Efficiency of a thermodynamic motor at maximum power

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Several recent theories address the efficiency of a macroscopic thermodynamic motor at maximum power and question the so-called Curzon-Ahlborn (CA) efficiency. Considering the entropy exchanges and productions in an n-sources motor, we study the maximization of its power and show that the controversies are partly due to some imprecision in the maximization variables. When power is maximized with respect to the system temperatures, these temperatures are proportional to the square root of the corresponding source temperatures, which leads to the CA formula for a bithermal motor. On the other hand, when power is maximized with respect to the transition durations, the Carnot efficiency of a bithermal motor admits the CA efficiency as a lower bound, which is attained if the duration of the adiabatic transitions can be neglected. Additionally, we compute the energetic efficiency, or "sustainable efficiency," which can be defined for n sources, and we show that it has no other universal upper bound than 1, but that in certain situations, which are favorable for power production, it does not exceed $\frac{1}{2}$.

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I. INTRODUCTION

The efficiency γ_C of a thermal motor was defined by Carnot as the ratio of the power produced over the heat received from the higher temperature source [1]. Carnot efficiency played a crucial role in theoretical and applied thermodynamics, which especially distinguished its upper bound η_C, i.e., the celebrated Carnot limit $\eta_C = 1 - T_2/T_1$, with T_1 and T_2 being the temperatures of the hot and cold sources. This upper bound can only be attained when reversibility is realized so that the power produced vanishes. Thus the Carnot limit is not appropriate for an actual motor, which must have a finite power production. Many authors over the years [2–17] have considered the Carnot efficiency of a bithermal motor when it produces its maximum power either for macroscopic motors or for microscopic systems. Clearly, this maximum depends on the parameters, which are supposed to be varied, and different responses can be found in different conditions. Nevertheless, the efficiency at maximum power has been found by many authors [2–7] to be the so-called Curzon-Ahlborn (CA) value, $\eta_{CA} = 1 - (T_2/T_1)^{1/2}$. The hypotheses used for deriving this formula became more and more sophisticated, and it was extended to broader situations [18–21]. Recently, a new energetic efficiency, called the "sustainable efficiency" γ_S, was proposed for stationary systems in the framework of stochastic thermodynamics [13–15]. According to this definition, which applies for an arbitrary number of temperature sources, γ_S is the ratio of the power produced over the maximum power which could be produced if the power dissipation could vanish. It can be shown that under specific but reasonably wide conditions, its maximum value is $\frac{1}{2}$: this conclusion implies that in this situation, the Carnot efficiency at maximum power of a stationary stochastic motor may be higher than the Curzon-Ahlborn value η_{CA}, with an upper bound $\bar{\eta}_C = (T_2 - T_1)/(T_2 + T_1)$. The same conclusion was obtained shortly afterward for classical thermodynamic motors [16] thanks to very general arguments.

The present work is devoted to macroscopic thermodynamic motors, including an arbitrary number of sources, in the context of endoreversible systems [8,22]: the entropy production is due to heat exchanges only, with all other sources of entropy creation being neglected. We study their efficiency at maximum power by computing the entropy exchanges and productions in each transition of a cycle, after discussing the assumptions currently used in similar studies, and avoiding some questionable hypotheses used in them. We carefully distinguish different kinds of power maximizations, showing that when the power is maximized with respect to the system temperatures (different from the source temperatures), the Carnot efficiency of a bithermal motor has the CA value, η_{CA}. On the contrary, if power is maximized with respect to the durations of the transitions, then η_{CA} is a lower bound of the efficiency, which can in principle attain the Carnot limit. Additionally, the energetic efficiency previously named sustainable efficiency for stochastic systems [15] is defined and computed for n-sources macroscopic motors. This quantity proves to have properties similar to those of its stochastic version, but with different consequences.

Before addressing these points, it is useful to discuss the pioneering derivations of the Curzon-Ahlborn bound [2–4], which used the simplest and perhaps clearest, although imperfect, method: this is the purpose of Sec. II. In Sec. III, we define a generalized cyclic motor, compute its power production, and maximize it with respect to the system temperatures. The maximization with respect to the transition durations is addressed and discussed in Sec. IV. Eventually, the sustainable efficiency of macroscopic motors is studied in Sec. V.

II. A SHORT DISCUSSION OF THE EARLIEST DERIVATIONS OF THE CURZON-AHLBORN BOUND

A. The principle of the derivations

The original derivation of Carnot efficiency at maximum power by Yvon [2] is not known in full detail, since this author only sketched it in a lecture at a Geneva conference in 1955 and, as far as we know, did not publish it completely elsewhere. Nevertheless, his reasoning was clearly described and apparently very simple. In later years, Chambadal [2] and Novikov [3] followed the same method, which can be summarized as follows. The engine performs a Carnot cycle between the hot and cold sources at respective temperatures T_1 and T_2, but the actual temperature of the system during its contact with the hot source should be $T_1 < T_2$ in order to have a finite heat flux input $\dot{Q}_1 = K_1(T_1 - T_1')$, where K_1 is a constant including the thermal conductivities and the areas of the walls allowing for the heat exchanges with the hot source. The efficiency of the engine has the Carnot value $(1 - T_2/T_1')$ and the power produced is $P = K_1(T_1 - T_1')(1 - T_2/T_1')$. It is then straightforwardly found that the power is maximum for $T_1' = (T_1T_2)^{1/2}$, which yields the Curzon-Ahlborn value of the Carnot efficiency,

$$\eta_{\mathrm{CA}} = 1 - (T_2/T_1)^{1/2}. \quad (1)$$

As mentioned above, this formula was rederived later by several authors [5–8] based on more sophisticated arguments, but the previous method may be the simplest one. Nevertheless, it is clearly imperfect for several reasons, in particular because, logically, one should also consider that the lowest temperature T_2' of the Carnot cycle is higher than the cold source temperature: $T_2' > T_2$. Furthermore, the durations of the various phases of the Carnot cycle are not taken into account, although they clearly play a role in power production. These points were addressed by subsequent researchers [5–8] in different formalisms, and they can be implemented in the original method, as shown in Appendix A.

Clearly, most actual motors do not follow a Carnot cycle (see, for instance, Ref. [22]). In fact, maintaining the system temperature constant during the heat exchanges is only approximately realized after transient regimes and requires special conditions that may be difficult to satisfy in practice. Nevertheless, it will be shown below that the maximum power is obtained, theoretically, in this situation, which, for a bithermal motor, corresponds to a Carnot cycle. This is why we focus on this cycle.

B. Temperatures and durations of the heat exchanges

The average power produced by the system can be maximized with respect, not to T_1', T_2', but to other variables, such as the durations τ_1, τ_2 of the exchanges with the reservoirs: this was the point of view considered, for instance, in Ref. [16]. It should be noted that the durations τ_1 and τ_2 are not independent of the internal temperatures T_1' and T_2'. As a matter of fact, the entropy of the system, being a state variable, must not change during a whole cycle: thus it satisfies the "closure relation"

$$\sum_i \delta S_i' = 0, \quad (2)$$

where $\delta S_i'$ is the variation of the system entropy during a transition i, which is the sum running on all transitions of the cycle. By the kinetic laws of irreversible thermodynamics, the increments can be expressed as functions of the other variables and of the transition durations, so that the previous equation relates the durations and the temperatures of the transitions.

Other such relations can be established, taking into account the exact mechanism of the transitions. Assume, for instance, that the system is a perfect gas performing a Carnot cycle including (i) the isothermal expansion (T_1') at temperature T_1' from the initial volume V_1 to volume V_2, (ii) the adiabatic expansion (A_1) from temperature T_1' and volume V_1 to temperature T_2' and volume V_3, (iii) the isothermal expansion (T_2') at temperature T_2' from volume V_3 to volume V_4, and (iv) the adiabatic expansion (A_2) from temperature T_2' and volume V_4 to temperature T_1' and volume V_1. Let τ_1, τ_{a1}, τ_2, and τ_{a2} be the respective durations of these successive steps and τ be the total duration of the cycle. If the molar heat capacities at constant volume, C_V, and at constant pressure, C_P, are constant, then it is well known that

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} = \left(\frac{T_1'}{T_2'}\right)^{C_V/R} \equiv \lambda, \quad (3)$$

where R is the perfect gas constant and $R/C_V = C_P/C_V - 1 \equiv \gamma - 1$. In the simplest circumstances, the gas is contained in a cylinder closed by a piston moving with the constant velocities v during the expansions and −v during the compressions. Then, the ratios τ_i/t are easily expressed in the function of λ and of the compression ratio $c = V_2/V_1$. In particular, we have

$$\tau_2 = \lambda\tau_1.$$

Moreover, the ratio of the total time of the adiabatic processes over the total time of the heat exchanges is

$$\frac{\tau_{a1} + \tau_{a2}}{\tau_1 + \tau_2} = \frac{(c+1)(\lambda-1)}{(c-1)(\lambda+1)} \quad (4)$$

and it also depends on the temperatures, which is contrary to the assumption used in Ref. [5] for deriving the famous value (1).

We remark that it is generally assumed in the literature [16] that the weak dissipation regime holds, implying that $\tau_{a1} + \tau_{a2} \ll \tau_1 + \tau_2$. However, formula (4) shows that this is only valid if $\lambda \approx 1$. In practical cases, a typical value of C_V is of the order of 2.5 R and T_1/T_2 ranges between 2 and 3 (see Table I). At the optimal regime of Curzon-Ahlborn, $T_1'/T_2' \approx (T_1/T_2)^{1/2} \approx 1.6$, and $\lambda \approx (1.6)^{2.5} \approx 3.1$: then the weak dissipation approximation is questionable.

Of course, relations (3) and (4) are only valid for this special, elementary model. In realistic cases, similar relations should hold, but they presumably become very complex and highly specific. Even if they could be written explicitly, they would not lead to any general relation. Thus, we will ignore them and consider that the temperatures and durations of the heat exchanges are only related by Eq. (10). Then we can obtain general results, but it is possible that the theoretical maximal power calculated later is not attained exactly for actual engines.

C. Some experimental data

It has been pointed out previously [5,16] that the efficiencies of actual power plants are generally much closer to the CA value than to the Carnot limit. Some examples are given in Table I: it is seen that in several cases, the experimental efficiency is larger than the CA value, and even larger than the upper bound of$\bar{\eta}_C$ predicted by some theories [14,16]. This is no surprise, in view of the previous remarks, since it is doubtful if the theoretical maximization conditions are actually satisfied. This remark, however, makes the interest of the previous values of efficiency questionable, and it shows the importance of specifying the conditions under

TABLE I. Experimental and theoretical efficiencies for some industrial power plants. The Carnot limit η_C, CA efficiency η_{CA}, and theoretical upper bound$\bar{\eta}_C$ can be compared to the experimental efficiency η_{exp}.

Plant	T_1 (K)	T_2 (K)	η_C	η_{CA}	$\bar{\eta}_C$	η_{exp}
Almaraz II (Nuclear, Spain) [16]	600	290	.52	.30	.35	.34
Calder Hall (Nuclear, UK) [16]	583	298	.49	.29	.32	.19
CANDU (Nuclear, Canada) [5]	573	298	.48	.28	.32	.30
Cofrentes (Nuclear, Spain) [16]	562	289	.49	.28	.32	.34
Doel 4 (Nuclear, Belgium) [16]	566	283	.50	.29	.33	.35
Heysham (Nuclear, UK) [16]	727	288	.60	.37	.43	.40
Larderello (Geothermal, Italy) [5]	523	353	.32	.18	.19	.16
Sizewell B (Nuclear, UK) [16]	581	288	.50	.30	.34	.36
West Thurrock (Coal, UK) [5]	838	298	.64	.40	.48	.36
Pressurized water nuclear reactor [24]	613	304	.50	.30	.34	.33
Boiling water nuclear reactor [24]	553	304	.45	.25	.29	.33
Fast neutron nuclear reactor [24]	823	296	.64	.40	.47	.40

which the theoretical maximization of the power production is performed.

III. ENTROPY PRODUCTION IN A GENERALIZED MOTOR

A. A general cyclic engine

Let us consider a system performing a cycle consisting of N, possibly infinitesimal, steps i = 1,2,...N, with respective positive durations $\delta t_1,\delta t_2,\ldots\delta t_N$. During step i, the system receives the heat δQ_i from a reservoir at fixed temperature T_i, whereas the system is at temperature T_i', which may be considered to be constant if δt_i is small enough. δQ_i can be 0, and several successive temperatures T_i may be identical, whereas the corresponding T_i' could in principle be different, thus describing a time-dependent system temperature.

Assuming that the laws of irreversible thermodynamics near equilibrium hold, we can write, by Fourier law,

$$\delta Q_i = K_i(T_i - T_i')\delta t_i. \tag{5}$$

It can be remarked that other expressions of the heat fluxes have been used [23], without definite theoretical support. Such expressions are equivalent to the Fourier law for a very small temperature difference between the source and the system, but not if this difference becomes significant while remaining relatively small. We assume that the Fourier law remains valid in this case. In fact, although this classical law is only approximate, its validity has been confirmed both experimentally and theoretically in many circumstances (see, for instance, among the abundant literature, the classical textbook [25] for gases, or [26,27] for solids, and references therein).

The adiabatic transitions are considered in the same formalism by taking $K_i = 0$ for them. According to elementary calculations, the entropy change of the reservoir during step i is

$$\delta S_i = \frac{-\delta Q_i}{T_i} \tag{6}$$

and the entropy change of the system is

$$\delta S_i' = \frac{\delta Q_i}{T_i'} \equiv -\delta S_i + \delta s_i, \tag{7}$$

with $-\delta S_i$ being the entropy received from the reservoir and δs_i being the entropy produced during the exchange, so

$$\delta s_i = \delta Q_i\left(\frac{1}{T'_i} - \frac{1}{T_i}\right) = K_i\frac{(T_i - T_i')^2}{T_iT_i'}\delta t_i. \tag{8}$$

The work produced during the whole cycle is

$$-W = \sum_i \delta Q_i = K_i(T_i - T_i')\delta t_i, \tag{9}$$

while the total entropy variation of the system vanishes,

$$0 = \sum_i \frac{\delta Q_i}{T_i'} = \sum_i K_i\frac{T_i - T_i'}{T_i'}\delta t_i. \tag{10}$$

We remark that for these classical formulas to be valid, the system temperature T_i' should be positive, so

$$\delta S_i' = K_i\frac{T_i - T_i'}{T_i'}\delta t_i > -K_i\delta t_i, \tag{11}$$

which we will always assume. If now we let $\delta t_i \to 0$, then the previous formulas apply, replacing the finite increments by differentials and the sums by time integrals.

B. Maximization of the power production with respect to the temperatures

If we assume that the durations of the transitions are fixed whereas the system temperatures T_i' are varied while respecting equality (10), then it is easily found that the maximum power per cycle is attained when for each nonadiabatic step

$$T_i' = (\mu T_i)^{1/2}, \quad \text{with} \quad \mu^{1/2} = \frac{\sum_i K_i(T_i)^{1/2}\delta t_i}{\sum_i K_i\delta t_i}, \tag{12}$$

the maximum power produced is

$$P_{\max} = \frac{\left(\sum_i K_iT_i\delta t_i\right)\left(\sum_i K_i\delta t_i\right) - \left(\sum_i K_i(T_i)^{1/2}\delta t_i\right)^2}{\left(\sum_i K_i\delta t_i\right)\left(\sum_i \delta t_i\right)} \geqslant 0. \tag{13}$$

In the case of infinitesimal time increments δt_i, Eqs. (11) and (12) apply for each nonadiabatic phase, replacing the sums by the corresponding integrals. Equation (12) implies that the maximum power is obtained when the system temperature T_i' is constant during the heat exchange, with source i at temperature T_i. In practice, this condition may be difficult to satisfy, and it certainly implies particular mechanisms, but for maximizing the power, it is favorable to approach it as far as possible. Thus, as most authors (see, in particular, [22]) have done, we assume from now on that this condition is realized: $\tau_i \equiv \delta t_i$ will represent the total duration of the exchanges with source i, where $\delta_i \equiv \delta S_i'$ is the corresponding variation of the system entropy.

If there are only two reservoirs with temperatures T_1 and T_2 ($T_1 > T_2$), then the heat received from reservoir i ($i = 1$ or 2) is $\delta Q_i = (T_i - T_i')\, K_i \tau_i = T_i' \delta_i$, and Eq. (10) indeed shows that the Carnot efficiency has the CA value $\eta_{\rm CA}$.

IV. MAXIMIZATION OF THE POWER PRODUCTION WITH RESPECT TO THE TRANSITION DURATIONS

In the previous multitemperature machine, let us now suppose that we can vary the durations of the different steps. We first remark that by Eq. (8), the entropy production during step i can be written for a nonadiabatic transition,

$$\delta s_i = K_i \frac{(T_i - T_i')^2}{T_i T_i'} \delta t_i = \frac{-\delta S_i \delta S_i'}{K_i \delta t_i}. \tag{14}$$

During an adiabatic transition i, we assume that $\delta S_i = \delta S_i' = \delta s_i = 0$. Using (14), if i is not adiabatic, then we obtain

$$\delta s_i = \frac{(\delta S_i')^2}{K_i \delta t_i} \left(1 + \frac{\delta S_i'}{K_i \delta t_i} \right)^{-1}. \tag{15}$$

Writing $\tau = \Sigma_{i=1,\dots N} \delta t_i$, the power produced after N steps, i.e., after a (pseudo) cycle, is

$$\begin{aligned} P &= \frac{-W}{\tau} = \frac{1}{\tau} \sum_i^{*} \delta Q_i = \frac{1}{\tau} \sum_i^{*} T_i(-\delta S_i) \\ &= \frac{1}{\tau} \sum_i^{*} T_i(\delta S_i' - \delta s_i) = \frac{1}{\tau} \sum_i^{*} T_i \frac{\delta S_i'}{1 + \frac{\delta S_i'}{K_i \delta t_i}}, \end{aligned} \tag{16}$$

where Σ^* denotes the sum over nonadiabatic transitions. From now on, we will use the previous formulas with the condensed notation $\delta_i \equiv \delta S_i'$ and $\tau_i \equiv \delta t_i$, assuming that, according to Sec. III B, $\delta_i > -K_i \tau_i$.

A. Weak dissipation regime

The weak dissipation regime (see, for instance, [16]) can be considered as the standard situation where the usual laws of irreversible thermodynamics hold. Then the heat exchanges are slow and we can consider that $\delta_i \ll K_i \tau_i$. So the power produced is

$$P \approx \frac{1}{\tau} \sum_i^{*} T_i \left(\delta_i - \frac{(\delta_i)^2}{K_i \tau_i} \right). \tag{17}$$

According to the methods of Ref. [16], we can maximize the power produced with respect to the τ_i, considering that the entropy variation $\delta_i = \delta S_i'$ of the system during each step i is fixed. Such a choice implies that the entropy changes of the system are relevant quantities, to be used in the best possible way: this arbitrary convention is contrary to the usual consideration that only the energy inputs import, but it is reasonable in the scope of sustainable development, now adopted in many circumstances. Then, we obtain, for nonadiabatic transitions,

$$\frac{T_i (\delta S_i')^2}{K_i (\tau_i)^2} = P, \tag{18}$$

where we consider only the situations when P is positive. Thus, if transition i is not adiabatic, then

$$\tau_i = |\delta_i|\, p^{-1/2} \left(\frac{T_i}{K_i} \right)^{1/2}, \tag{19}$$

whereas the adiabatic transitions should obviously be as short as possible in order to maximize the power produced. The maximum power and the heat received from each source can be expressed (Appendix B) in terms of the δ_i and the other parameters.

In the case of a bithermal motor operating successively with two sources at temperatures T_1 and T_2 ($T_1 > T_2$), we obtain (Appendix B)

$$P^{1/2} = \frac{1}{2}(T_1 - T_2) \left[\left(\frac{T_1}{K_1} \right)^{1/2} + \left(\frac{T_2}{K_2} \right)^{1/2} \right]^{-1} \tag{20}$$

and the Carnot efficiency is found to be

$$\begin{aligned} \lambda_{\rm C} &= 1 + \frac{\delta Q_2}{\delta Q_1} \\ &= \frac{T_1 - T_2}{2T_1 - (T_1 - T_2)[1 + (T_2/T_1)^{1/2}(K_1/K_2)^{1/2}]^{-1}}. \end{aligned} \tag{21}$$

We recover the main result of Ref. [16] as well as its consequences. $\lambda_{\rm C}$ is a decreasing function of K_1/K_2: if this ratio tends to ∞, then one obtains a lower bound of $\lambda_{\rm C}$,

$$\lambda_{\rm C\,min} = \frac{1}{2} \left(1 - \frac{T_2}{T_1} \right) = \frac{1}{2} \eta_{\rm C},$$

whereas if $K_1/K_2 \to 0$, then one finds the upper bound $\bar{\eta}_{\rm C}$ which was previously obtained for stationary stochastic motors [14,15],

$$\lambda_{\rm Cmax} = \frac{T_1 - T_2}{T_1 + T_2} = \bar{\eta}_{\rm C}.$$

Eventually, if $K_1/K_2 = 1$, then the Curzon-Ahlborn value $\eta_{\rm CA}$ is recovered. We will see, however, that these conclusions are not preserved in a more general regime, when the weak dissipation approximation does not hold.

B. A generalized regime

We now maximize the general expression (16) of the power produced with respect to the τ_i, again considering that the entropy variation $\delta_i \equiv \delta S_i'$ of the system during each step i is fixed. Thus, we admit that the Fourier law remains valid outside of the weak dissipation regime, when $\delta S_i'$ is not necessarily much smaller than $K_i \tau_i$, which implies that $|T_i' - T_i|$ can be of the order of T_i'. In fact, this is currently realized in actual heat engines where, nevertheless, the Fourier law is assumed to be valid. In this case, maximizing expression (15) of $P > 0$ yields, for a nonadiabatic transition,

$$\frac{T_i(\delta_i)^2}{K_i(\tau_i)^2\left[1+\frac{\delta_i}{K_i\tau_i}\right]^2} = P, \tag{22}$$

if $\tau_i > 0$. It results from (22) that

$$\frac{\frac{\delta_i}{K_i\tau_i}}{1+\frac{\delta_i}{K_i\tau_i}} = \varepsilon_i\left(\frac{P}{K_iT_i}\right)^{1/2}, \tag{23}$$

where ε_i is the sign of the left-hand side. We have seen that $d_i > -K_i\tau_i$, so that $\varepsilon_i = \text{sign}(\delta_i)$. Thus,

$$\tau_i = \frac{|\delta_i|}{K_i}\left[\left(\frac{P}{K_iT_i}\right)^{-1/2} - \varepsilon_i\right], \tag{24}$$

with the right-hand side being positive, provided that $P < K_iT_i$ for each step with positive entropy variation. It is easily seen from (16) that this inequality is satisfied for at least one of these steps. If the inequality was not satisfied for some step j with positive entropy variation, then this step j should be skipped ($\tau_j = 0$) in order to maximize the power production. We assume that such steps, if any, have been suppressed. On the other hand, it can be checked by (24) that τ_i satisfies inequality (11), as it should be. From (16), (22), and (24), we obtain

$$\begin{aligned} \tau P &= A - P^{1/2}B \\ \tau &= \tau_a + P^{-1/2}B - C \end{aligned} \tag{25}$$

with

$$A = \sum_i^* T_i\delta_i, \quad B = \sum_i^*\left(\frac{T_i}{K_i}\right)^{1/2}|\delta_i|, \quad C = \sum_i^* \frac{1}{K_i}\delta_i. \tag{26}$$

It is clear that $B > 0$ and that by (25), $P > 0$ implies $A > 0$ (so, obviously, all of the temperatures T_i cannot be equal), but C may be negative as well as positive. Equations (25) yield

$$(C-\tau_a)P - 2BP^{1/2} + A = 0, \tag{27}$$

which has (at least) one positive solution if

$$\tau_a \geqslant C - B^2/A. \tag{28}$$

It is always possible to satisfy (28) if τ_a is large enough, but then the power production is small. In order that no minimum value is assigned to τ_a, it is desirable that $B^2 - AC \geqslant 0$: we assume that we are in this situation. Then, Eq. (27) has one positive solution such that $\tau > 0$,

$$P^{1/2} = \frac{A}{B + (B^2 - AC + \tau_a A)^{1/2}}. \tag{29}$$

This formula shows that in order to maximize P, the duration τ_a of the adiabatic transitions should be as small as possible, as already pointed out. We see that the power production P can be higher than the value $A^2/(2B)^2$ obtained in the standard weak dissipation regime (see Appendix B), provided that $C > \tau_a$. So, in order to consider a favorable situation, we may assume that C is positive and that τ_a, if not completely negligible, is at least less than $C : \tau_a < C$.

The heat exchanged with source i is, by (17),

$$\delta Q_i = T_i\frac{\delta_i}{1+\frac{\delta S_i}{K_i\tau_i}} = T_i\delta S_i - P^{1/2}\varepsilon_i\delta S_i\left(\frac{T_i}{K_i}\right)^{1/2}, \tag{30}$$

and the temperature T_i' of the system during the heat exchanges with source i is

$$T_i' = T_i - P^{1/2}\varepsilon_i\left(\frac{T_i}{K_i}\right)^{1/2}, \tag{31}$$

which implies that $P < K_iT_i$ for each step i such that $\delta S_i' > 0$, as noticed previously. It can be shown that this solution is indeed a maximum of the power production.

As an example, let us consider a bithermal motor with sources at temperatures T_1 and T_2 ($T_1 > T_2$). We have $\delta S_1' = -\delta S_2' > 0$, and

$$\begin{aligned} \delta Q_1 &= T_1\delta S_1'\left(1 - \frac{P^{1/2}}{(K_1T_1)^{1/2}}\right), \\ \delta Q_2 &= -T_2\delta S_1'\left(1 + \frac{P^{1/2}}{(K_2T_2)^{1/2}}\right). \end{aligned}$$

It is found that $B^2 - AC = (T_1/K_2 + T_2/K_1)^2 > 0$, so that τ_a may indeed be arbitrarily small. Equation (29) yields

$$P^{1/2} = \frac{(T_1 - T_2)}{[(T_1/K_1)^{1/2} + (T_2/K_2)^{1/2}] + \{[(T_1/K_2)^{1/2} + (T_2/K_1)^{1/2}]^2 + \tau_a(T_1 - T_2)\}^{1/2}}. \tag{32}$$

If τ_a is small enough to be neglected, we obtain

$$P^{1/2} \approx \frac{(T_1)^{1/2} - (T_2)^{1/2}}{(K_1)^{-1/2} + (K_2)^{-1/2}} \equiv (P_0)^{1/2}. \tag{33}$$

${P_0}^{1/2}$ is, furthermore, an upper bound for $P^{1/2}$. The Carnot efficiency is

$$\gamma_C = 1 + \frac{\delta Q_2}{\delta Q_1} = \frac{(T_1 - T_2) - P^{1/2}[(T_1/K_1)^{1/2} + (T_2/K_2)^{1/2}]}{T_1 - P^{1/2}(T_1/K_1)^{1/2}} \equiv \gamma_C(P), \tag{34}$$

which is a decreasing function of P. Cumbersome but elementary calculations show that if $\tau_a \to 0$, then

$$\gamma_C \to \gamma_C(P_0) = 1 - \left(\frac{T_2}{T_1}\right)^{1/2} = \eta_{CA}. \quad (35)$$

Thus, when the total duration of the adiabatic steps tends to 0, the Curzon-Ahlborn efficiency is recovered for all values of K_1 and K_2, and not only when $K_1 = K_2$, as found in the weak dissipation regime. Moreover, for finite τ_a, $P < P_0$, so that

$$\gamma_C(P) > \gamma_C(P_o) = \eta_{CA}, \quad (36)$$

which is one of our main results: the Curzon-Ahlborn efficiency is a lower bound of Carnot efficiency at maximum power in the present maximization conditions. A similar conclusion was obtained, in certain circumstances, from the explicit study of the so-called three-level model motor [17].

V. MULTISOURCES MOTOR AND SUSTAINABLE EFFICIENCY

A. Sustainable efficiency of a macroscopic thermodynamic motor

The notion of sustainable efficiency was introduced in stochastic thermodynamics for a mesoscopic motor operating in a nonequilibrium stationary state in Ref. [14], noticing that the power produced P can be written as

$$P = A - D_P,$$

where D_P is the power dissipation, i.e., the energetic equivalent of the entropy production rate, which is always positive out of equilibrium. Here, A is the power which would be produced if the power dissipation could vanish. The sustainable efficiency γ_S was defined [14] as

$$\gamma_S = \frac{P}{A} = \frac{P}{P + D_P}$$

Thus, the sustainable efficiency makes sense for any multisource motor without favoring one the sources, and it may be appropriate for an unbiased estimation of the motor performances. For these reasons, and its relations with Carnot efficiency [14,15], it is interesting to extend this concept to the present macroscopic formalism. The heat δQ_i received by the system during step i can be written, by (15),

$$\delta Q_i = -T_i \delta S_i = T_i \delta S'_i - T_i \delta s_i,$$

where δS_i and $\delta S'_i$ are the entropy variations of reservoir i and of the system, respectively, and δs_i is the entropy production during step i. The work produced during the cycle is

$$-W = \sum_i \delta Q_i = \sum_i T_i \delta S'_i - \sum_i T_i \delta s_i. \quad (37)$$

The energy dissipation during step i is $T_i \delta s_i$, and the total energy dissipation during the cycle is

$$D_W = \sum_i T_i \delta s_i, \quad (38)$$

whereas $A \equiv \Sigma_i T_i \delta S'_i = -W + D_W$ is the maximum work that could be produced during a cycle if all dissipation could be avoided. In fact, during step i, the temperature of the heat source is T_i and the exergy [28] of the system (in the absence of a pressure reservoir) is $\mathrm{Ex}_i = E'_i - T_i S'_i$, E'_i, with S'_i being the internal energy and the entropy of the system, respectively. According to classical engineering thermodynamics, the maximum work that can be produced by the system during step i is $-\delta \mathrm{Ex}_i$, and the maximum work produced during a complete cycle is $-\Sigma_i \mathrm{Ex}_i \equiv \Sigma_i T_i S'_i = A$, in agreement with definitions (26). Thus, for a macroscopic engine, the sustainable efficiency can be defined by

$$\gamma_S = \frac{-W}{-W + D_W} = \frac{\sum_i \delta Q_i}{\sum_i T_i \delta S'_i} \leqslant 1. \quad (39)$$

In similarity with the stochastic case, it is interesting to consider the value of the sustainable efficiency at maximum power. In this situation, we use formulas (29), (30), and (39) and obtain

$$\gamma_S = \frac{A - P^{1/2} B}{A} = 1 - \frac{B}{B + (B^2 - AC + \tau_a A)^{1/2}}. \quad (40)$$

It is clear that if $\tau_a < C$, and in particular if $C > 0$ and $\tau_a \to 0$, then $\gamma_S \leqslant \frac{1}{2}$. This situation is not general: the sustainable efficiency, as defined above, can be larger than $\frac{1}{2}$ at maximum power, and it even tends to 1 if the adiabatic transitions are infinitely slow, as shown by (40): then the Carnot efficiency reaches the classical Carnot limit, as shown below by formula (41), but the power produced vanishes. On the other hand, we have seen that in order to maximize the power production, it is desirable to minimize the duration of the adiabatic phases and to have a positive C coefficient larger that τ_a: in such situations, the maximum value of the thermodynamic sustainable efficiency is $\frac{1}{2}$. This upper bound is attained if $C = \tau_a$ and, in particular, when all Fourier coefficients K_i are equal and $\tau_a \to 0$.

It can be remarked that the stochastic sustainable efficiency defined in Refs. [14] and [15] also admit the upper bound $\frac{1}{2}$ in certain situations; essentially, when the stochastic dynamics is varied while maintaining constant the stationary probability distribution. In the absence of any specific condition, however, the stochastic sustainable efficiency has no general upper bound lower that 1, like its macroscopic version.

B. Macroscopic and stochastic sustainable efficiencies

At this point, it can be useful to summarize the analogies and differences between the macroscopic and stochastic sustainable efficiencies. Clearly, the macroscopic sustainable efficiency considered in Sec. V A has much similarity with the stochastic sustainable efficiency of Refs. [14] and [15], by its definition and it properties. Nevertheless, an important difference between these efficiencies is that the first one only depends on a few macroscopic parameters, whereas the stochastic efficiency depends on the complete dynamics of the system and implies a large number of microscopic parameters. For this reason, the constraint of a constant stationary probability, imposed in maximizing the stochastic sustainable efficiency, can hardly be transposed to the macroscopic case.

These analogies and differences are also manifested in the relation existing between the Carnot and sustainable efficiencies. It can be shown (see Appendix C) that the macroscopic sustainable efficiency satisfies the relation already found [15] in the stochastic case, which reads (if the only sources of entropy production are the heat exchanges, as assumed here)

$$\gamma_C = \left(1 - \frac{T_2}{T_1}\right) \times \left[1 - T_2 \frac{\delta s_1 + \delta s_2}{(1-\gamma_S)^{-1}(T_1\delta s_1 + T_2 \delta s_2) - (T_1 - T_2)\delta s_1}\right]. \quad (41)$$

Thus, γ_C is an increasing function of γ_S. It tends to the Carnot limit η_C if $\gamma_S \to 1$, which is in principle possible. In practice, we saw that in favorable situations, we should have $\gamma_S \leqslant \frac{1}{2}$. In the macroscopic theory, however, this inequality, combined with (41), does not imply any new effective upper bound for γ_C, because the parameters in Eq. (41) cannot be varied independently (Appendix C), which is contrary to the stochastic case [15]. Then the Carnot efficiency can in principle attain the Carnot limit, as found in the complete analytical study of the three-level stochastic motor [17], and developed more generally in a recent paper [29].

VI. CONCLUSION

The efficiency of a thermal motor, conditioned on maximal power production, has been discussed intensively from more than 50 years. The original derivations were completed and generalized, and alternative derivations were proposed, until recent papers contested these apparently well-established results, either proposing a higher upper bound on the efficiency, or even asserting that there is no general upper bound other than the Carnot value. These discussions concern the classical engine considered in macroscopic thermodynamics, as well as the mesoscopic motors introduced in stochastic thermodynamics. In the present paper, focused on macroscopic thermodynamics, we have shown that the controversies are partly due to the imprecise definition of the maximization conditions. When the maximization is taken over the system temperatures during the nonadiabatic transitions (or equivalently over the corresponding system entropy variations), the Curzon-Ahlborn efficiency is obtained for a bithermal motor. On the other hand, if the maximization is taken over the durations of the transitions, then the CA efficiency is a lower bound of the actual efficiency: it is attained if the total duration of the adiabatic transitions can be neglected with respect to the duration of the other transitions.

In analogy with a definition given for stochastic motors, we have also introduced the notion of "sustainable efficiency" to macroscopic motors, i.e., the ratio of the power produced to the maximum power which could be produced with the same resources if all irreversible effects could be suppressed. This efficiency γ_S not only can be used for any number of reservoirs without favoring one of them, but also gives a new light on the more usual Carnot efficiency. The only general upper bound of the sustainable efficiency is 1, implying that in the most general situation, the Carnot efficiency at maximum power can in principle approach the Carnot limit. Practically, however, maximizing the power is preferably obtained in situations where $\gamma_S \leqslant \frac{1}{2}$, and the energy exchanges with the reservoir obey the law of heat diffusion: these conditions lead to the Curzon-Ahlborn efficiency found by direct calculations.

In conclusion, the concept of efficiency at maximum power can be misleading, because it depends on the kind of maximization which is considered. As well as the other values that have been proposed, the Curzon-Ahlborn efficiency is not "universal." However, it plays a crucial role when the energy exchanges with the reservoirs are governed by the Fourier law: in certain situations, it represents the lowest bound of the efficiency at maximum power, which is attained in specific circumstances. This is why it remains an essential value in the theory of macroscopic motors.

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APPENDIX A: COMPLEMENTS ON THE EARLIEST DERIVATION OF CURZON-AHLBORN BOUND

These derivations can be completed to include ingredients which were discarded initially, mainly, the durations of the exchanges. These durations were considered, for instance, in Ref. [5], with questionable assumptions on the durations of the adiabatic transitions, which can be avoided. With the notations of Sec. II, we again study a bithermal motor operating with a hot source at temperature T_1 and a cold source at temperature T_2. Assume that during the exchanges with source i ($i = 1, 2$), the heat flux is finite and given by $\dot{Q}_i = K_i(T_i - T_i') < 0$, with K_i being a constant. We now take into account the durations of the different transitions occurring during a Carnot cycle. Let the isothermal phase (T_1') of exchange with the hot source 1 have a duration τ_1, and the isothermal exchange phase (T_2') with the cold source 2 have a duration τ_2, whereas the total duration of the cycle is τ. Considering a stationary statistical ensemble of identical, independent systems operating according to the same Carnot cycle, at a given time the proportion of systems undergoing a transition (T_i') is τ_i/τ, so that the ensemble average of the heat flux received by a system from source (i) is

$$\langle \dot{Q}_i \rangle = K_i(T_i - T_i')\tau_i/\tau \equiv \kappa_i(T_i - T_i'), \quad (A1)$$

which is clearly identical to the time average of this heat flux over a period much larger than τ for a unique system.

It results from classical thermodynamics that the entropy of the system should not change during a whole cycle, so that if the transitions connecting the isothermal phases are rigorously adiabatic, then

$$\frac{\langle \dot{Q}_1 \rangle}{T_1'} + \frac{\langle \dot{Q}_2 \rangle}{T_2'} = 0. \quad (A2)$$

The average work $\langle \dot{W} \rangle$ received by the system by unit time results from the energy conservation

$$\langle \dot{W} \rangle + \langle \dot{Q}_1 \rangle + \langle \dot{Q}_2 \rangle = 0. \quad (A3)$$

Thus, the average power produced by the system is $\langle P \rangle \equiv -\langle \dot{W} \rangle = \langle \dot{Q}_1 \rangle + \langle \dot{Q}_2 \rangle$ and its Carnot efficiency is

$$\gamma_C \equiv \frac{\langle P \rangle}{\langle \dot{Q}_1 \rangle} = 1 - \frac{\langle \dot{Q}_2 \rangle}{\langle \dot{Q}_1 \rangle} = 1 - \frac{T_2'}{T_1'}. \quad (A4)$$

The temperatures T_1' and T_2' cannot be chosen independently, since they must satisfy the relation resulting from (A1) and (A2),

$$\frac{\kappa_1(T_1 - T_1')}{T_1'} + \frac{\kappa_2(T_2 - T_2')}{T_2'} = 0, \tag{A5}$$

where (A2) and (A3) obviously imply that $T_1' < T_1$ and $T_2' > T_2$. It is easily shown that if the temperatures T_1' and T_2' are varied while respecting condition (A2), then the power produced $\langle P \rangle$ is maximum when

$$\begin{aligned} T_1' &= (\alpha_1\sqrt{T_1} + \alpha_2\sqrt{T_2})\sqrt{T_1}, \\ T_2' &= (\alpha_1\sqrt{T_1} + \alpha_2\sqrt{T_2})\sqrt{T_2}, \end{aligned} \tag{A6}$$

with $\alpha_i = \frac{\kappa_i}{\kappa_1+\kappa_2}(i = 1,2)$. It results from (A3) and (A4) that the Carnot efficiency has the expected CA value of

$$\eta_{\mathrm{CA}} = 1 - \frac{\sqrt{T_2}}{\sqrt{T_1}}. \tag{A7}$$

APPENDIX B: POWER MAXIMIZATION IN THE WEAK DISSIPATION APPROXIMATION

In the weak dissipation regime, the transitions are slow [16] and we have $\delta_i \ll -K_i\tau_i$. Then the power produced is

$$P \approx \frac{1}{\tau}\sum_i^{*} T_i\left(\delta_i - \frac{(\delta_i)^2}{K_i\tau_i}\right). \tag{B1}$$

Following the methods of Ref. [16], we now maximize the power produced with respect to the τ_i considering that the entropy variation $\delta_i = \delta S_i'$ of the system during each step i is fixed. To be physically meaningful, this assumption implies that the important quantities to consider when running a motor are the entropy inputs from or to the reservoirs, rather than the energy inputs. This is not the usual point of view, which focuses on the fuel consumptions or energies rejected to the environment. Nevertheless, we think that the entropy changes could be more significant than the energy variations, since energy is in principle conserved, even if it can hardly be used in certain forms, whereas entropy is not. In any case, this maximization condition can be considered as a mathematical condition but, once more, it is connected with considerations that are not purely scientific. Then, the δ_i should satisfy the constraint

$$\sum_i \delta_i = 0, \tag{B2}$$

which, nevertheless, does not affect the variables τ_i. Maximizing P gives, for nonadiabatic transitions,

$$\frac{T_i(\delta S_i')^2}{K_i(\tau_i)^2} = P, \tag{B3}$$

where we consider only the situations when P is positive. Thus, if transition i is not adiabatic, then

$$\tau_i = |\delta_i|\, p^{-1/2}\left(\frac{T_i}{K_i}\right)^{1/2}, \tag{B4}$$

whereas the adiabatic transitions should obviously be as short as possible in order to maximize the power produced. So,

$$P = \frac{1}{\tau}\sum_i^{*} T_i\left(\delta_i - \frac{(\delta_i)^2}{K_i\tau_i}\right) = \frac{1}{\tau}A - \frac{P^{1/2}}{\tau}B, \tag{B5}$$

where we have defined

$$A = \sum_i T_i\delta_i, B = \sum_i^{*}\left(\frac{T_i}{K_i}\right)^{1/2}|\delta_i|. \tag{B6}$$

On the other hand, if τ_a is the total duration of the adiabatic steps, then we have

$$\tau = \tau_a + \tau_b, \quad \text{where} \quad \tau_b = \sum_i^{*}\tau_i = P^{-1/2}B. \tag{B7}$$

In the weak dissipation approximation, τ_a should be small with respect to τ: $\tau_a \ll \tau$. So, combining (B5) and (B7) yields

$$P^{1/2} = \frac{A}{2B}, \tau = \frac{2B^2}{A}. \tag{B8}$$

The heat received from source (i) at temperature T_i during step i is, in the present approximation,

$$\delta Q_i \approx T_i\left(\delta_i - \frac{(\delta_i)^2}{K_i\tau_i}\right). \tag{B9}$$

In the case of a bithermal motor operating successively with two sources at temperatures T_1 and T_2 ($T_1 > T_2$),

$$P^{1/2} = \frac{1}{2}(T_1 - T_2)\left[\left(\frac{T_1}{K_1}\right)^{1/2} + \left(\frac{T_2}{K_2}\right)^{1/2}\right]^{-1}, \tag{B10}$$

and the Carnot efficiency is

$$\begin{aligned} \gamma_{\mathrm{C}} &= 1 + \frac{\delta Q_2}{\delta Q_1} \\ &= \frac{T_1 - T_2}{2T_1 - (T_1 - T_2)[1 + (T_2/T_1)^{1/2}(K_1/K_2)^{1/2}]^{-1}}. \end{aligned} \tag{B11}$$

We recover, with different notations, the main result of Ref. [13], as well as the following conclusions. γ_{C} is a decreasing function of K_1/K_2. Its lower bound is obtained if $K_1/K_2 \to \infty$,

$$\gamma_{\mathrm{C\,min}} = \frac{1}{2}\left(1 - \frac{T_2}{T_1}\right) = \frac{1}{2}\eta_C,$$

whereas if $K_1/K_2 \to 0$, then one finds the upper bound $\bar{\eta}_{\mathrm{C}}$, which was also obtained for stationary stochastic motors [14,15],

$$\gamma_{\mathrm{C\,max}} = \frac{T_1 - T_2}{T_1 + T_2} \equiv \bar{\eta}_C.$$

Eventually, if $K_1/K_2 = 1$, then the Curzon-Ahlborn value η_{CA} is recovered. These conclusions, however, essentially depend on the weak dissipation assumption, as shown in Sec. IV B.

APPENDIX C: CARNOT EFFICIENCY AND SUSTAINABLE EFFICIENCY FOR A BITHERMAL MOTOR

The Carnot efficiency γ_{C} of a bithermal motor exchanging heat with a hot source at temperature T_1 and a cold source at temperature T_2, as considered in Sec. III, can be expressed in terms of its sustainable efficiency γ_S defined by (39), as done in Refs. [14,15] for the stochastic efficiency. Using (36), we obtain (40),

$$\gamma_C = \left(1 - \frac{T_2}{T_1}\right) \times \left[1 - T_2 \frac{\delta s_1 + \delta s_2}{(1-\gamma_S)^{-1}(T_1\delta s_1 + T_2\delta s_2) - (T_1 - T_2)\delta s_1}\right], \tag{C1}$$

where we supposed that there is rigorously no entropy production during the adiabatic phases nor during the energy exchanges with mechanical systems (whereas the contrary hypothesis was considered in the stochastic version of this problem [15]). Then, γ_C is an increasing function of γ_S if the other parameters remain constant. If $\gamma_S \leqslant \frac{1}{2}$, as discussed above, then we have

$$\gamma_C \leqslant [1 - (T_2/T_1)]\left[1 - T_2 \frac{1 + \delta s_2/\delta s_1}{2(T_1 + T_2\delta s_2/\delta s_1) - (T_1 - T_2)}\right]. \tag{C2}$$

As in the case of stochastic efficiency [15], the right-hand side of (C2) increases from $\frac{1}{2}(1 - T_2/T_1)$ to $\bar{\eta}_C = (T_1 - T_2)/(T_1 + T_2)$ when $\delta s_2/\delta s_1$ decreases from ∞ to 0, but now $\bar{\eta}_C$ is not an effective upper bound of γ_S, since in practice $\delta s_2/\delta s_1$ cannot vanish, nor vary independently of the other parameters. In fact, as an example, let us consider the symmetric case, where the coefficients K_1 and K_2 of the Fourier law (5) are equal,

$$K_1 = K_2 \equiv K.$$

Then it is seen from (26) and (40) that $\gamma_S = 1/2$ (if the duration of the adiabatic phases is neglected). From (15) and the maximization condition (23), it is found that $\delta s_2/\delta s_1 = (T_1/T_2)^{1/2}$, and (C1) implies that

$$\gamma_C = 1 - (T_2/T_1)^{1/2} = \eta_{CA},$$

in agreement with (35).

This result is also valid for a stochastic motor if $\gamma_S \leqslant \frac{1}{2}$ and if $\delta s_2/\delta s_1 = (T_1/T_2)^{1/2}$. The last relation holds whenever the heat exchanges are governed by diffusion phenomena and obey the Fourier law, but it is not necessarily true if the energy is transferred differently, as it may happen in molecular motors or in complex biochemical phenomena.

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