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Hydrodynamics of fractal continuum flow

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A model of fractal continuum flow employing local fractional differential operators is suggested. The generalizations of the Green-Gauss divergence and Reynolds transport theorems for a fractal continuum are suggested. The fundamental conservation laws and hydrodynamic equations for an anisotropic fractal continuum flow are derived. Some physical implications of the long-range correlations in the fractal continuum flow are briefly discussed. It is noteworthy to point out that the fractal (quasi)metric defined in this paper implies that the flow of an isotropic fractal continuum obeying the Mandelbrot rule of thumb for intersection is governed by conventional hydrodynamic equations.

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Fluid flow in porous media has tremendous importance in fields as diverse as soil science and hydrology, petroleum and construction industries, and drug release from porous reservoirs in pharmaceutics, among others [1]. In addition, hydrodynamics in porous media brings forward challenging questions from a fundamental point of view, in particular, related to the fractal geometry of pore space [2–6]. Accordingly, there have been many attempts to account for the effects of fractal properties of porous media on the fluid flow (see, for example, Refs. [2–11] and references therein). The great variety of methods and models employed in this task can be grouped into two basic categories associated either with a flow described by conventional hydrodynamic equations through a fractal pore network, or with the continuum flow governed by nonconventional (fractal) hydrodynamic equations. In this way, one can construct a prefractal model of a porous medium and then numerically solve the hydrodynamic equations using, for example, the finite element method. While this approach permits to obtain some relevant information regarding the flow through fractal porous media (see, for example, Refs. [2] and [7]), it cannot be used for practical purposes, e.g., in the hydrology or petroleum industry. Consequently, there have been many efforts to modify the classical hydrodynamic equations in order to account for the fractal geometry of actual flow fields (see, for example, Refs. [8–11]). These efforts involve either the replacement of the topological dimension in the hydrodynamic equations with the fractal one and the use of the space-time-dependent permeability (e.g., Ref. [8]), or the use of nonlocal fractional derivatives with respect to the space and/or time variables (e.g., Refs. [9] and [10]). However, the modifications of classic hydrodynamic equations inspired by the anomalous diffusion on fractals do not provide a consistent description of fractal flow and often violate the fundamental conservation laws (for review of fractional conservations laws, see Refs. [10] and [12]). On the other hand, Tarasov [11] has suggested mapping the flow in fractal pore space, which is essentially discontinuous in the embedding Euclidean space, into a continuous flow governed by conventional partial differential equations. However, the model of fractal continuum introduced in Ref. [11] leads to some contradictions (see Refs. [13] and [14]) associated with inconsistent definitions of the material time derivative and the Jacobian for the transformation between the current and initial configurations in the fractal continuum. Some other inconsistencies [15] are associated with the obscure definition of fractional differential operators [16].

In this Rapid Communication, we present the hydrodynamics of fractal flow based on a self-consistent model of fractal continuum employing the local fractional differential operators allied to the Hausdorff derivative introduced in Ref. [17].

It is worth noting that a fractal with metric (mass) dimension $D < 3$ cannot continuously fill the embedding Euclidean space E^3. Nonetheless, we can define the three-dimensional fractal continuum Φ_D^3 as a region of Euclidean space E^3 filled with continuous matter (leaving no pores or empty spaces) such that its properties, e.g., density $\rho(x_i)$ and flow velocity $u_j(x_i)$, are describable by continuous functions of the Euclidean coordinates $x_i \in E^3$ $(i = 1,2,3)$, whereas the mass of any cubic (or spherical) region $W \subset \Phi_D^3$ scales with the region size L as

$$m(L) = m_0(L/\ell_0 + 1)^D, \tag{1}$$

where ℓ_0 is the lower cutoff of scaling behavior, m_0 is a proportionally constant, and D is the mass fractal dimension related to some kind of box-counting quasimeasure (see Ref. [18]). To meet this definition, the mass of three-dimensional region $W \subset \Phi_D^3$ occupying the volume V_3 in E^3 should be defined as

$$m = \int_W \rho(x_i) dV_D = \int_W \rho(x_i) c_3(x_i, D) dV_3, \tag{2}$$

where $dV_D = c_3(x_i, D) dV_3$ is the infinitesimal volume element of Φ_D^3, while $dV_3(dx_i)$ is the infinitesimal volume element in E^3, such that the function providing transformation between the Euclidean and fractal (quasi)measures is defined as $c_3(x_i, D) = dV_D/dV_3$. Physically, the function $c_3(x_i, D)$ plays the role of the density of states in the fractal continuum, i.e., describes how permitted states of particles forming the fractal continuum are closely packed in the Euclidean space (see Ref. [19]).

The symmetry and functional form of the transformation function is defined by the symmetry of the fractal continuum. Using the Cartesian coordinates, the infinitesimal volume element in E^3 can be defined as $dV_3 = \prod_i^3 dx_i$. Accordingly, the infinitesimal volume element in the fractal continuum is commonly defined as $dV_D = \prod_i^3 dx_i^{\alpha_i}$ (see Refs. [13,14,19]). However, the presentation of fractal measure as the product of infinitesimal elements along the fractional Cartesian coordinates implies that the fractal continuum can be treated as the Cartesian product of three fractals with the fractal dimensions $0 < \alpha_i \leqslant 1$, such that $D = \sum_i^3 \alpha_i$ and the fractal dimensions of intersections with Cartesian planes are $d_k = \sum_{i \neq k}^2 \alpha_i$ [13]. Therefore, in the case of an isotropic fractal continuum ($\alpha_i = \alpha$ for all i) it is expected that $D = 3\alpha$ and $d_k = 2\alpha = (2/3)D$, whereas many isotropic fractals (e.g., percolation clusters and porous media) obey a so-called Mandelbrot's rule of thumb for intersections [20], according to which $d_k = D - 1$ (see also Ref. [21]). Moreover, more usually, D and d_k are independent characteristics of an anisotropic fractal [22]. So, more generally, the infinitesimal volume element of Φ_D^3 can be presented in the form

$$dV_D = d^\zeta x_k dA_d^{(k)}(x_{i\neq k}) = c_1^{(k)}(x_{k\neq i,j},\zeta_k)c_2^{(k)}(x_i,x_j,d_k)dx_k dA_2^{(k)}, \tag{3}$$

where $dA_d^{(k)} = c_2^{(k)}(x_{i\neq k},d_k)dA_2^{(k)}$ is the infinitesimal area element on the intersection of fractal continuum with the Cartesian plane $(x_i,x_j) \in E^2$ normal to axes k in Φ_D^3, while $dA_2^{(k)}$ is the infinitesimal area of this element in E^2, and the transformation function $c_2^{(k)}(x_{i\neq k},\ell_{i\neq k},d_k) = dA_d^{(k)}/dA_2^{(k)}$ represents the density of states on the intersection, and $d^\zeta x_k = c_1^{(k)}(x_k,\zeta_k)dx_k$ is the infinitesimal length element along the normal to the intersection and $c_1^{(k)}(x_k,\zeta_k)$ is the density of states along this normal.

Specifically, in the case of homogeneous fractal continuum $\rho(x_i) = \rho_c = \text{const}$ in E^3, and so from Eqs. (1)–(3) follows that the density of states in the fractal continuum can be represented in the following form:

$$c_3(x_i,D) = \kappa_k\,(x_k/\ell_k + 1)^{\zeta_k - 1}\,c_2^{(k)}(x_{i\neq k},\ell_{i\neq k},d_k), \tag{4}$$

where ℓ_i is the lower cutoff along the axis i, $\kappa_k = \ell_k^{\zeta_k - 1}$, while the scaling exponent ζ_k characterizing the density of states along the direction of the normal to the intersection is defined as

$$\zeta_k = D - d_k. \tag{5}$$

It should be pointed out that, generally, ζ_k is not equal to the fractal dimension α_k of the intersection between the modeled fractal and the normal to its intersection with a two-dimensional plane. For example, the mass dimension of the fractal continuum used to model the Menger sponge [see Figs. 1(a) and 1(b)] should be equal to the mass dimension of the sponge, i.e., $D = \ln 20/\ln 3$, while the intersection of the fractal continuum with a plane is characterized by the fractal dimension $d_k = \ln 8/\ln 3$ equal to the fractal dimension of the Menger sponge intersection with the two-dimensional plane [see Figs. 1(c) and 1(d)]. Hence, to assure scaling behavior (1), from Eq. (5) it follows that $\zeta_k = \ln(2.5)/\ln 3$, whereas the fractal dimension of the intersection of the Menger sponge with a line is equal to $\alpha_k = \ln 2/\ln 3 < \zeta_k$. Notice also that $dA_d^{(k)} = d^\zeta x_i d^\zeta x_j = dx_i^{d_k/2}dx_j^{d_k/2}$, where $d_k/2 = \ln 8/\ln 9 > D/3 > \zeta_k > \alpha_k$. Another pertinent example is the invasive stochastic fractals constructed by employing the random midpoint displacement algorithm linked to the fractional Brownian walk with the Hurst exponent $0 < H < 1$ [23]. In E^3 the mass dimension of these fractals is $D = 3 - H$ and the fractal dimensions of intersections are $d_k = D - 1 = 2 - H$ and $\alpha_k = D - 2 = 1 - H < 1$, respectively [22] and [23], whereas the scaling exponent (5) is independent on H and is equal to $\zeta_k = 1$ for any intersection of a two-dimensional surface with the continuum model of an isotropic invasive stochastic fractal characterized by the mass fractal dimension $2 < D = 3 - H < 3$.

Furthermore, in the case of isotropic fractal continuum Eq. (4) can be rewritten in the cylindrical coordinates as $c_3 = \kappa_{rz}(z/\ell_0 + 1)^{D-d-1}(r/\ell_d + 1)^{d-2}$, where $r = \sqrt{x^2 + y^2}$, $\kappa_{rz} = \ell_z^{D-d-1}\ell_r^{d-2}$, and the fractal dimensions D and d can be independent. In the spherical coordinates the densities of states in the fractal continuum and on its intersection with a sphere of radius $r = \sqrt{x^2 + y^2 + z^2}$ can be expressed as $c_3 = (r + \ell_0)^{D-3}$ and $c_2 = (r + \ell_0)^{d-2}$, respectively, $c_1 = (r + \ell_0)^{D-d-1}$. On the other hand, if the modeled fractal medium can be treated as the Cartesian product of three fractals with fractal dimensions $0 < \alpha_k \leqslant 1$, such that $d_k = \sum_{i\neq k}^2 \alpha_k$, Eq. (4) can be rewritten in the form $c_3 = \sum_k^3 \kappa_k(x_k/\ell_k + 1)^{\alpha_k - 1}$, similar to the one used in Refs. [13] and [14], and so the right-hand side of Eq. (2) represents the Riemann-Liouville fractional integral up to a numerical factor $8\pi^{D/2}/\Gamma(D/2)$, where $\Gamma(\cdots)$ denotes the gamma function. Here, it is pertinent to note that regardless of the homogeneity of Φ_D^3, the density distribution in E^3 possesses the long-range correlations characterized by the power-law scaling behavior of the density-density correlation function $\langle\rho(x_i)\rho(x_i + \lambda)\rangle = V_3^{-1}\int_W \rho(x_i)\rho(x_i + \lambda)c_3(x_i,D)dV_3 \propto |\lambda|^{D-3}$ for any $\lambda \gg \ell_0$.

From Eqs. (2)–(5) it follows that the fractal continuum flow through the intersection with a two-dimensional Euclidean plane obeys the following generalization of the Green-Gauss divergence theorem:

$$\begin{aligned}\int_A u_k n_k dA_d^{(k)} &= \int_A u_k c_2^{(k)}(x_{i\neq k},d_k)dA_2^{(k)} = \int_W c_2^{(k)}(x_{i\neq k},\ell_{i\neq k},d_k)\frac{\partial u_k}{\partial x_k}dV_3\\ &= \int_W c_3^{-1}(x_i/\ell_i,D)c_2^{(k)}(x_{i\neq k},\ell_{i\neq k},d_k)\frac{\partial u_k}{\partial x_k}dV_D\\ &= \int_W \nabla_k^H u_k dV_D,\end{aligned} \tag{6}$$

where $\vec{u} = u_k\vec{e}_k$ is a velocity field [24], $\vec{n} = n_k\vec{e}_k$ is a vector of normal (see Fig. 1), and the summation over repeated indexes is presumed [25], while the symbol ∇_i^H denotes the local partial fractional derivative

$$\nabla_k^H = \left(\frac{x_k}{\ell_k} + 1\right)^{1-\zeta_k}\frac{\partial}{\partial x_k}, \tag{7}$$

associated with the Hausdorff derivative defined in Ref. [17] as

$$\frac{d^H}{dx^\zeta}f = \lim_{x\to x'}\frac{f(x') - f(x)}{x'^\zeta - x^\zeta}, \tag{8}$$

where the exponents ζ_k are defined by Eq. (5). It is straightforward to verify that the Hausdorff derivative (8) is the inverse to the Riemann-Liouville fractional integral up to a constant ζ^{-1} [26]. In addition, $\nabla_i^H\text{const} = 0$,

$\nabla_i^H(\psi\varphi) = \psi\nabla_i^H\varphi + \varphi\nabla_i^H\psi$, while $\nabla_i^H\nabla_j^H\psi = \nabla_j^H\nabla_i^H\psi = \chi^{(i)}(x_i)\chi^{(j)}(x_j)\frac{\partial^2}{\partial x_i \partial x_j}\psi$, for $i \neq j$, where

$$\chi^{(i)} = (x_i/\ell_i + 1)^{1-D+d_i}, \tag{9}$$

and so the fractional (Hausdorff) Laplacian Δ_H has the following form:

$$\nabla_i^H\nabla_i^H\psi = \left(\chi^{(i)}\right)^2\left[\left(\frac{\partial^2\psi}{\partial x_i^2}\right) + \frac{1-D+d_k}{x_i+\ell_i}\left(\frac{\partial\psi}{\partial x_i}\right)\right], \tag{10}$$

which differs from the local fractional Laplacian introduced in Ref. [27] and later used in Ref. [28].

Furthermore, we can construct the local fractional (Hausdorff) operators for the vector calculus on the fractal continuum. Specifically, the Hausdorff divergence of vector field $\vec{\Psi} = (\psi_1,\psi_2,\psi_3)$ is defined as

$$\mathrm{div}_H\vec{\Psi} = \sum_i^3 \nabla_i^H\Psi_i, \tag{11}$$

while the Hausdorff gradient vector of scalar function $\psi(x_i)$ is $\mathrm{grad}_H\psi = \vec{\nabla}^H\psi = (\nabla_1^H\psi)\vec{e}_1 + (\nabla_2^H\psi)\vec{e}_2 + (\nabla_3^H\psi)\vec{e}_3$. This leads to the Hausdorff curl operator of a vector field in the following form

$$\mathrm{rot}_H\vec{\Psi} = \nabla^H \times \vec{\Psi}, \text{or } \nabla_i^H\Psi_j = \varepsilon_{kij}\nabla_i^H\Psi_j, \tag{12}$$

where ε_{ijk} is the Levi-Civita symbol. It is straightforward to verify that the Hausdorff operators (10)–(12) obey the identities which resemble the fundamental identities of the conventional vector calculus, i.e., $\mathrm{div}_H\mathrm{rot}_H\vec{\Psi} = 0$, $\mathrm{rot}_H\mathrm{grad}_H\psi = 0$, $\mathrm{div}_H\mathrm{grad}_H\psi = \Delta_H\psi$. Notice also that the fractal (quasi)metric defined by Eqs. (1)–(5) implies the conversion of fractional differential operators defined by Eqs. (8)–(12) into the conventional ones in the case of the isotropic fractal continuum obeying the Mandelbrot rule of thumb for intersection.

It should be emphasized that the geometric framework in which the hydrodynamics of the fractal continuum is developed is the three-dimensional Euclidean space. Hence, the fractal metric defined by Eqs. (1)–(5) implies that the fractal Jacobian matrix of the transformation between the initial $x_i \in \Phi_D^3$ and current coordinates $X_i(t) = X_i \in \Phi_D^3$ has the form $J_D^3 = [\nabla_i^H X_j]$ and so the determinant of the fractal Jacobian is defined as

$$J_D = \varepsilon_{ijk}\nabla_1^H X_i \nabla_2^H X_j \nabla_3^H X_k, \tag{13}$$

where ε_{ijk} is the Levi-Civita symbol. Consequently, the fractal material time derivative should be defined as

$$\left(\frac{d}{dt}\right)_D \psi = \frac{\partial}{\partial t}\psi + u_k\nabla_k^H\psi = \frac{\partial}{\partial t}\psi + u_k\left(\frac{x_k}{\ell_k}+1\right)^{1-\zeta_k}\frac{\partial}{\partial x_k}\psi, \tag{14}$$

such that $(\frac{d}{dt})_D J_D = J_D\nabla_i^H u_i$, and so the generalization of the Reynolds transport theorem for a fractal continuum reads as follows:

$$\left(\frac{d}{dt}\right)_D\int_{W_t}\psi dV_D = \left(\frac{d}{dt}\right)_D\int_{W_0}\psi J_D dV_D^0 = \int_{W_0}\left[\left(\frac{d}{dt}\right)_D\psi J_D + \psi\left(\frac{d}{dt}\right)_D J_D\right]dV_D^0 = \int_{W_0}\left[\left(\frac{d}{dt}\right)_D\psi + \psi\nabla_k^H u_k\right]J_D dV_D^0$$
$$= \int_{W_t}\left[\left(\frac{d}{dt}\right)_D\psi + \psi\nabla_k^H u_k\right]dV_D = \int_{W_t}\left(\frac{\partial}{\partial t}\psi + \nabla_k^H(\psi u_k)\right)dV_D = \int_{W_t}\frac{\partial}{\partial t}\psi dV_D + \int_A \psi u_k n_k dA_d^{(k)}, \tag{15}$$

where $\psi(x_i,t)$ is any quantity accompanied by a moving material system W_t [29].

Furthermore, using Eqs. (3)–(15), it is straightforward to derive the fundamental conservation laws for fractal continuum. Specifically, the equation of continuity can be written in the form

$$\frac{\partial\rho_c}{\partial t} = -\mathrm{div}_H(\rho_c\vec{u}), \tag{16}$$

where the Hausdorff divergence is defined by Eq. (11). Taking into account the Green-Gauss divergence theorem for fractal continuum (6), from Eq. (16) it follows that the velocity field in a stationary flow of an incompressible fractal continuum is solenoidal in the sense that for any closed surface ∂W the net total flux through the surface is equal to zero, in contrast to the opposite statement in Ref. [11]. It is pertinent to note that the solenoidal velocity field in the fractal continuum can be expressed as the Hausdorff curl (12) of a vector potential $\vec{\Phi}$, i.e., $\vec{u} = \mathrm{rot}_H\vec{\Phi}$, such that $\mathrm{div}_H\vec{u} = 0$, whereas $\sum_i^3 \frac{\partial u_i}{\partial x_i} \neq 0$.

The equation of balance of energy density $e(x_i,t)$ in the fractal continuum flow has the form

$$\rho_c\left(\frac{d}{dt}\right)_D e = \rho_c\frac{\partial e}{\partial t} + u_i\rho_c\nabla_i^H e = \sigma_{ij}\nabla_j^H u_i + \rho_c\nabla_i^H q_i, \tag{17}$$

where σ_{ij} is the stress tensor and $\vec{q} = q_i n_i$ is the density of heat flux. Finally, the balance of density of momentum in the fractal continuum is governed by the following equation:

$$\left(\frac{d}{dt}\right)_D u_k = \frac{\partial}{\partial t}u_k + u_i\nabla_i^H u_k = f_k + \rho_c^{-1}\nabla_i^H\sigma_{ki}, \tag{18}$$

where f_k is the density of volume forces, e.g., the gravitational constant $g = f_z$. It is imperative to point out that the forms of conservation equations (16)–(18) are determined by the fractal (quasi)metric defined by Eqs. (1)–(5). Hence Eqs. (16)–(18) can be used to model any type of mechanical behavior of fractal media within the fractal continuum framework, e.g., elastic or plastic deformations and flow of Newtonian or non-Newtonian fluids.

From Eq. (18) it immediately follows that the generalization of the Bernoulli integral for a steady incompressible flow of inviscid fractal continuum in the gravitational field can be represented as

$$\sum_k^3 \frac{u_k^2}{2} + \frac{p}{\rho_c} + g(D-d_z)\ell_z\left(\frac{z}{\ell_z}+1\right)^{D-d_z} = h = \text{const}, \tag{19}$$

where the notations $z = x_3$, $\ell_z = \ell_3$, and $d_z = d_3$ are used, while h is the total hydraulic head. Notice that if $0 < D - d_z \leqslant 1$, the gravitational head [third term on the left-hand side of Eq. (19)] increases with the fluid elevation more slowly than in the Euclidean case. Accordingly, in the hydrostatic equilibrium, the pressure distribution in a homogeneous fractal continuum has the form

$$p(z) = p_0 - g(D-d_z)\rho_c\ell_z\left(\frac{z}{\ell_z}+1\right)^{D-d_z}, \tag{20}$$

where $p_0 = p(z=0)$ is the pressure on the free surface normal to the gravitational field. Notice the difference between Eqs. (19) and (20) and the corresponding equations derived in Ref. [11]. It is pertinent to point out that Eq. (20) can be relatively easily verified in laboratory experiments with prefractal reservoirs, e.g., the Menger sponge, because if $D - d_z < 1$, Eq. (20) predicts that the rate of fluid discharge from a fractal reservoir is considerable less than the discharge rate expected according to the Torricelli equation (see Ref. [30]).

To develop the hydrodynamics of fractal continuum flow, we need to define the constitutive relations between the displacements, flow velocities, and external forces. Experimental observations suggest that many fluids, such as water and oils, obey the linear relation between the shear stresses and the pure shear strain rates. Fluids possessing such behavior are called Newtonian fluids [30]. In the case of fractal continuum the strain tensor is defined by the fractal Jacobian J_D^3. Hence, the tensor of stresses in an incompressible Newtonian fractal continuum flow ($\nabla_i^H u_i = 0$) should be written in the form

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\nabla_i^H u_j + \nabla_j^H u_i\right), \tag{21}$$

where p is the fluid pressure, μ is the dynamic viscosity, and δ_{ij} is Kronecker's delta. Notice that Eq. (21) expresses the linear relation between the shear stresses and the pure shear strain rates in the fractal continuum, whereas the conventional form of the constitutive equation used the hydrodynamic of the Euclidean flow in the case of fractal continuum flow describes a non-Newtonian fluid behavior.

Substituting Eq. (21) into Eq. (18) we obtain the generalization of Navier-Stokes equations for incompressible fractal continuum flow represented by the following set of no linear partial differential equations:

$$\frac{\partial}{\partial t}u_k + \chi^{(i)}u_i\frac{\partial u_k}{\partial x_i} = f_k - \rho_c^{-1}\chi^{(k)}\frac{\partial p}{\partial x_k} + \nu\chi^{(i)}\frac{\partial}{\partial x_i}\left(\chi^{(i)}\frac{\partial u_k}{\partial x_i} + \chi^{(k)}\frac{\partial u_i}{\partial x_k}\right), \tag{22}$$

where $\nu = \mu/\rho_c$ is the kinematic viscosity and functions $\chi^{(i)}(x_i)$ are defined by Eq. (9). Notice that Eqs. (22) are converted into the classical Navier-Stokes equations when $d_k = d = D - 1$, whereas if $d_k > D - 1$, the long-range density-density correlations in E^3 introduce the long-range correlations in the fractal continuum flow.

Equations (16)–(22) describe the hydrodynamic of fractal continuum flow. Here, it is noteworthy to emphasize that

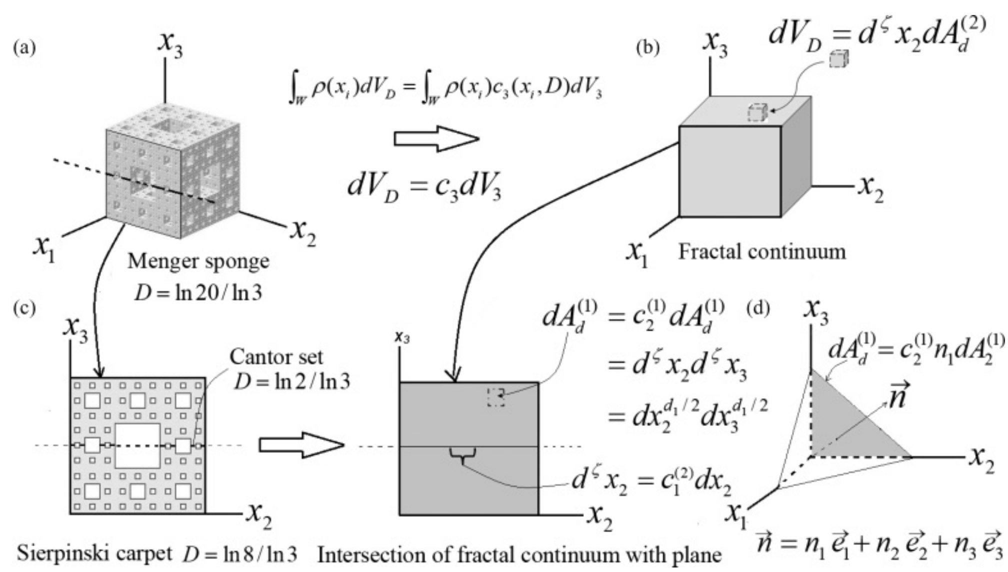

FIG. 1. Illustration of the mapping of discontinuous prefractal Menger sponge (a) into the fractal continuum with the mass fractal dimension $D = \ln 20/\ln 3$ (b). Notice that the intersection of the Menger sponge with a plane is the Sierpinski carpet of the fractal dimension $d = \ln 8/\ln 3$ (c), while the intersection Menger sponge with a line is the Cantor set with the fractal dimension $\alpha = \ln 2/\ln 3$ [dashed line in (c)], whereas the density of states along the normal to the intersection of the fractal continuum with the plane [see (d)] is characterized by the scaling exponent $\zeta = \ln(2.5)/\ln 3 > \alpha$ [see Eqs. (3)–(5)].

the flow of an isotropic fractal continuum with $d = D - 1$ is governed by conventional hydrodynamic equations for the three-dimensional flow, while the mapping of fluid flow in a fractal pore space to the fractal continuum flow implies that the density of fractal continuum is equal to $\rho_c = \phi\rho_f$, where ρ_f is the fluid density and ϕ is the medium porosity (see Fig. 1). Furthermore, it is easy to understand that the mass fractal dimension of the fractal continuum flow should be equal to the fractal dimension of the backbone of the pore space D_{bb}, rather than to the mass fractal dimension of pores $D_p \geqslant D_{bb}$ (see Ref. [31]), and so the intersections of the fractal continuum with the Cartesian planes are characterized by the fractal dimensions of the backbone intersections with two-dimensional Cartesian planes.

The hydrodynamics of fractal continuum flow developed in this work permits to improve the fractal approach to model the pumping well pressure response that is widely used in petroleum engineering (see Refs. [8–11,32,33] and references therein). This requires the definition of an analog of the Darcy law for fractal continuum flow, instead of the commonly used Darcy law with the spatially or/and time-dependent permeability (see Refs. [32] and [33]). Furthermore, the fractal continuum approach employing the local fractional differential operators can be used to model the fractal systems of diverse nature within a continuum framework, e.g., mechanics of elastic or viscoelastic fractal continuum and electrodynamics of fractal continuum.

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[15] E.g., the failure of the Reynolds transport theorem for fractal continuum flow (see Ref. [13]), the conclusion that the velocity field of stationary incompressible fractal continuum flow is nonsolenoidal, the statement that the Bernoulli integral does not exists for the fractal continuum flow in the gravitational (potential) field, among others misstatements.

[16] In fact, the author of Ref. [11] has postulated the expressions of fractional differential operators in terms of conventional partial derivatives without any definition of the fractional derivative. Moreover, the local character of the fractional operators was not outlined explicitly. This has led to some confusion in subsequent works devoted to the fractal continuum. For example, in Refs. [13] and [14] a somewhat modified form of the fractional partial derivative introduced in Ref. [11] is explicitly referred to as a nonlocal fractional differential operator related to the right Jumarie fractional derivative. Consequently, the authors

of Refs. [13] and [14] have introduced the upper limit of the Jumarie integral in the relation for density of states. However, it is worth noting that the operators related to the nonlocal Jumarie fractional derivative cannot be expressed in terms of the conventional partial derivatives as used in Refs. [13] and [14].

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[24] It is worth noting that any quantity accompanied by flow should satisfy the generalized Green-Gauss divergence theorem (6).

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[26] Accordingly, the numerical factor $\zeta^{-1}\ell_i$ is not used in definition (7).

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[29] Notice the difference between Eqs. (14) and (15) and the forms of the material time derivative and the Reynolds transport theorem adopted in the fractal continuum model of Ref. [13].

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