Percolation in a kinetic opinion exchange model

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We study the percolation transition of the geometrical clusters in the square-lattice LCCC model [a kinetic opinion exchange model introduced by Lallouache, Chakrabarti, Chakraborti, and Chakrabarti, Phys. Rev. E **82**, 056112 (2010)] with the change in conviction and influencing parameter. The cluster is comprised of the adjacent sites having an opinion value greater than or equal to a prefixed threshold value of opinion (Ω). The transition point is different from that obtained for the transition of the order parameter (average opinion value) found by Lallouache *et al.* Although the transition point varies with the change in the threshold value of the opinion, the critical exponents for the percolation transition obtained from the data collapses of the maximum cluster size, the cluster size distribution, and the Binder cumulant remain the same. The exponents are also independent of the values of conviction and influencing parameters, indicating the robustness of this transition. The exponents do not match any other known percolation exponents (e.g., the static Ising, dynamic Ising, and standard percolation). This means that the LCCC model belongs to a separate universality class.

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I. INTRODUCTION

The geometrical percolation transition has been a long-studied subject [1,2]. It is characterized by a set of universal critical exponents, which describe the fractal properties of the percolating medium at large scales and sufficiently close to the transition. The exponents only depend on the type of percolation model and on the spatial dimension. The occupancy of the sites or bonds of a percolating system is controlled by a parameter, and at a critical value of that parameter the cluster sizes (defined by the number of adjacent sites possessing a predefined common feature) go to infinity, which we call the percolation transition. This phenomenon has been studied extensively for the thermal excitation of the two-dimensional (2D) Ising model, and in this case the system undergoes a percolation transition [3–6] at the same critical temperature as the magnetization [7,8]. In the case of the Ising model, the geometrical cluster is defined by the adjacent sites consisting of parallel spins. The transition point differs in the case of higher dimensions [3,9]. The percolation exponents of the geometrical clusters are identical for the models belonging to the same universality class [Ising and $Z(3)$ symmetric models [10,11]]. Recently, the dynamical percolation transition was studied for the 2D Ising model by applying a pulsed magnetic field [12]. The critical exponents were different from that of the static percolation transition associated with the thermal transition of the Ising model, indicating a different universality class. The distinct crossing point of the Binder cumulant of the order parameter for different system sizes at the transition point also characterizes the dynamical percolation transition as being in a different universality class.

The study of social dynamics has been very popular recently. To capture the basic idea of consensus formation, the concepts of statistical physics have been largely applied [13,14]. A large number of models have been studied (e.g., the voter model [15,16], the Sznajd model [17], etc.) so far. In some models, opinions have been considered as a continuous variable [18–23]. The spreading of an opinion through society may be compared with the percolation problem in physics. This was studied for a nonconsensus opinion model earlier [24], and it was found to belong to the same universality class as invasion percolation. In this paper, we have studied the percolation transition of geometrical clusters in a recently proposed opinion model called the Lallouache-Chakrabarti-Chakraborti-Chakrabarti (LCCC) model [22,25], in which individuals exchange opinions controlled by an influencing parameter and a conviction parameter, the values of which are equal. The opinion of an individual is taken uniformly between -1 and $+1$, which changes by binary interactions, where an individual stays with his own opinion up to a certain fraction λ and takes a random fraction of a part of another agent's opinion determined by the same parameter (a detailed discussion is given in the next section). By Monte Carlo simulation, it was found that below a critical value ($\lambda_c \approx 2/3$) of the conviction parameter, the average opinion value remains zero, whereas above the critical value the average opinion value becomes nonzero. Some critical exponents characterizing the transition in the LCCC model and some variants of the LCCC model were studied numerically [26]. A generalized version of this model was introduced by Sen [27], in which the influencing parameter and the conviction parameter were different. A discrete version of the LCCC model has also been studied [28].

In this paper, we have investigated the percolation transition of the geometrical clusters of the LCCC model assuming individuals are located on the sites of a square lattice. We have defined *clusters* as a group of adjacent sites with an opinion value equal to or above a preassigned threshold value (Ω). The cluster sizes are controlled by the influencing parameter λ, and for a fixed Ω, at a critical value of the influencing parameter λ_c^p, the percolation transition occurs. We determine the critical exponents by finite-size scaling analysis of the maximum cluster size. The value of the critical point decreases with a decrease of Ω and coincides with that for the transition point of $\lambda_c = 2/3$ (at which the average opinion diverges) as $\Omega \to 0.0$. But the critical exponents remain unaltered with a change in Ω and also differ with the exponents known for the

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previously known models, indicating that the square-lattice LCCC belongs to a separate university class. We have also investigated this percolation transition in the case of the generalized LCCC model [27], where the conviction parameter (λ) is different from the influencing parameter (μ), and once again we found that although the critical point shifts depending on the values of Ω, λ, and μ, the critical exponents remain the same.

The paper has been organized in the following manner: In Sec. II, we give a brief description of both the LCCC model and the generalized LCCC model. In Sec. III, we present the description of clusters and measure the critical exponents for the square-lattice LCCC model. In Sec. IV, we measure the exponents for the generalized LCCC model, and finally in Sec. V we discuss this transition and we present some conclusions.

II. A BRIEF DESCRIPTION OF THE LCCC AND GENERALIZED LCCC MODEL

The origin of this model is a multiagent statistical model of closed economy [29] in which N agents exchange a fixed wealth through pairwise interaction controlled by a "saving" parameter. Lallouache *et al.* [22,25] proposed a similar multiagent model to describe the dynamics of opinion formation. The basic difference in this model is that there is no constraint regarding the conservation of opinion. There are N agents and each agent i begins with an individual opinion $o_i \in [-1,+1]$. They exchange opinions between each other by binary interactions as follows:

$$\begin{aligned} o_i(t+1) &= \lambda[o_i(t) + \epsilon o_j(t)], \\ o_j(t+1) &= \lambda[o_j(t) + \epsilon' o_i(t)], \end{aligned} \tag{1}$$

where ϵ, ϵ' are drawn randomly from uniform distributions in $[0,1]$. In this model, the opinions are bounded, i.e., $-1 \leqslant o_i \leqslant +1$ for all i. Here the parameter λ is interpreted as "conviction," i.e., the power to retain one's own opinion. The second term signifies the extent to which somebody gets influenced by another. Here both the conviction parameter and the influencing parameters are the same, and moreover they are identical for every individual. The opinion exchange for the generalized LCCC model was as follows [27]:

$$\begin{aligned} o_i(t+1) &= \lambda o_i(t) + \epsilon \mu o_j(t), \\ o_j(t+1) &= \lambda o_j(t) + \epsilon' \mu o_i(t), \end{aligned} \tag{2}$$

where λ is the conviction parameter and μ is the influencing parameter with $o_i \in [-1,+1]$. The special case of $\lambda = \mu$ is the LCCC model. The order parameter is the average opinion $O = |\sum_i o_i|/N$. Numerical simulations show that the system stabilizes into two possible phases: for any $\lambda \leqslant \lambda_c$, $o_i = 0$ $\forall i$, while for $\lambda > \lambda_c$, $O > 0$ and $O \to 1$ as $\lambda \to 1$. In the LCCC model, $\lambda_c \approx 2/3$ is the critical point. In the case of the generalized model, λ_c depends on the value of μ and the mean-field phase boundary is given by $\lambda = 1 - \mu/2$. If we study these models on a square lattice, then also the critical points do not change. Some critical exponents characterizing the transition in the LCCC model and some variants of the LCCC model were also studied numerically [26].

III. PERCOLATION ON A SQUARE-LATTICE LCCC MODEL

In this section, we will discuss the percolation behavior of the geometrical clusters formed on a square-lattice LCCC model. Here we assume that the agents are placed on the sites of a square lattice and follow the LCCC dynamics. We define a geometrical cluster as consisting of the adjacent sites having an opinion value greater than or equal to a prefixed threshold opinion value (Ω). In our numerical simulation, we have used the random sequential updating rule. For each value of λ and Ω, when the system reaches a steady state, we measure the percolation order parameter $P_{\max} = S_L/L^2$ (where S_L is the size of the largest cluster and L is the linear size of the system). The value of $P_{\max}$ increases with λ, and at some λ_c^p the system undergoes a percolation transition (Fig. 1). The value of λ_c^p decreases with a decrease in Ω, approaching the value λ_c as $\Omega \to 0.0$ (Figs. 2 and 3). Moreover, it is also evident from Fig. 2 that the finite-size effect diminishes with a decrease in Ω.

The percolation transition is characterized by power-law variation of different quantities. The order parameter, i.e., the relative size ($P_{\max}$) of the largest cluster, varies as

$$P_{\max} \sim \left(\lambda_c^p - \lambda\right)^{\beta} \tag{3}$$

and the correlation length diverges near the percolation transition point as

$$\xi \sim \left(\lambda_c^p - \lambda\right)^{-\nu}, \tag{4}$$

where λ_c^p is the critical conviction parameter. The values of the critical exponents β and ν specify the universality class of the transition.

However, the exponents are not determined from these definitions due to finite-size effects. The critical exponents are determined from the finite-size scaling relations [12,30].

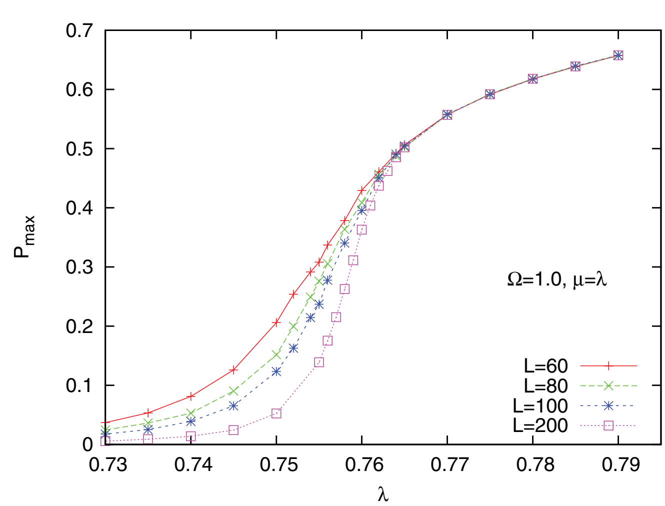

FIG. 1. (Color online) Maximum cluster size as a function of the conviction parameter for four different system sizes ($L = 60, 80, 100$, and 200) for the LCCC model, i.e., $\mu = \lambda$ and the threshold opinion value $\Omega = 1.0$.

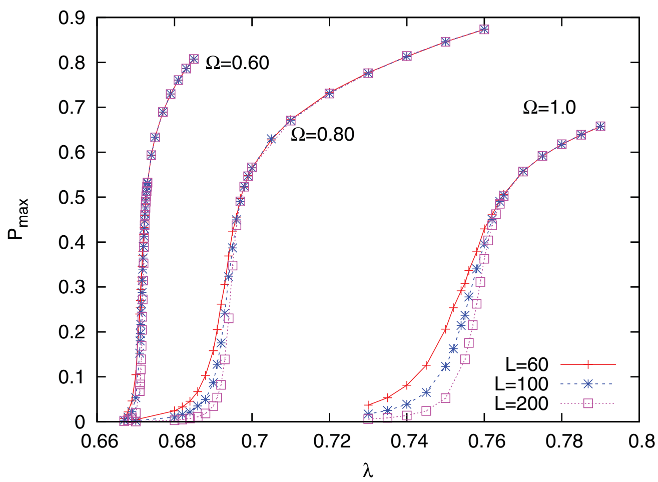

FIG. 2. (Color online) Comparative plots for the largest cluster size with conviction parameter for three different system sizes and at three various values of the opinion threshold ($\Omega = 1.0$, 0.80, and 0.60).

For example, the order parameter is expected to follow the scaling form

$$P_{\max} = L^{-\beta/\nu} \mathcal{F}\left[L^{1/\nu}\left(\lambda_c^p - \lambda\right)\right], \tag{5}$$

where $\mathcal{F}$ is a suitable scaling function. If we plot $P_{\max}L^{\beta/\nu}$ against λ for different system sizes but fixed Ω, then by tuning β/ν, all the curves can be made to cross at a single point. The value of λ for which this happens is the critical conviction parameter (λ_c^p). To estimate ν, $P_{\max}L^{\beta/\nu}$ should be plotted against $(\lambda_c^p - \lambda)L^{1/\nu}$, and by tuning $1/\nu$, the curves are made to collapse, giving an accurate estimate of the exponent ν. The other exponents can be obtained from scaling relations [1].

For $\Omega = 1.0$, we plot $P_{\max}L^{\beta/\nu}$ against λ (Fig. 4). The curves for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at a point when $\beta/\nu = 0.130 \pm 0.005$, and the crossing point ($\lambda_c^p = 0.760 \pm 0.001$) gives the critical conviction parameter. Now to determine ν, we plot $P_{\max}L^{\beta/\nu}$

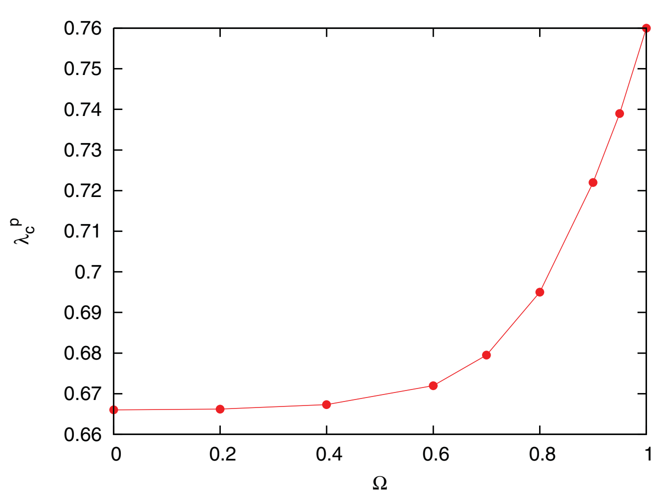

FIG. 3. (Color online) Plot of the critical conviction parameter (λ_c^p) with the threshold opinion value (Ω).

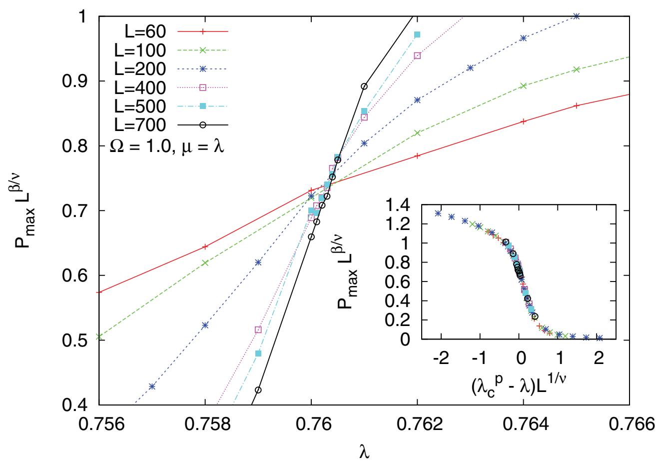

FIG. 4. (Color online) $P_{\max}L^{\beta/\nu}$ plotted against the conviction parameter λ for $\Omega = 1.0$ and $\mu = \lambda$. The curves for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at $\lambda_c^p = 0.760 \pm 0.001$ for $\beta/\nu = 0.130 \pm 0.005$. In the inset, the data collapse for $P_{\max}$ with $(\lambda_c^p - \lambda)$ has been shown for $\Omega = 1.0$ giving $1/\nu = 0.80 \pm 0.01$ and $\beta/\nu = 0.130 \pm 0.005$.

against $(\lambda_c^p - \lambda)L^{1/\nu}$, and by tuning the value of $1/\nu$ all three plots are made to collapse on a single curve (inset of Fig. 4), yielding an estimate of $1/\nu = 0.80 \pm 0.01$. Although with a decrease of Ω the critical point for percolation approaches λ_c, the exponents remain the same. We have shown the same plots for $\Omega = 0.80$ in Fig. 5. The corresponding critical point is $\lambda_c^p = 0.6955 \pm 0.0005$, but $\beta/\nu = 0.130 \pm 0.005$ and $1/\nu = 0.80 \pm 0.01$. The values of β/ν and $1/\nu$ are different from that obtained for the percolation transition in the case of static Ising ($\beta_s/\nu_s = 0.052 \pm 0.002$, $1/\nu_s = 0.996 \pm 0.009$) [10], dynamic Ising ($\beta_d/\nu_d = 0.20 \pm 0.05$, $1/\nu_d = 0.85 \pm 0.05$) [12], and standard percolation ($\beta/\nu = 5/48, 1/\nu = 3/4$) [1] for the two-dimensional system.

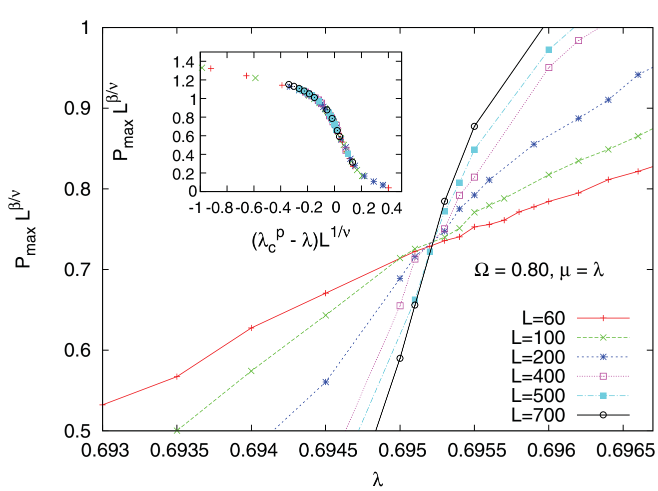

FIG. 5. (Color online) $P_{\max}L^{\beta/\nu}$ plotted against the conviction parameter λ, where $\Omega = 0.80$ and $\mu = \lambda$. The curves for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at $\lambda_c^p = 0.6955 \pm 0.0005$ for $\beta/\nu = 0.130 \pm 0.005$. In the inset, the data collapse for $P_{\max}$ with $\lambda_c^p - \lambda$ has been shown for $\Omega = 0.80$ giving $1/\nu = 0.80 \pm 0.01$ and $\beta/\nu = 0.130 \pm 0.005$.

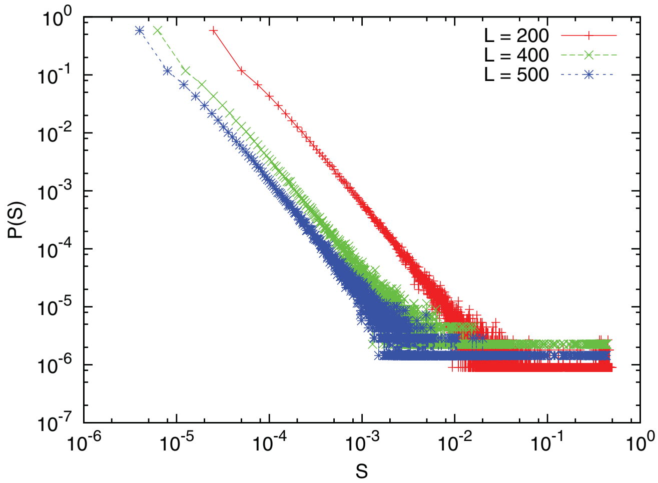

FIG. 6. (Color online) The cluster size distribution for the LCCC model for three different system sizes at $\Omega = 1.0$ and corresponding critical conviction parameter $\lambda_c^p = 0.760$. All the curves decay algebraically with an exponent 1.82 ± 0.01.

We have also studied the cluster size distribution for a fixed value of Ω and for three different system sizes ($L = 200$, 400, and 500). For $\Omega = 1.0$, at the critical point (i.e., $\lambda = 0.760$), all the curves decay algebraically as $P(S) \sim S^{-\tau}$ (where S denotes the sizes of the cluster) with $\tau = 1.82 \pm 0.01$ (Fig. 6). The value of τ remains the same for other values of Ω (at corresponding critical values of λ).

For further verification of the critical point and the universality class, we have studied the reduced fourth-order Binder cumulant of the order parameter, defined as [31]

$$U = 1 - \frac{\langle P_{\max}^4 \rangle}{3\langle P_{\max}^2 \rangle^2}, \tag{6}$$

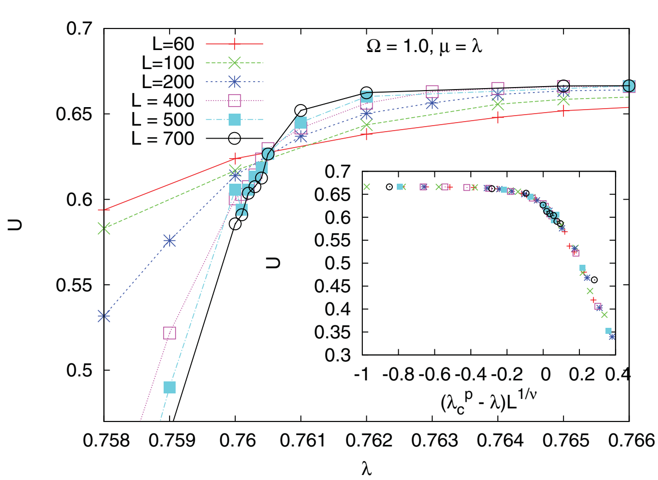

FIG. 7. (Color online) Fourth-order reduced Binder cumulant of percolation order parameter ($P_{\max}$) for six different system sizes ($L = 60$, 100, 200, 400, 500, and 700) at $\Omega = 1.0$ and $\mu = \lambda$; the crossing point determines the critical point ($\lambda_c^p = 0.760 \pm 0.001$). The critical Binder cumulant value is $U^\star = 0.62 \pm 0.01$. Inset shows the data collapse for the same value of $1/\nu$ as obtained for the data collapse of $P_{\max}$.

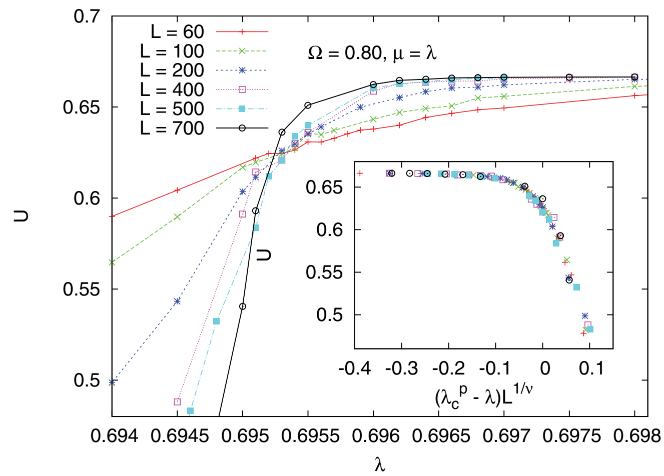

FIG. 8. (Color online) Fourth-order reduced Binder cumulant of percolation order parameter ($P_{\max}$) for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) at $\Omega = 0.80$ and $\mu = \lambda$; the crossing point determines the critical point ($\lambda_c^p = 0.6955 \pm 0.0005$). The critical Binder cumulant value is $U^\star = 0.62 \pm 0.01$. Inset shows the data collapse for the same value of $1/\nu$ as obtained for the data collapse of $P_{\max}$.

where $P_{\max}$ is the percolation order parameter (as defined before) and the angular brackets denote ensemble average. $U \to \frac{2}{3}$ deep inside the ordered phase and $U \to 0$ in the disordered phase when the fluctuation is Gaussian. The crossing point of the different curves ($U - \lambda$) for different system sizes gives the critical point ($\lambda_c^p = 0.760 \pm 0.001$) for $\Omega = 1.0$, which is in good agreement with the previous estimation from finite-size scaling of the corresponding Ω (Fig. 7). The value of U at the critical point for any value of Ω is $U^\star = 0.624 \pm 0.002$ (see Fig. 7). The Binder cumulant also follows the scaling form

$$U = \mathcal{U}\left[(\lambda_c^p - \lambda)L^{1/\nu}\right], \tag{7}$$

where $\mathcal{U}$ is a suitable scaling function. The data collapse for $\Omega = 1.0$ has been shown in the inset of Fig. 7, and the value of $1/\nu$ is 0.80 ± 0.01, which is in good agreement with the value of $1/\nu$ obtained from the finite-size scaling of the largest cluster size. The same plot has been shown in Fig. 8 for $\Omega = 0.80$, which also gives the same value of $1/\nu$, which indicates that the critical exponents are independent of Ω.

IV. PERCOLATION IN THE GENERALIZED LCCC MODEL

We have also investigated the percolation transition in the case of the generalized LCCC model in which the conviction parameter (λ) and the influencing parameter (μ) are different. We have studied the percolation transition for two sets of parameters: $\Omega = 1.0, \mu = 0.50$ and $\Omega = 1.0, \mu = 1.0$. In both cases, the plots of $P_{\max} L^{\beta/\nu}$ with λ for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at a single point for $\beta/\nu = 0.130 \pm 0.005$ (Figs. 9 and 10), which is the same as that obtained for the LCCC model. The critical points are different ($\lambda_c^p = 0.842 \pm 0.001$ for $\mu = 0.50$ and $\lambda_c^p = 0.687 \pm 0.001$ for $\mu = 1.0$). The value of $1/\nu$ is obtained from the finite-size scaling of the largest cluster size (inset of Figs. 9

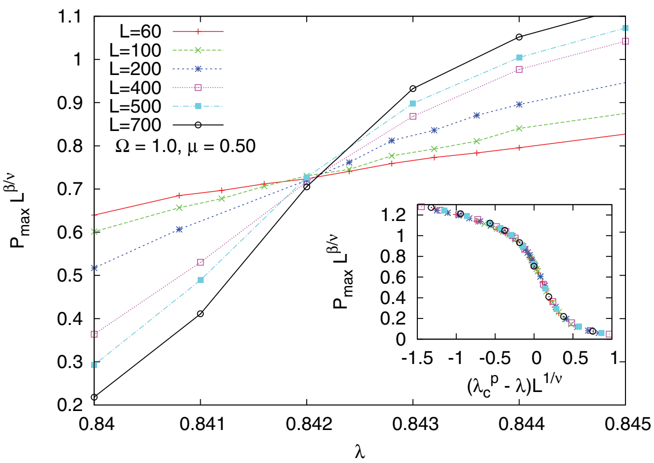

FIG. 9. (Color online) $P_{\max} L^{\beta/\nu}$ plotted against the conviction parameter λ, where $\Omega = 1.0$ and $\mu = 0.50$. The curves for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at $\lambda_c^p = 0.842 \pm 0.001$ for $\beta/\nu = 0.130 \pm 0.005$. In the inset, we have shown the data collapse for $P_{\max}$ with $\lambda_c^p - \lambda$ for $\Omega = 1.0$ and $\mu = 0.50$ giving $1/\nu = 0.80 \pm 0.01$ and $\beta/\nu = 0.130 \pm 0.005$.

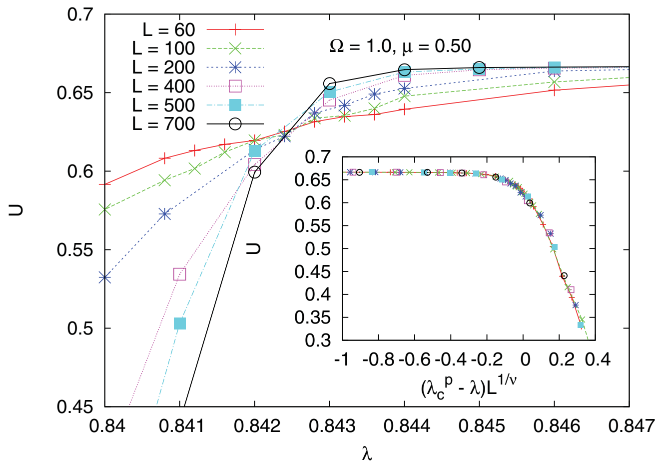

FIG. 11. (Color online) Fourth-order reduced Binder cumulant of percolation order parameter ($P_{\max}$) for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) at $\Omega = 1.0$ and $\mu = 0.50$; the crossing point determines the critical point ($\lambda_c^p = 0.842 \pm 0.001$). The critical Binder cumulant value is $U^\star = 0.624 \pm 0.002$. Inset shows the data collapse for the same value of $1/\nu$ as obtained for the data collapse of $P_{\max}$.

and 10), and the estimated values of $\beta/\nu = 0.130 \pm 0.005$ and $1/\nu = 0.80 \pm 0.01$ are the same as that obtained for the LCCC model and are independent of the value of μ, which is in contrast with the results obtained for the transition of the average opinion value, where the critical exponents change with μ. This implies that the percolation transition is much more robust than the average opinion transition. The plots for the Binder cumulant also satisfy the crossing point and the critical exponents as obtained previously (Figs. 11 and 12). The cluster size distribution also decays algebraically with an exponent 1.82 ± 0.01 for $\mu = 0.50$, which is the same as that obtained for the LCCC model.

V. DISCUSSION

We have investigated the geometrical percolation transition of the square-lattice LCCC model and have found the critical points and the critical exponents ($\beta/\nu = 0.130 \pm 0.005, 1/\nu = 0.80 \pm 0.01, \tau = 1.82 \pm 0.01$) characterizing the transition. Although the system does not show any finite-size effect in the case of the transition of the average opinion, the percolation transition shows a prominent finite-size effect for a given threshold opinion value (Ω). The finite-size effect diminishes gradually as we decrease the value of Ω. The transition point also decreases with Ω, but the change

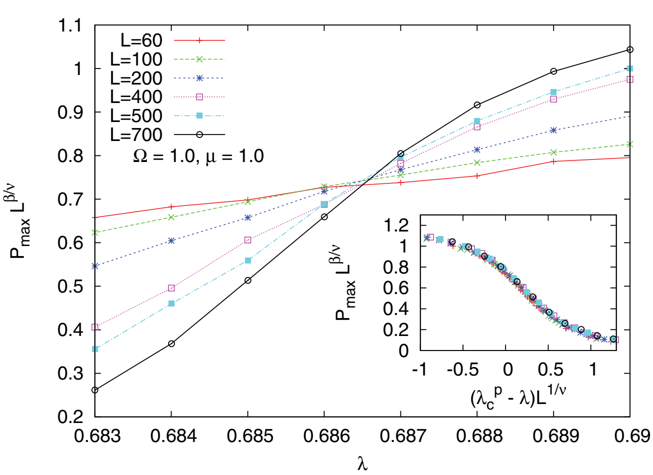

FIG. 10. (Color online) $P_{\max} L^{\beta/\nu}$ plotted against the conviction parameter λ, where $\Omega = 1.0$ and $\mu = 1.0$. The curves for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) cross at $\lambda_c^p = 0.687 \pm 0.001$ for $\beta/\nu = 0.130 \pm 0.005$. In the inset, we have shown the data collapse for $P_{\max}$ with $\lambda_c^p - \lambda$ for $\Omega = 1.0$ and $\mu = 1.0$ giving $1/\nu = 0.80 \pm 0.01$ and $\beta/\nu = 0.130 \pm 0.005$.

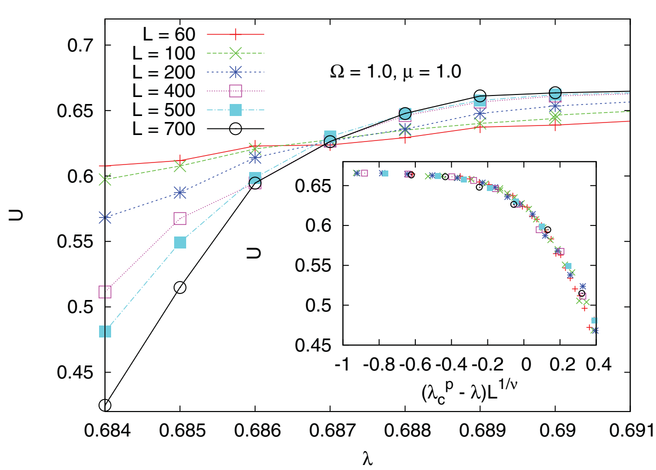

FIG. 12. (Color online) Fourth-order reduced Binder cumulant of percolation order parameter ($P_{\max}$) for different system sizes ($L = 60$, 100, 200, 400, 500, and 700) at $\Omega = 1.0$ and $\mu = 1.0$; the crossing point determines the critical point ($\lambda_c^p = 0.687 \pm 0.001$). The critical Binder cumulant value is $U^\star = 0.624 \pm 0.002$. Inset shows the data collapse for the same value of $1/\nu$ as obtained for the data collapse of $P_{\max}$.

is continuous. The critical exponents are independent of the value of the threshold opinion value as well as the value of the conviction and influencing parameter, which shows the robustness of this percolation transition in this system. The critical exponents are significantly different from those obtained in the case of the static and dynamic Ising system and standard percolation. These exponents suggest that this LCCC model belongs to a separate universality class from the viewpoint of percolation transition.

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[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).
[2] G. Grimmett, *Percolation* (Springer-Verlag, Berlin, 1999).
[3] H. Müller-Krumbhaar, Phys. Lett. A **48**, 459 (1974).
[4] E. Stoll, K. Binder, and T. Schneider, Phys. Rev. B **6**, 2777 (1972).
[5] H. Müller-Krumbhaar and E. P. Stoll, J. Chem. Phys. **65**, 4294 (1976).
[6] K. Binder and D. Stauffer, J. Stat. Phys. **6**, 49 (1972).
[7] A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo, Commun. Math. Phys. **51**, 315 (1976).
[8] A. Coniglio and W. Klein, J. Phys. A **13**, 2775 (1980).
[9] D. W. Heermann and D. Stauffer, Z. Phys. B **44**, 339 (1981).
[10] S. Fortunato, Phys. Rev. B **66**, 054107 (2002).
[11] S. Fortunato, Phys. Rev. B **67**, 014102 (2003).
[12] S. Biswas, A. Kundu, and A. K. Chandra, Phys. Rev. E **83**, 021109 (2011).
[13] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[14] *Econophysics and Sociophysics: Trends and Perspectives*, edited by B. K. Chakrabarti, A. Chakraborti, and A. Chatterjee (Wiley-VCH, Berlin, 2006).
[15] R. A. Holley and T. M. Liggett, Ann. Probab. **3**, 643 (1975).
[16] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Springer, Berlin, 1999); R. Lambiotte and S. Redner, Europhys. Lett. **82**, 18007 (2008).
[17] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[18] R. Hegselmann and U. Krause, J. Artif. Soc. Soc. Simul. **9**, 10 (2006) http://jasss.soc.surrey.ac.uk/9/3/10.html.
[19] G. Deffuant, D. Neau, D. Amblard, and G. Weisbuch, Adv. Complex Syst. **3**, 87 (2000).
[20] S. Fortunato, Int. J. Mod. Phys. C **16**, 17 (2005).
[21] S. Biswas, A. Chatterjee, and P. Sen, Physica A: Stat. Mech. Appl., doi:10.1016/j.physa.2012.01.046.
[22] M. Lallouache, A. Chakraborti, and B. K. Chakrabarti, Sci. Cult. **76**(9-10), 485 (2010) http://www.scienceandculture-isna.org/sep-oct-2010.htm.
[23] A. Chakraborti and B. K. Chakrabarti, in *Econophysics of Order-driven Markets*, edited by F. Abergel, B. K. Chakrabarti, A. Chakraborti, and M. Mitra (Springer-Verlag, Milan, 2011).
[24] J. Shao, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **103**, 018701 (2009).
[25] M. Lallouache, A. S. Chakrabarti, A. Chakraborti, and B. K. Chakrabarti, Phys. Rev. E **82**, 056112 (2010).
[26] S. Biswas, A. K. Chandra, A. Chatterjee, and B. K. Chakrabarti, J. Phys. Conf. Ser. **297**, 012004 (2011).
[27] P. Sen, Phys. Rev. E **83**, 016108 (2011).
[28] S. Biswas, Phys. Rev. E **84**, 056106 (2011).
[29] A. Chakraborti and B. K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000); A. Chatterjee, B. K. Chakrabarti, and S. S. Manna, Physica A **335**, 155 (2004); A. Chatterjee and B. K. Chakrabarti, Eur. Phys. J. B **60**, 135 (2007).
[30] N. Tsakiris, M. Maragakis, K. Kosmidis, and P. Argyrakis, Phys. Rev. E **82**, 041108 (2010).
[31] K. Binder and D. Heermann, *Monte Carlo Simulations in Statistical Physics* (Springer, Berlin, 1988).