$$G_i = \begin{bmatrix} \zeta_i^{N-1}\psi_i & \cdots & \psi_i[N-1,m_i] & -\zeta_i^{*(N-1)}\phi_i^* & \cdots & -\phi_i[N-1,m_i]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_i & \cdots & \psi_i[0,m_i] & -\phi_i^* & \cdots & -\phi_i[0,m_i]^* \\ \zeta_i^{N-1}\phi_i & \cdots & \phi_i[N-1,m_i] & \zeta_i^{*(N-1)}\psi_i^* & \cdots & \psi_i[N-1,m_i]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \phi_i & \cdots & \phi_i[0,m_i] & \psi_i^* & \cdots & \psi_i[0,m_i]^* \end{bmatrix},$$

$$H_i = \begin{bmatrix} \zeta_i^{N}\phi_i & \cdots & \phi_i[N,m_i] & \zeta_i^{*N}\psi_i^* & \cdots & \psi_i[N,m_i]^* \\ \zeta_i^{N-2}\psi_i & \cdots & \psi_i[N-2,m_i] & -\zeta_i^{*(N-2)}\phi_i^* & \cdots & -\phi_i[N-2,m_i]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \psi_i & \cdots & \psi_i[0,m_i] & -\phi_i^* & \cdots & -\phi_i[0,m_i]^* \\ \zeta_i^{N-1}\phi_i & \cdots & \phi_i[N-1,m_i] & \zeta_i^{*(N-1)}\psi_i^* & \cdots & \psi_i[N-1,m_i]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \phi_i & \cdots & \phi_i[0,m_i] & \psi_i^* & \cdots & \psi_i[0,m_i]^* \end{bmatrix}.$$

We point out that, applied to special seed solutions, (17) enables us to have a determinant form for higher order rogue wave solutions. We consider

$$\psi_1 = i(C_1 e^A - C_2 e^{-A}), \quad \phi_1 = (C_2 e^A - C_1 e^{-A}), \tag{18}$$

where

$$C_1 = \frac{(1 + f^2 - f\sqrt{2+f^2})^{1/2}}{f\sqrt{2+f^2}}, \quad C_2 = \frac{(1 + f^2 + f\sqrt{2+f^2})^{1/2}}{f\sqrt{2+f^2}},$$

$$A = f\sqrt{2+f^2}[x + i(1+f^2)t + \Phi(f)], \quad \Phi(f) = \sum_{i=0}^{N} s_i f^{2i}, \quad s_i \in \mathbb{C}.$$

The associated Taylor expansions are

$$i^j(1+f^2)^j\psi_1(f) = i^j\psi_1(0) + \psi_1[j,1]f^2 + \cdots + \psi_1[j,N]f^{2N} + \cdots,$$

$$\psi_1[j,n] = \frac{1}{(2n)!}\frac{\partial^{2n}}{\partial f^{2n}}[i^j(1+f^2)^j\psi_1(f)]|_{f=0},$$

$$i^j(1+f^2)^j\phi_1(f) = i^j\phi_1(0) + \phi_1[j,1]f^2 + \cdots + \phi_1[j,N]f^{2N} + \cdots,$$

$$\phi_1[j,n] = \frac{1}{(2n)!}\frac{\partial^{2n}}{\partial f^{2n}}[i^j(1+f^2)^j\phi_1(f)]|_{f=0}$$

$(j = 0,1,\ldots,N, n = 1,2,3,\ldots)$. It follows that the Nth-order rogue wave solution for the NLS equation (1), reads

$$q[N] = \left[1 - 2\frac{D_2}{D_1}\right]e^{it}, \tag{19}$$

where

$$D_1 = \begin{vmatrix} i^{N-1}\psi_1 & \cdots & \psi_1[N-1,N-1] & -(-i)^{(N-1)}\phi_1^* & \cdots & -\phi_1[N-1,N-1]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_1 & \cdots & \psi_1[0,N-1] & -\phi_1^* & \cdots & -\phi_1[0,N-1]^* \\ i^{N-1}\phi_1 & \cdots & \phi_1[N-1,N-1] & -i^{(N-1)}\psi_1^* & \cdots & \psi_1[N-1,N-1]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \phi_1 & \cdots & \phi_1[0,N-1] & \psi_1^* & \cdots & \psi_1[0,N-1]^* \end{vmatrix},$$

$$D_2 = \begin{vmatrix} i^N\phi_1 & \cdots & \phi_1[N,N-1] & (-i)^N\psi_1^* & \cdots & \psi_1[N,N-1]^* \\ i^{N-2}\psi_1 & \cdots & \psi_1[N-2,N-1] & -(-i)^{(N-2)}\phi_1^* & \cdots & -\phi_1[N-2,N-1]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \psi_1 & \cdots & \psi_1[0,N-1] & -\phi_1^* & \cdots & -\phi_1[0,N-1]^* \\ i^{N-1}\phi_1 & \cdots & \phi_1[N-1,N-1] & (-i)^{(N-1)}\psi_1^* & \cdots & \psi_1[N-1,N-1]^* \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \\ \phi_1 & \cdots & \phi_1[0,N-1] & \psi_1^* & \cdots & \psi_1[0,N-1]^* \end{vmatrix}.$$

For the case where $N = 2$, the above formula may provide the second-order rogue wave solution with two free parameters for the NLS equation, which was analyzed in detail in Ref. [18]. It was shown that this solution splits into three first-order rogue waves rather than two. Indeed, our results supply the high-order rogue wave solutions with more free parameters, which determine the spatial-temporal structures of the solutions. In particular, the third rogue wave solution, possessing four free parameters, may be worked out by setting $N = 3$ and $\Phi(f) = (b + ic)f^2 + (e + ig)f^4$ in our formula, whose explicit expression is omitted since it is rather cumbersome. In the following, we consider two special cases, which have different spatial-temporal patterns.

(1) *Case A*. In this case, we assume $b = c = 0$, $|e| \gg 0$, or $|g| \gg 0$. The corresponding third-order rogue wave solution is composed of six first-order rogue waves, which array a regular pentagon. Interestingly, among the six first-order rogue waves, one sits in the center and the rest are located on the vertices of the pentagon. After some calculations, we find that the radial distance from the center of the pentagon approximately

equals $\frac{1}{2}360^{1/5}(e^2 + g^2)^{1/10}$. For the case $g = 0$ and $e \gg 0$, one of the vertices is located along the negative direction of the x axis and the corresponding quadrantal angle for the (e,g) plane is assumed to be 0 (Fig. 1). For the general nonzero g, the pentagon will rotate $\frac{1}{5}\theta$ along the anticlockwise direction, where θ is the associated quadrantal angle for the (e,g) plane.

(2) *Case B*. For the second case, we take the parameters $e = g = 0$, $|b| \gg 0$, or $|c| \gg 0$. The corresponding third-order rogue wave consists of the six first-order rogue waves as well, which array an equilateral triangle. The distance between the center and any vertex approximately equals $90^{1/6}(b^2 + c^2)^{1/6}$. For the case $c = 0$ and $b \gg 0$, one of the vertices is located along the positive direction of the x axis and the corresponding quadrantal angle for the (b,c) plane is assumed to be 0 (Fig. 2). For the general nonzero c, the triangle will rotate $\frac{1}{3}\theta$ in the anticlockwise direction, where θ is the related quadrantal angle for the (b,c) plane. When $b = c = e = g = 0$, the rogue wave is the one considered by Akhmediev *et al.* [19].

Let (x_i, t_i) $(i = 1, \ldots, 6)$ be the coordinates of the six peaks; then we find that the third-order rogue wave may be
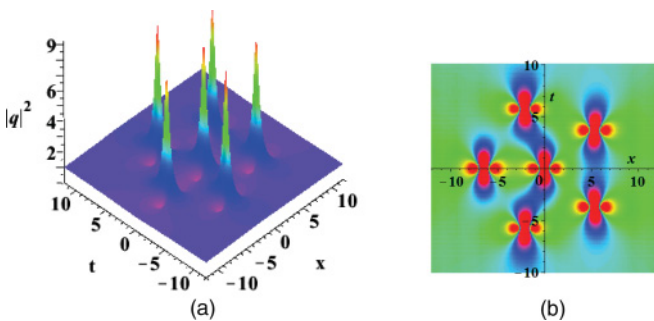

FIG. 1. (Color online) (a) Third-order rogue wave solution $|q|^2$. (b) Density plot for the third-order rogue wave solution $|q|^2$. Parameters $e = 1000$ and $g = 0$. The third-order rogue wave exhibits a regular pentagon spatial symmetry structure.
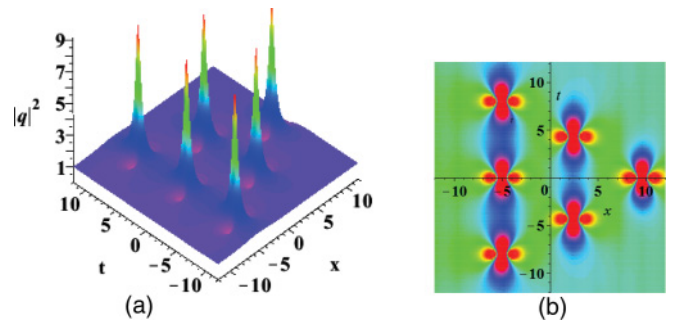

FIG. 2. (Color online) (a) Third-order rogue wave solution $|q|^2$. (b) Density plot for the third-order rogue wave solution $|q|^2$. Parameters $b = 100$ and $c = 0$. The third-order rogue wave is seen to possess a regular triangle spatial symmetry structure.
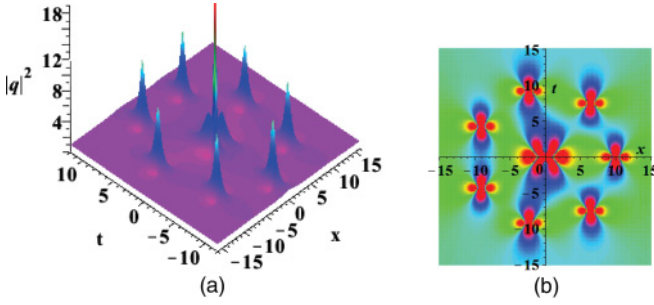
FIG. 3. (Color online) (a) Fourth-order rogue wave solution $|q|^2$. (b) Density plot for the fourth-order rogue wave solution $|q|^2$. Parameters $\Phi(f) = 5 \times 10^5 i f^6$. The fourth-order rogue wave possesses a regular heptagonal spatial-temporal structure with a second-order rogue wave located at the center.

approximated by

$$q[3] \approx \sum_{i=1}^{6} \left[-1 + 4\frac{1 + 2i(t - t_i)}{d_i} \right] e^{it},$$

$$d_i = 1 + 4(x - x_i)^2 + 4(t - t_i)^2,$$

where the "center of mass" is at the origin (0,0) in both cases. With raising of the order, the rogue solution contains more free parameters and exhibits more interesting spatial-temporal structures. For instance, choosing the proper parameters, a fourth-order rogue wave possesses the regular heptagon spatial-temporal pattern (Fig. 3), and a second-order rogue wave is located in the center. Naturally, we conjecture that the spatial-temporal pattern of an Nth-order rogue wave possesses a $2N - 1$-gon spatial-temporal pattern upon choosing the parameters $\Phi(f) = cf^{2(N-1)}$ ($|c| \gg 0$) in formula (19), where c is a complex number.

We conclude this section with the following remarks in order.

(1) If we choose the seed solution $q[0] = 0$, we may obtain the higher soliton solution [21,22].

(2) The integrable Hirota equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q - i\alpha(q_{xxx} + 6|q|^2 q_x) = 0,$$
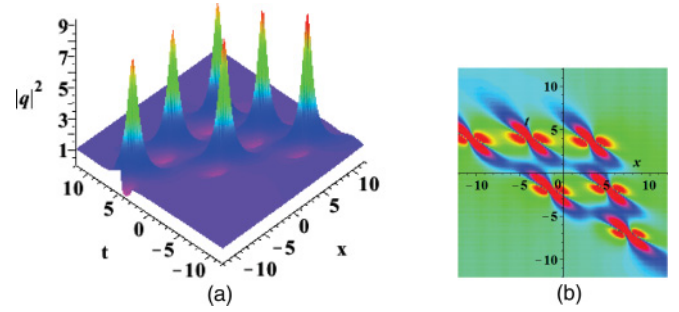$$\alpha \text{ is a real constant,} \tag{20}$$

FIG. 4. (Color online) (a) Third-order rogue wave solution $|q|^2$. (b) Density plot for the third-order rogue wave solution $|q|^2$. Parameters $\alpha = \frac{1}{6}$, $\Phi(f) = 50 i f^2$. The third-order rogue wave of the Hirota equation exhibits a triangular temporal-spatial structure.

is the third flow of the NLS hierarchy. Its rogue wave solutions and rational solutions are discussed in Ref. [20]. To obtain the Nth-order rogue wave solution for Hirota equations (20) from (19), we merely need to modify the A as $A = f\sqrt{2 + f^2}[x + i(1 + f^2)t + \alpha(2 + 4(1 + f^2)^2)t + \Phi(f)]$ in Eq. (18). Rather than giving the explicit expressions, which is lengthy, we plot the third-order rogue wave solution for the Hirota equation in Fig. 4. Due to the third-order dispersion and time-delay correction to the cubic term [20], the solution displays a different behavior from the NLS equation.

IV. CONCLUSION

Through a limit procedure, we have generalized the original Darboux transformation for the NLS equation. This Darboux transformation, in particular, allows us to calculate higher order rogue wave solutions in a unified way. We believe that the idea is rather general and could be applied to other physically interested models as well. These spatial-temporal structures for Nth-order rogue waves may be useful for study of the spatial-temporal distribution of rogue waves in deep water.

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