Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation

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(Received 13 November 2011; published 13 February 2012)

The determinant representation of the n-fold Darboux transformation of the Hirota equation is given. Based on our analysis, the 1-soliton, 2-soliton, and breathers solutions are given explicitly. Further, the first order rogue wave solutions are given by a Taylor expansion of the breather solutions. In particular, the explicit formula of the rogue wave has several parameters, which is more general than earlier reported results and thus provides a systematic way to tune experimentally the rogue waves by choosing different values for them.

DOI: 10.1103/PhysRevE.85.026601 PACS number(s): 42.65.Tg, 47.35.Fg, 05.45.Yv

I. INTRODUCTION

It is well known that the completely integrable nonlinear Schrödinger equation (NLSE)

$$iq_t + 2|q|^2 q + q_{xx} = 0 \tag{1}$$

plays an important role in many branches of physics and applied mathematics, such as nonlinear optics [1,2], plasma physics [3], and nonlinear quantum field theory [4]. Especially in nonlinear optics, the propagation of a picosecond optical pulse in an optical fiber is governed by the NLSE. After the theoretical prediction of the existence of solitary waves [5] and the experimental demonstration of the optical solitons [6], the research on optical solitons is more and more fascinating since it may be applied as bit rates in the next generation of optical communication systems.

The NLSE has been used successfully to describe the propagation of a picosecond optical pulse. However, for the propagation of a subpicosecond or femtosecond pulse, the higher order effects should be taken into account, and one version of the higher-order nonlinear Schrödinger equation (HNLSE) is of the form

$$iq_t + \alpha_1 q_{xx} + \alpha_2 q|q|^2 + i\alpha_3 q + i\alpha_4 q_{xxx} + \alpha_5 q(|q|^2)_x + i\alpha_6(q|q|^2)_x = 0. \tag{2}$$

This equation was first proposed by Hasegawa and Kodama [7]. Mathematically, for Eq. (2), many authors have obtained the following four completely integrable cases:

(1) α_1:α_2:α_3:α_4:α_6:Im$(\alpha_5) + \alpha_6 = \frac{1}{2}$:1:0:0:1:1,

(2) α_1:α_2:α_3:α_4:α_6:Im$(\alpha_5) + \alpha_6 = \frac{1}{2}$:1:0:0:1:0,

(3) α_1:α_2:α_3:α_4:α_6:Im$(\alpha_5) + \alpha_6 = \frac{1}{2}$:1:0:1:6:0, which implies the Hirota equation [8,9],

(4) α_1:α_2:α_3:α_4:α_6:Im$(\alpha_5) + \alpha_6 = \frac{1}{2}$:1:0:1:6:3, which implies the Sasa-Satsuma equation [10,11],

by using different approaches like the Painlevé test [12], the Galilean transformation [13], and the Wahlquist-Estabrook prolongation method [14]. There are multicomponent extensions [15–17] of the above NLSE.

In recent years, a new wave called a rogue wave has attracted much attention. It was observed in many fields, such as oceanics [18–22] and nonlinear optics [23–25]. Though rogue waves have caused many marine disasters, fortunately, there are already some achievements in understanding this natural phenomenon. In [24], a system with an extremely steep and large wave has been studied, and the observation of a rogue wave has been reported in an optical fiber. In [25], a mathematical solution called the Peregrine soliton as a prototype of an ocean rogue wave has been observed in a physical system. In [26], the authors have used an experimental setup to observe a Peregrine soliton in a water wave tank.

The rogue wave of the Hirota equation is given by a very simple and powerful Darboux transformation (DT) with the help of the authors' very rich empirical ideas [27]. However, there are two unusual points in this work, i.e., (1) the Lax pair does not contain spectral parameters and (2) the seed solution $\psi = e^{ix}$ is too special, such that its rogue wave is not universal enough. Considering the wide applicability of the Hirota equation, we shall try to find a more general form of the rogue wave of the Hirota equation by the DT [28–31] from a general seed solution. Specifically, we follow the Ablowitz-Kaup-Newell-Segur (AKNS) procedure [32] to construct the Lax pair with spectral parameters, and the corresponding Hirota equation takes the form

$$iq_t + \alpha(2|q|^2 q + q_{xx}) + i\beta(q_{xxx} + 6|q|^2 q_x) = 0, \tag{3}$$

with the choice of coefficients $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 :$ Im$(\alpha_5) + \alpha_6 = 1 : 2 : 0 : 1 : 6 : 0$. If we let $\alpha = 1$, $\beta = 0$, Eq. (3) reduces to Eq. (1). Note that Eq. (3) is another equivalent form of the Hirota equation [27]. This Lax pair is more convenient for constructing the DT due to its parameters. Furthermore, solitons are derived from zero seed, and breathers are derived from a periodic seed with a constant amplitude. At last, the rogue wave of Eq. (3) is given by the Taylor expansion of the breather, which implies the rogue wave [18,19] of NLSE (1).

II. LAX PAIR OF THE HIROTA EQUATION

The Lax pair assures the complete integrability of a nonlinear system and is often used to obtain explicit solutions by the DT. In this section, we use the AKNS procedure [32] to get the Lax pair with the spectral parameters of the Hirota equation (3).

By a similar way to the AKNS system, the Lax pair for Eq. (3) can be expressed as follows:

$$\varphi_x = M\varphi, \varphi_t = N\varphi, \tag{4}$$

where $\varphi=(\varphi_1,\varphi_2)^T$, and

$$M=\begin{pmatrix}-i\lambda & q\\ -q^* & i\lambda\end{pmatrix},\quad N=\lambda^3\begin{pmatrix}-4\beta i & 0\\ 0 & 4\beta i\end{pmatrix}+\lambda^2\begin{pmatrix}-2\alpha i & 4\beta q\\ -4\beta q^* & 2\alpha i\end{pmatrix}+\lambda\begin{pmatrix}2\beta i|q|^2 & 2\beta i q_x+2\alpha q\\ 2\beta i q_x^*-2\alpha q^* & -2\beta i|q|^2\end{pmatrix}$$
$$+\begin{pmatrix}i\alpha|q|^2+\beta(qq_x^*-q^*q_x) & i\alpha q_x-\beta(q_{xx}+2|q|^2q)\\ i\alpha q_x^*+\beta(q_{xx}^*+2|q|^2q^*) & -i\alpha|q|^2-\beta(qq_x^*-q^*q_x)\end{pmatrix},$$

λ is a complex spectral parameter, and "*" denotes the complex conjugate. One can verify that the compatibility condition $M_t-N_x+[M,N]=0$ gives rise to Eq. (3), where the bracket represents the usual matrix commutator.

III. DARBOUX TRANSFORMATION

The DT [28–31] is an effective method to construct solutions, including the n-soliton and breather solutions. In this section, we would like to introduce a simple gauge transformation of spectral problems (4) as follows:

$$\varphi^{[1]}=T\varphi. \tag{5}$$

It can transform linear problems (4) into new one possessing the same matrix form, namely,

$$\varphi_x^{[1]}=M^{[1]}\varphi^{[1]},\ \varphi_t^{[1]}=N^{[1]}\varphi^{[1]}, \tag{6}$$

where $M^{[1]},N^{[1]}$ have the same forms with M,N except that of the q,q^* in the matrices M,N are replaced with $q^{[1]},q^{[1]*}$ in the matrices $M^{[1]},N^{[1]}$. It is easy to obtain the equations

$$M^{[1]}T=T_x+TM, \tag{7}$$
$$N^{[1]}T=T_t+TN. \tag{8}$$

In general, if the transformation T is a polynomial of the parameter λ, according to the Hirota equation (3), we can start from

$$T=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\lambda+\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \tag{9}$$

where a_1,b_1,c_1,d_1,a,b,c,d are all functions of the variables x and t.

From Eqs. (7) and (9), it is easy to have

$$\begin{aligned}&\begin{pmatrix}a_{1x} & b_{1x}\\ c_{1x} & d_{1x}\end{pmatrix}\lambda+\begin{pmatrix}a_x & b_x\\ c_x & d_x\end{pmatrix}\\ &=\begin{pmatrix}c_1q^{[1]}\lambda-ia_1\lambda^2 & d_1q^{[1]}\lambda-ib_1\lambda^2\\ ic_1\lambda^2-a_1q^{[1]*}\lambda & id_1\lambda^2-b_1q^{[1]*}\lambda\end{pmatrix}\\ &\quad+\begin{pmatrix}cq^{[1]}-ia\lambda & dq^{[1]}-ib\lambda\\ ic\lambda-aq^{[1]*} & id\lambda-bq^{[1]*}\end{pmatrix}\\ &\quad-\begin{pmatrix}-ia_1\lambda^2-b_1q^*\lambda & a_1q\lambda+ib_1\lambda^2\\ -ic_1\lambda^2-d_1q^*\lambda & c_1q\lambda+id_1\lambda^2\end{pmatrix}\\ &\quad-\begin{pmatrix}-ia\lambda-q^*b & aq+ib\lambda\\ -ic\lambda-q^*d & qc+id\lambda\end{pmatrix},\end{aligned} \tag{10}$$

and comparing the coefficients of $\lambda^k(k=0,1,2)$ of the above formula gives

$$b_1=c_1=0\quad\text{for}\quad k=2, \tag{11}$$

$$\begin{aligned}a_{1x}=d_{1x}=0,\quad -2ib+q^{[1]}d_1-qa_1=0,\\ 2ic-q^{[1]*}a_1+q^*d_1=0\quad\text{for}\quad k=1,\end{aligned} \tag{12}$$

$$\begin{aligned}a_x=q^{[1]}c+q^*b,\quad b_x=q^{[1]}d-qa,\\ c_x=-q^{[1]*}a+q^*d,\quad d_x=-q^{[1]*}b-qc\ \text{for}\ k=0.\end{aligned} \tag{13}$$

By using the calculation above, it is obvious that a_1,d_1 can be made into constants and allowed to equal 1 without loss of generality, so the DT for Eq. (3) could be in the form of

$$\varphi^{[1]}=T\varphi=(\lambda I-S)\varphi, \tag{14}$$

where λ is a complex spectral parameter, I is a 2×2 identity matrix, and S is a nonsingular matrix.

Substituting the expressions of $M,M^{[1]}$ and T into Eq. (7), the coefficients of λ become

$$\begin{pmatrix}0 & q^{[1]}\\ -q^{[1]*} & 0\end{pmatrix}=\begin{pmatrix}0 & q\\ -q^* & 0\end{pmatrix}+i[S,\sigma],$$

where $\sigma=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $S=\begin{pmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix}$. Therefore, the new solutions are given by

$$q^{[1]}=q-2is_{12},\ -q^{[1]*}=-q^*+2is_{21}, \tag{15}$$

under a constraint

$$s_{12}^*=-s_{21}. \tag{16}$$

Similar to the case of the NLSE [28,29], to obtain the explicit formula of S by the solutions of the Lax pair, we introduce

$$S=H\Lambda H^{-1}, \tag{17}$$

with

$$H=\begin{pmatrix}f_1 & g_1\\ f_2 & g_2\end{pmatrix},\quad \Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},$$

where $(f_1,f_2)^T$ is a solution of the eigenvalue equation of the Lax pair (4) when $\lambda=\lambda_1$. It is useful to know that $(g_1,g_2)^T=(-f_2^*,f_1^*)^T$ is a solution of (4) when $\lambda=\lambda_1^*$. In order to satisfy the constraint of S, let $\lambda_2=\lambda_1^*$ and $(g_1,g_2)^T=(-f_2^*,f_1^*)^T$; then

$$S=\frac{1}{\Delta}\begin{pmatrix}\lambda_1|f_1|^2+\lambda_1^*|f_2|^2 & (\lambda_1-\lambda_1^*)f_1f_2^*\\ (\lambda_1-\lambda_1^*)f_1^*f_2 & \lambda_1|f_2|^2+\lambda_1^*|f_1|^2\end{pmatrix}, \tag{18}$$

where $\Delta=|f_1|^2+|f_2|^2$. By a direct calculation, constraint (16) of the S can be verified. So from (15) and (18), the DT generates a new solution of the Hirota equation as

$$q^{[1]} = q - \frac{2i}{\Delta}(\lambda_1 - \lambda_1^*) f_1 f_2^*. \tag{19}$$

In fact, as in the case of the NLSE [28,29,33], the DT of the Hirota equation also has determinant representation, which is convenient for getting the solutions generated by the higher order transformation. Here we rewrite the one-fold DT (19) in the form of a determinant as

$$q^{[1]} = q - 2i\frac{S_2}{W_2} = q - 2i\frac{\begin{vmatrix} f_1 & \lambda_1 f_1 \\ g_1 & \lambda_2 g_1 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}, \tag{20}$$

under the reductions $g_1 = -f_2^*, g_2 = f_1^*, \lambda_2 = \lambda_1^*$. For the two-fold DT, we obtain

$$q^{[2]} = q - 2i\frac{S_4}{W_4}, \tag{21}$$

where

$$S_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1^2 f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2^2 g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3^2 f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4^2 g_3 \end{vmatrix}, \quad W_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 \end{vmatrix},$$

and under the reductions $g_1 = -f_2^*, g_2 = f_1^*, g_3 = -f_4^*, g_4 = f_3^*, \lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*$. Similarly, the n-fold DT could be written in determinant form as

$$q^{[n]} = q - 2i\frac{S_{2n}}{W_{2n}}, \tag{22}$$

where

$$S_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \ldots & \lambda_1^{n-1} f_1 & \lambda_1^n f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \ldots & \lambda_2^{n-1} g_1 & \lambda_2^n g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \ldots & \lambda_3^{n-1} f_3 & \lambda_3^n f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \ldots & \lambda_4^{n-1} g_3 & \lambda_4^n g_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \ldots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^n g_{2n-1} \end{vmatrix},$$

$$W_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \ldots & \lambda_1^{n-1} f_1 & \lambda_1^{n-1} f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \ldots & \lambda_2^{n-1} g_1 & \lambda_2^{n-1} g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \ldots & \lambda_3^{n-1} f_3 & \lambda_3^{n-1} f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \ldots & \lambda_4^{n-1} g_3 & \lambda_4^{n-1} g_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \ldots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^{n-1} g_{2n} \end{vmatrix}.$$

It is convenient to calculate the multisolitons, multibreathers, and higher order rogue waves of the Hirota equation. This result under $\alpha = 1$ and $\beta = 0$ is consistent with the corresponding determinant representation of Refs. [28,29,33].

IV. SOLITON AND BREATHER SOLUTIONS

In this section, we start from a zero seed solution and a periodic seed solution to construct new solutions (including soliton and breather solutions) by the DT obtained above; then the first order rogue wave could be obtained by a Taylor expansion from the breather solution.

(1) Now let the seed $q = 0$ and $\lambda_1 = \xi + i\eta$; then

$$\begin{aligned} f_1 &= e^{-i(\xi+i\eta)x-(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t}, \\ f_2 &= e^{i(\xi+i\eta)x+(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t}. \end{aligned} \tag{23}$$

Taking f_1, f_2 given by Eq. (23) back into the DT (20), we can get 1-soliton solution (see Fig. 1)

$$\begin{aligned} q^{[1]}_{\text{soliton}} &= 2\eta e^{2i(-\xi x-4\beta\xi^3 t-2\alpha\xi^2 t+12\beta\xi\eta^2 t+2\alpha\eta^2 t)} \\ &\quad \times \sec h(-2\eta x - 24\beta\eta\xi^2 t + 8\beta\eta^3 t - 8\alpha\eta\xi t). \end{aligned} \tag{24}$$

(2) Let the seed $q = 0$ and $\lambda_1 = \xi + i\eta$, $\lambda_3 = \theta + i\vartheta$, by solving linear problems (4); the eigenfunctions can be obtained

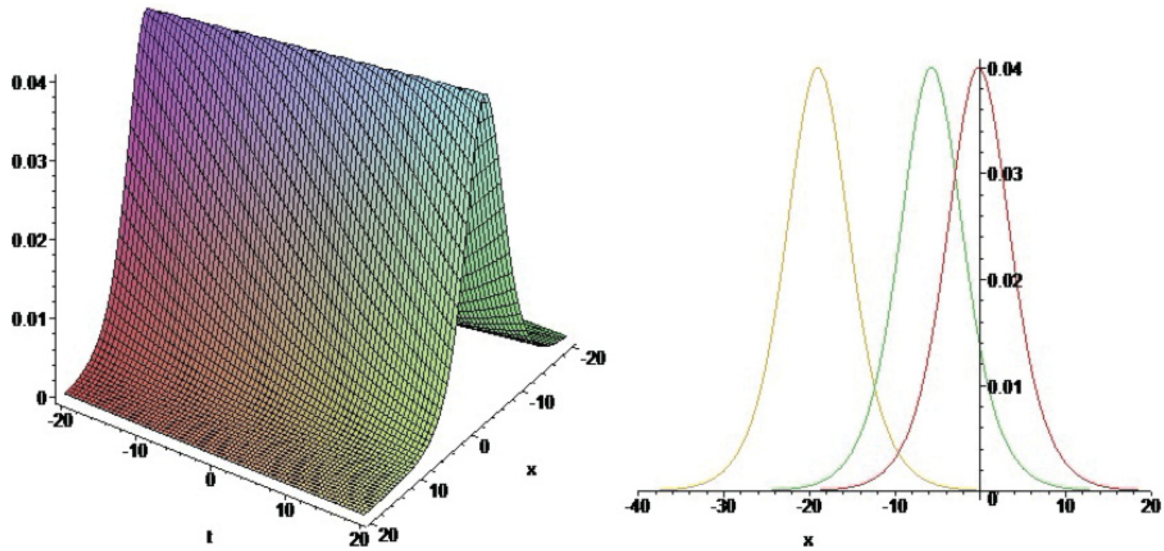

FIG. 1. (Color online) (Left panel) The 1-soliton solution of the Hirota equation with $\eta = 0.1$, $\xi = 0.05$, $\alpha = 1$, $\beta = 1$ and (Right panel) its profiles at different times $t = 1$(red/right), $t = 30$(green/middle), $t = 100$ (yellow/left).

as follows:

$$
\begin{aligned}
f_1 &= e^{-i(\xi+i\eta)x-(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t},\\
f_2 &= e^{i(\xi+i\eta)x+(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t},\\
f_3 &= e^{-i(\theta+i\vartheta)x-(4\beta i(\theta+i\vartheta)^3+2\alpha i(\theta+i\vartheta)^2)t},\\
f_4 &= e^{i(\theta+i\vartheta)x+(4\beta i(\theta+i\vartheta)^3+2\alpha i(\theta+i\vartheta)^2)t}.
\end{aligned}
$$

According to the reductions $g_1 = -f_2^*, g_2 = f_1^*, g_3 = -f_4^*, g_4 = f_3^*, \lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*$, the 2-soliton solution is given explicitly by the DT (21), which is plotted in Fig. 2.

(3) In order to get non-trivial periodic solutions, we set seed $q = ce^{i\rho}$ with $\rho = ax + bt$, where a,b,c are all real constants under a condition $b = \alpha(2c^2 - a^2) + \beta(a^3 - 6ac^2)$. The corresponding solutions of the eigenvalue equations of the Lax pair are given by

$$
\begin{aligned}
f_1 &= ce^{i[(\frac{1}{2}a+c_1)x+(\frac{1}{2}b+2c_1c_2)t]},\\
f_2 &= i\left(\tfrac{1}{2}a + \lambda_1 + c_1\right) e^{i[(-\frac{1}{2}a+c_1)x+(-\frac{1}{2}b+2c_1c_2)t]},
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
c_1 &= \tfrac{1}{2}\sqrt{4c^2 + 4\lambda_1^2 + 4\lambda_1 a + a^2},\\
c_2 &= \left(\alpha\lambda_1 + 2\beta\lambda_1^2 - \tfrac{1}{2}a\alpha - \beta c^2 + \tfrac{1}{2}\beta a^2 - \lambda_1 a\beta\right).
\end{aligned}
$$

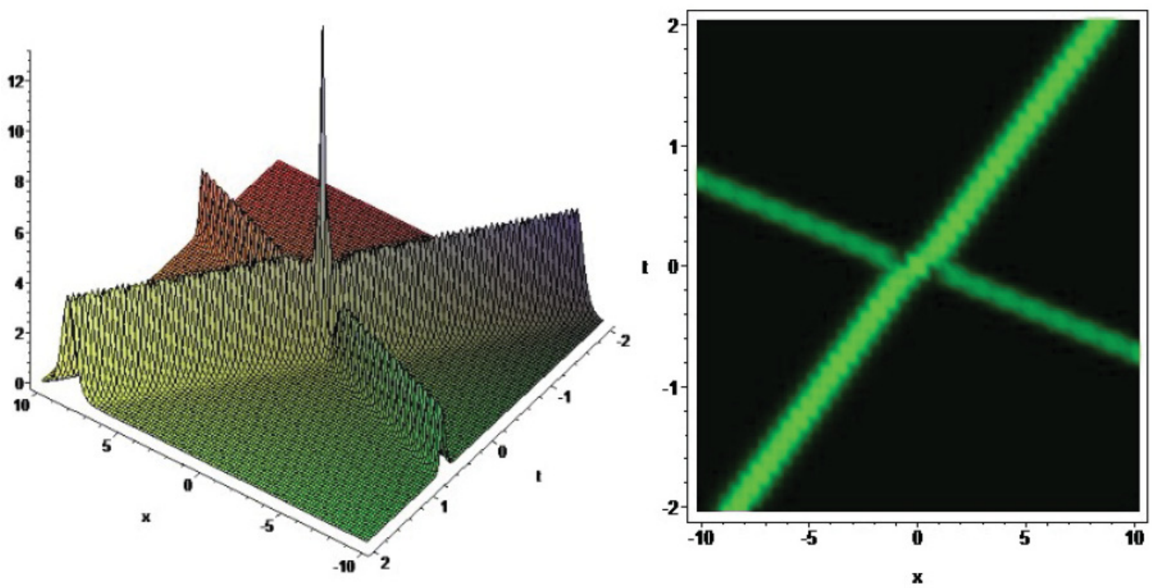

FIG. 2. (Color online) (Left panel) The 2-soliton solution of the Hirota equation with $\eta = 0.1$, $\xi = 0.8$, $\theta = 0$, $\vartheta = 1$, $\alpha = 1$, $\beta = 1$ and (Right panel) its trajectory lines.

By the principle of the superposition of the linear differential equation, the new eigenfunctions associated with λ_1 can be expressed by

$$F_1 = f_1 - f_2^*, \quad F_2 = f_2 + f_1^*;$$

then we use them to get following breather solution

$$q^{[1]} = q - \frac{2i}{\Delta}(\lambda_1 - \lambda_1^*)F_1 F_2^*, \tag{26}$$

and $\Delta = |F_1|^2 + |F_2|^2$ by the DT (20). By a tedious calculation, we finally get the breather solution under $a = -2\mathrm{Re}(\lambda_1)$ (see Fig. 3):

$$q^{[1]}_{\text{breather}} = e^{i\rho}\left[c - \frac{2\eta[\eta\cosh(2d_2) - i\sigma\sinh(2d_2) - c\cos(2d_1)]}{c\cosh(2d_2) - \eta\cos(2d_1)}\right], \tag{27}$$

where

$$\lambda_1 = \xi + i\eta, \quad a = -2\xi, \quad \rho = ax + bt = -2\xi x + bt,$$
$$d_1 = \sigma x + (4\sigma\alpha\xi + 12\sigma\beta\xi^2 - 4\sigma\beta\eta^2 - 2\sigma^3\beta - 2\sigma\beta\eta^2)t,$$
$$d_2 = (2\sigma\alpha\eta + 12\sigma\beta\xi\eta)t,$$
$$\sigma = \sqrt{\frac{-b - 4\alpha\xi^2 - 8\beta\xi^3}{-2\alpha - 12\beta\xi} - \eta^2}.$$

V. ROGUE WAVE SOLUTIONS

There are at least two examples—the NLSE [18] and the derivative NLSE [34] to get a rogue wave by the Taylor expansion of the breather solutions. Here we shall use this approach again to get the rogue wave of the Hirota equation from the breather solution (27).

The Taylor expansion at $\eta = \sqrt{\frac{-b-4\alpha\xi^2-8\beta\xi^3}{-2\alpha-12\beta\xi}}$ of the breather solution (27) implies a general form of the first order rogue wave of the Hirota equation:

$$q_{\text{roguewave}} = ke^{i(-2\xi x + bt)}\left(1 - \frac{2k_1 + 2k_2 + ik_3 t}{k_1 - k_2}\right), \tag{28}$$

where

$$k = \sqrt{\frac{b + 4\alpha\xi^2 + 8\beta\xi^3}{2\alpha + 12\beta\xi}},$$
$$k_1 = v_1 t^2 + v_2 xt + v_3 x^2,$$
$$k_2 = \alpha^3 + 18\alpha^2\beta\xi + 108\alpha\beta^2\xi^2 + 216\beta^3\xi^3,$$
$$k_3 = 32\xi^2\alpha^4 + 864\alpha\beta^2\xi^2 b + 144\alpha^2\beta\xi b + 13824\alpha\beta^3\xi^5 + 13824\beta^4\xi^6 + 1728\beta^3\xi^3 b + 8\alpha^3 b + 4608\alpha^2\beta^2\xi^4 + 640\alpha^3\beta\xi^3,$$
$$v_1 = -79872\beta^3\xi^7\alpha^2 - 13824\beta^3\xi^5 b\alpha - 832\beta\xi^3\alpha^3 b - 4\alpha^3 b^2 - 22528\beta^2\xi^6\alpha^3 - 92160\beta^5\xi^9 - 216\beta^2 b^2\alpha\xi^2 - 13824\beta^4\xi^6 b - 432\beta^3\xi^3 b^2 - 3200\beta\xi^5\alpha^4 - 138240\beta^4\xi^8\alpha - 64b\alpha^4\xi^2 - 24\alpha^2\xi\beta b^2 - 192\alpha^5\xi^4 - 18\beta^2 b^3 - 4992\beta^2\xi^4\alpha^2 b,$$
$$v_2 = -9216\beta^4\xi^7 - 144\alpha^2\beta\xi^2 b - 64\alpha^4\xi^3 - 576\beta^3\xi^4 b - 384\alpha\beta^2\xi^3 b + 12\alpha\beta b^2 - 10752\beta^3\xi^6\alpha - 896\alpha^3\xi^4\beta - 16\alpha^3\xi b - 4608\alpha^2\beta^2\xi^5 + 72\beta^2\xi b^2,$$
$$v_3 = -8\alpha^3\xi^2 - 576\beta^3\xi^5 - 72\beta^2\xi^2 b - 2\alpha^2 b - 24\alpha\beta\xi b - 480\alpha\beta^2\xi^4 - 112\alpha^2\beta\xi^3.$$

It is not difficult to verify the validity of this solution. Obviously, this form of the rogue wave $q_{\text{roguewave}}$ is more general than the known result [27] because of the appearance of several parameters related to the background and the eigenvalue of the Lax pair; and thus it also provides a possible

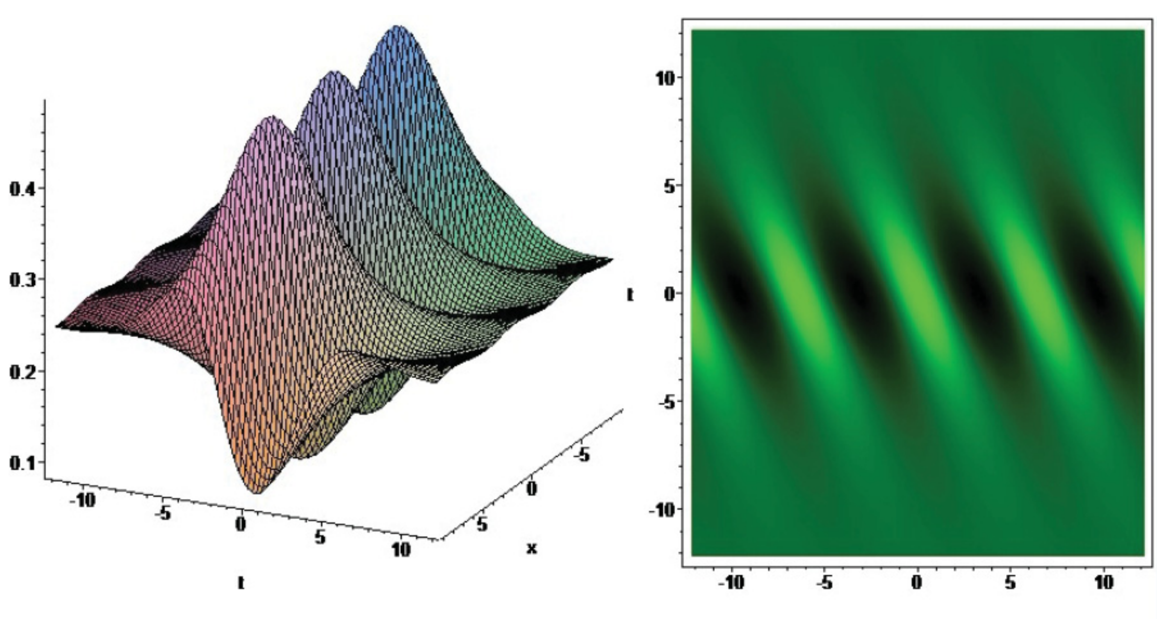

FIG. 3. (Color online) (Left panel) Breather solution (27) of the Hirota equation with $\alpha = 1$, $\beta = 1$, $\xi = -0.5$, $\eta = 0.1$, $b = 1$ and (Right panel) its density plot.

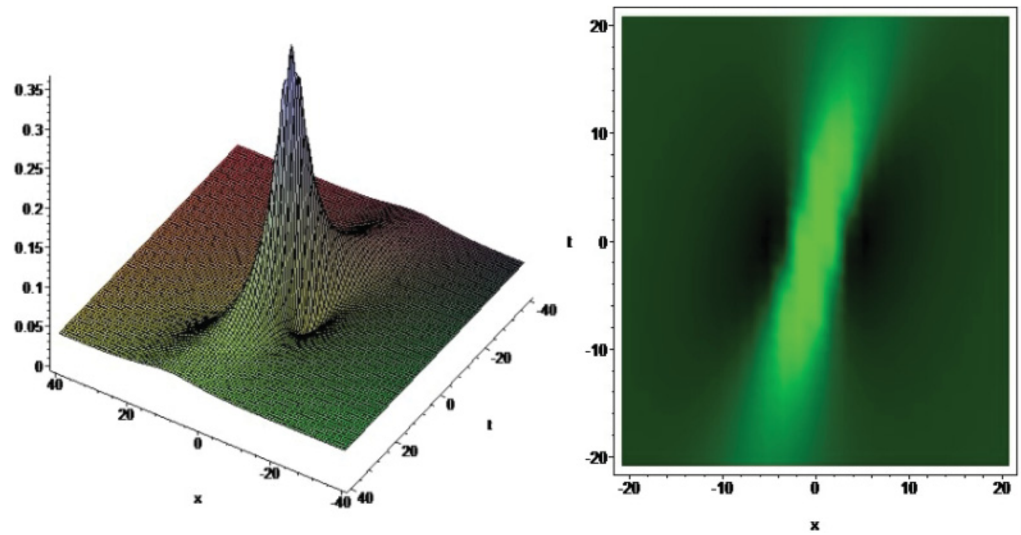

FIG. 4. (Color online) (Left panel) Rogue wave (29) of the Hirota equation with $\alpha = 1$, $\beta = 1$, $b = 0.08$ and (Right panel) its density plot.

way to tune experimentally the rogue wave by choosing different values of them. Moreover, this controllability of the rogue wave highly improves the possibility of observing it in a laboratory. Set $\xi = 0$ in (28), then a simple rogue wave

$$q^{[1]}_{\text{roguewave}} = e^{ibt}\frac{\sqrt{\frac{b}{2\alpha}}(-2b\alpha^2x^2 + 12b^2\alpha\beta xt - 18b^3\beta^2t^2 - 4b^2\alpha^3t^2 + 8ib\alpha^3t + 3\alpha^3)}{4b^2\alpha^3t^2 + 2b\alpha^2x^2 - 12b^2\alpha\beta xt + 18b^3\beta^2t^2 + \alpha^3} \tag{29}$$

is obtained, which is plotted in Fig. 4. Furthermore, the above rogue wave (29) reduces to the known result given by Ref. [27]. Moreover, setting $\alpha = 1$, $\beta = 0$, our rogue wave (29) reduces to the simplest form

$$q^{[11]}_{\text{roguewave}} = e^{ibt}\frac{\sqrt{\frac{b}{2}}(-2bx^2 - 4b^2t^2 + 8ibt + 3)}{4b^2t^2 + 2bx^2 + 1}, \tag{30}$$

which is an equivalent formula of the rogue wave [18] of the NLSE (1) as expected and plotted in Fig. 5. As a final remark of this paper, we would like to stress that the higher order rogue wave of the Hirota equation can be calculated from the determinant representation (22) of the DT, which will be done in a separate paper soon.

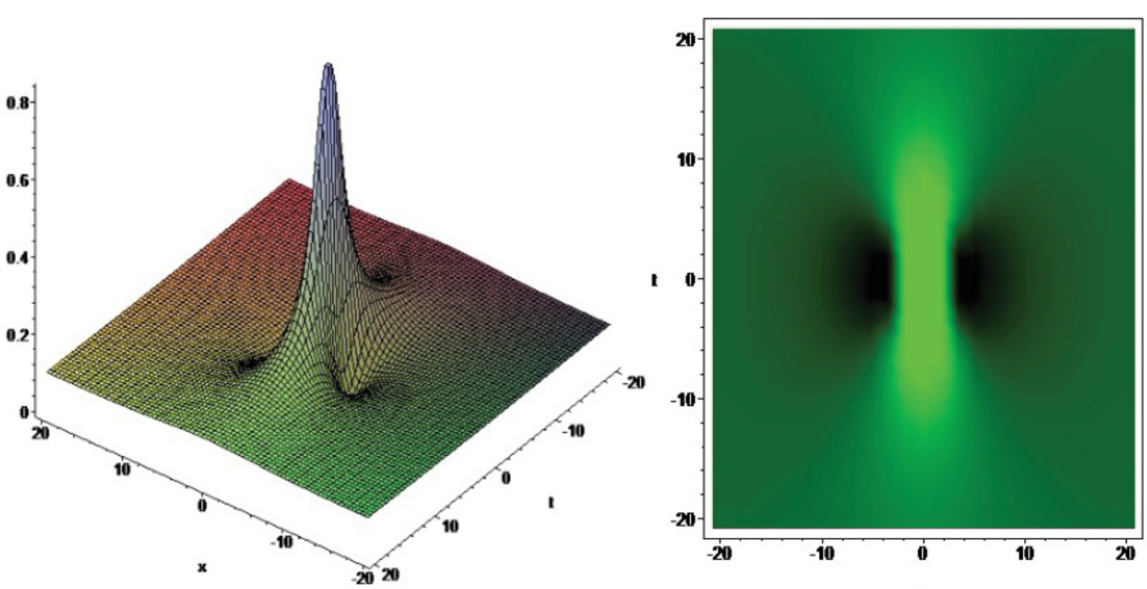

FIG. 5. (Color online) (Left panel) Rogue wave (30) of the NLSE (1) with $b = 0.2$ and (Right panel) its density plot.

ACKNOWLEDGMENTS

This work is supported by the NSF of China under Grant No. 10971109 and the K. C. Wong Magna Fund in Ningbo University. J.H. is also supported by the Program for NCET under Grant No. NCET-08-0515 and the Natural Science Foundation of Ningbo under Grant No. 2011A610179. We thank Professor Yishen Li (USTC, Hefei, China) for his useful suggestions on the rogue wave.

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