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Nonequilibrium phase transition of contact processes with the Kauffman NK model

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We consider a multistate contact process (CP) in which new particles are created with probabilities that depend on the fitness of the parent particle and with mutations that occur at the time of creation. The fitness is determined by the Kauffman NK model. Using Monte Carlo simulations, we show that such an evolutional CP exhibits critical behaviors that differ from the basic CP. In addition, we present numerical results suggesting that the fitness averaged over surviving particles exhibits a maximum value at the critical point.

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I. INTRODUCTION

The contact process (CP), which was originally introduced as a prototype model for the spread of epidemics [1], is one of the best known nonequilibrium many-particle systems that exhibit a dynamical phase transition [1–4]. In a CP, a particle either creates a new particle with probability p at a randomly chosen nearest-neighbor site or annihilates with probability $1 - p$, independent of other particles. The purpose of this paper is to study the CP of particles having multistates, in which the creation probability of a new particle depends on the fitness that is a function of the multistate. In particular, we examine the CP whose fitness is described by the Kauffman NK model [5].

The Kauffman NK model has been used to study evolution in many branches of science [6–11]. Thus, we may obtain an evolutional CP (ECP) by combining a CP with the Kauffman NK model. In the NK model and its variants, the emphasis is on changes in internal states of particles. A particle represents features of a species in biological applications [8], or it represents decisions in an organization in management applications [12]. Internal states of particles change over time: species evolve, or organizations learn and change their decisions accordingly. In the NK model, the dynamics is explicitly defined to change the internal states of particles to achieve greater fitness (particles "search" for fitter states in the vicinity of their current states), with fitness defined as a function of the set of all possible particle states. Depending on the values of the model parameters N and K, the fitness function ranges from smooth to rugged, with a large number of local maximums; therefore, these model parameters control the difficulty of searching for higher levels of fitness.

In the ECP model that we introduce in this paper, the state of each particle remains constant over its lifetime, and the dynamics is achieved by forming new particles through imperfectly copying existing particles. The probability of a particle to be copied is higher if it is fitter; the definition of fitness in our proposed model is the same as in the NK model. In addition to creating new particles via copying, particles can disappear, with less fit particles being more likely to disappear than fitter particles. In terms of management applications, our model can be viewed as an extreme case of imitation and exploitation. New organizations are formed by imperfectly copying existing organizations (giving preference to more successful organizations); organizations do not search (explore) for better sets of decisions, but they exploit the sets of decisions that were adopted when they were created, leading to a discussion of exploration versus exploitation in organizational behavior [13]. Even though there is no search, there is a trend toward greater fitness, because fitter organizations are more likely to be copied and less likely to disappear (to fail).

Although extensions of the NK model involving imperfect imitation have been previously considered [14–18], we introduce a model in which imitation at an organization's inception is the only force driving the system to achieve overall better fitness. Because the ECP model is close to the basic CP, there is hope that the approaches developed to analyze the dynamics of the CP and other interacting particle systems are applicable to our model.

Similar to the CP model, in the ECP model, parameter p controls the ratio of the rate at which new particles are created via imitation to the rate at which particles disappear. By analogy with the basic CP, a question arises whether in an ECP, at a certain value of parameter p, there is a phase transition between an extinction state in which all particles eventually disappear and a survival state in which the dynamics continues indefinitely. A related question is how the average fitness of surviving particles depends on the value of p. In terms of management applications, p can be viewed as a level of subsidy provided to an organization by some central authority. Then, these questions can be restated as determining the level of subsidy that is sufficient for organizational activities to survive, and understanding the effect of this level of subsidy on organizations' ability to function effectively and efficiently. We focus on two main problems for this ECP: (1) whether a dynamical phase transition occurs, and, if it occurs, whether it is the same as that for the CP, and (2) how the averaged fitness depends on the creation probability.

In the ECP, the particle labeled by an integer i with a fitness $\phi(i)$ $(0 \leqslant \phi(i) \leqslant 1)$ creates a new particle at a randomly

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chosen nearest-neighbor site with probability $p_{\mathrm{B}} \equiv \phi(i)p$ or annihilates with probability $1 - p_{\mathrm{B}}$. Accordingly, particles with high fitness can create more particles in comparison to those with low fitness. In addition to these two actions, we add mutations into the evolution rule. Fitness is characterized by two numbers, N and K. Each state of the particle is given by a point in the N-dimensional Boolean hypercube, and the parameter K determines how richly interconnected are the parts of the system. We show that particles in an ECP extinguish for small p and survive for large p, similar to the behavior in a CP. However, the critical behavior in an ECP is significantly different from that in a CP.

The structure of the paper is as follows. In Sec. II, we specify the rule of an ECP and show the survival probability obtained by a series expansion method and a Monte Carlo simulation for $N = 1$ and $K = 0$. In Sec. III, we use a time-dependent simulation method to determine the critical point and the dynamical exponent for different combinations of N and K. In this section, we show that an ECP does not belong to the directed percolation (DP) universality class [19–23]. In Sec. IV, we show that the fitness averaged over surviving particles takes a maximum value near the critical point after a sufficiently long duration. In Sec. V, we present our conclusions and discuss the possibility for optimizing the NK model using an ECP.

II. SPECIFICATIONS OF THE MODEL

We consider an ECP on a one-dimensional lattice $\{k : k \in Z^1\}$. Each site is either empty or occupied by a particle. The state of the occupied site labeled by k is described by N bit digits $\mu_k \equiv \{\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\}$, where $\sigma_j \in \{0,1\}$. A particle at the site k creates a new particle at a vacant site at $k+1$ or $k-1$ with probability $p_{\mathrm{B}} \equiv \phi(\mu_k)p$ in a unit time interval, where $\phi(\mu_k)$ is the fitness of the particle at site k. The fitness $\phi(\mu_k)$ is defined by a summation of the local fitness. The local fitness, which is a map from a $(K+1)$-dimensional Boolean hypercube to non-negative real numbers is given as

$$\begin{aligned} f_K : \Omega_K &\equiv \{(b_0, b_1, \ldots, b_K) : b_i \in \{0,1\}\} \\ &\to \{x_0, x_1 \ldots, x_j, \ldots, x_{2^K} : x_j \in [0,1]\}, \end{aligned} \tag{1}$$

where $j \equiv \sum_{i=0}^{K} 2^i b_i$. Then, the fitness at the site k is defined as

$$\phi(\mu_k) = \frac{1}{N}\sum_{i=0}^{N-1} f_K(\sigma_{i \bmod \mathrm{N}}, \sigma_{i+1 \bmod \mathrm{N}}, \ldots, \sigma_{i+K \bmod \mathrm{N}}), \tag{2}$$

where σ_j denotes the jth component of μ_k. In this paper, we assume that f_K is assigned by selecting from a uniform probability distribution between 0 and 1. Since the fitness takes a value between 0 and 1, the value of p is restricted between 0 and 1.

To evolve the fitness, we add mutations that always occur by flipping a randomly chosen bit in one state when a new particle is created. Accordingly, the state of the new particle is different from that of the parent particle at a single bit. As stated previously, a particle annihilates with probability $1 - p_{\mathrm{B}}$ in a unit time interval. We assume that only one site is occupied by a particle at its origin at the initial time $t = 0$, and the local fitness f_K does not change over the sequence of the time step, i.e., at one trial. The dynamics in an ECP with $N = 3$ are illustrated in Fig. 1. Here, the total number of annihilation and creation events is denoted by n. If the interaction distance $K = 1$, the fitness at $t = 0$ is given by $\phi(\{0,0,1\}) = (f_1(0,0) + f_1(0,1) + f_1(1,0))/3 = (x_0 + x_1 + x_2)/3$, where x_0, x_1, and x_2 are random numbers selected between 0 and 1.

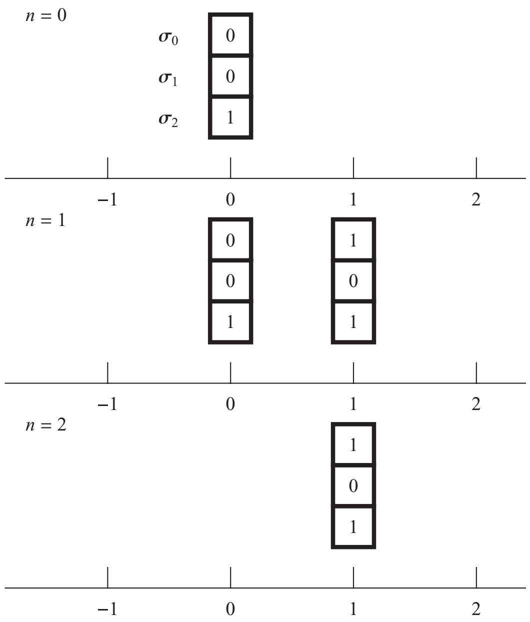

FIG. 1. Evolution rule of ECP for $N = 3$. A particle whose state is $\{0,0,1\}$ initially occupies the origin and creates a new particle on the vacant site at $k = 1$. The bit reverse occurs at $j = 0$ and the new particle is in the state $\{1,0,1\}$. An annihilation occurs at $n = 2$ and the particle at $k = 0$ is removed.

Next, we consider the probability of finding at least one particle at continuous time t of the ECP averaged over possible fitness functions. We define the survival probability $P_{N,K}(t)$ as

$$\begin{aligned} P_{N,K}(t,p) \equiv \frac{1}{2^N} \sum_{\sigma_0=\{0,1\}} \sum_{\sigma_1=\{0,1\}} \cdots \sum_{\sigma_{N-1}=\{0,1\}} \\ \times \int_0^1 dx_0 \int_0^1 dx_1 \cdots \int_0^1 dx_{2^K} P(t, f_K), \end{aligned} \tag{3}$$

where $P(t, f_K)$ is the survival probability of the ECP with fitness f_K defined in Eq. (1). Similarly, we define the survival probability under the condition that the total number of events is n as $\mathcal{P}_{N,K}(n,p)$.

We exactly calculate $\mathcal{P}_{1,0}(n,p)$ for small n. Since the state of a particle is either "0" or "1", the local function is completely described by $f_0(0) = x_0$ and $f_0(1) = x_1$, where x_0 and x_1 are randomly chosen numbers between 0 and 1. If the initial state of the particle is 0 and a new particle is created at the site $k = 1$, then the configuration of particles at $t = 1$ can be described as $\{\sigma_0, \sigma_1\} = \{0,1\}$, because the mutation will always occur at the time of the creation of a particle. If a new particle is created at the site $k = -1$, then the configuration at $t = 1$ is described as $\{\sigma_{-1}, \sigma_0\} = \{1,0\}$. The probabilities of these events are the same and are given by $x_0 p/2$. Conversely, if the initial state of the particle is 1, then the configuration at $t = 1$ is expressed by $\{\sigma_0, \sigma_1\} = \{1,0\}$ and $\{\sigma_{-1}, \sigma_0\} = \{0,1\}$ with probability $x_1 p/2$. Since the probability of the initial state is randomly chosen to be $1/2$, we find the sequence of the states $\{0,1\}$ with probability $x_0 p/4 + x_1 p/4$, and the configuration $\{1,0\}$ with the same probability. Accordingly, we obtain the survival probability as

$$\mathcal{P}_{1,0}(1,p) = \frac{1}{2} \sum_{\sigma_0=\{0,1\}} \sum_{\sigma_1=\{0,1\}} \int_0^1 dx_0 \int_0^1 dx_1 \left(x_0 \frac{p}{4} + x_1 \frac{p}{4} \right), \tag{4}$$

$$= \frac{p}{2}. \tag{5}$$

Similarly, we calculate the survival probability as a series of p for $n \leqslant 5$:

$$\mathcal{P}_{1,0}(2,p) = \frac{p}{2}, \tag{6}$$

$$\mathcal{P}_{1,0}(3,p) = \frac{7p^2}{12} - \frac{p^3}{6}, \tag{7}$$

$$\mathcal{P}_{1,0}(4,p) = \frac{7p^2}{16} + \frac{p^3}{48} - \frac{17p^4}{288}, \tag{8}$$

$$\mathcal{P}_{1,0}(5,p) = \frac{151p^3}{192} - \frac{757p^4}{1440} + \frac{33p^5}{320}. \tag{9}$$

Figure 2 shows the exact survival probabilities (solid line) up to $n = 15$ and values at $n = 10^5$ obtained by a Monte Carlo simulation, which is averaged over 10^6 trials, i.e., 10^6 combinations of initial states and local fitness functions. We find that the survival probability is zero for small p and it becomes positive for p near 0.8 for large t. Thus, we conjecture that the phase transits from an extinction state to a survival state near the critical point. The critical point is very close to that of the basic CP, $p_{\mathrm{CP}} = 0.7673$. However, the critical behavior is significantly different from that of a CP. The survival probability of a CP near the critical point can be expressed by $P_{\mathrm{CP}}(p) \propto (p - p_{\mathrm{CP}})^{0.277}$. Thus, the survival probability increases rapidly above the critical point, and the derivative coefficient of the survival probability with respect to p is discontinuous. If we use an analogy to phase transitions in equilibrium systems, the type of the phase transition in the ECP for the survival probability is classified as a second-order phase transition. In contrast to a CP, the survival probability of the ECP increases more smoothly. Below, we show that the critical exponent of the survival probability is greater than 1. Thus, the phase transition of the ECP is a higher-order transition than that of the CP, which undergoes a second-order phase transition.

III. TIME-DEPENDENT SIMULATION

To discuss the critical behavior, it is important to first determine the critical point of the ECP defined by

$$p_c(N,K) \equiv \inf\{p > 0 : \lim_{t\to\infty} P_{N,K}(p,t) > 0\}. \tag{10}$$

Although we estimated that the critical point $p_c(1,0)$ exists near $p = 0.8$, it is difficult to determine the critical point precisely from the survival probability at fixed t, because a smooth change in the survival probability is observed for a CP with a small t. Thus, we determine the critical point using a time-dependent simulation [24–26]. According to the scaling hypothesis [24], the survival probability at the critical point obeys the power law for large t given by

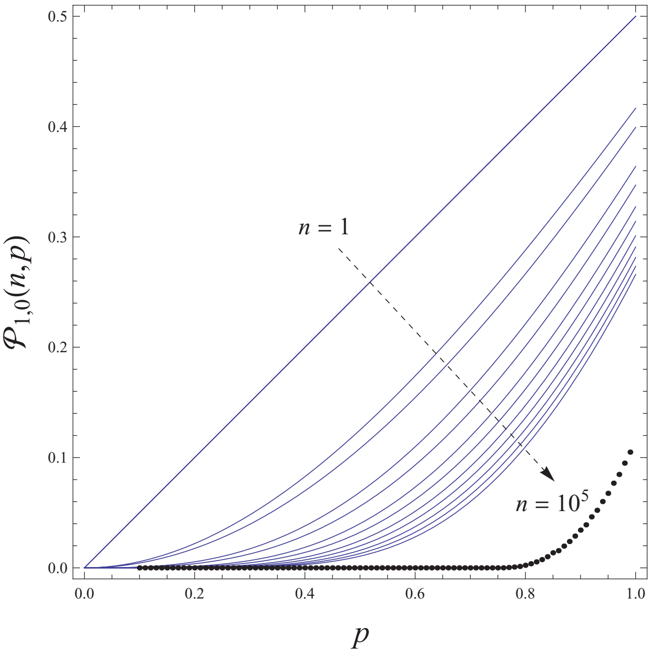

FIG. 2. (Color online) Plots of the survival probabilities for $N = 1$ and $K = 0$ as functions of p. Solid lines are the exact survival probabilities from $n = 1$ to $n = 15$. Solid circles denote the survival probabilities at $n = 10^5$ obtained by Monte Carlo simulations.

$$P_{N,K}(t, p_c(N,K)) \approx t^{-\delta}, \quad t \to \infty. \tag{11}$$

Figure 3 shows the survival probability for $N = 1$ and $K = 0$, which was obtained by the Monte Carlo simulation over 10^7 realizations, on a log-log scale near the critical point of the basic CP, $p_{\mathrm{CP}} = 0.7673$ [25]. In this simulation, time is advanced by $1/N(n)$, where $N(n)$ is the total number of particles at the nth trial. The power-law decay is clearly observed for the ECP near p_{CP} similar to that observed for the CP, but the slope is very different from the values of the CP, $\delta_{\mathrm{CP}} = 0.160$ [25]. The critical behavior of the ECP is clearly different from that of a CP; however, it is difficult to accurately estimate the critical exponent due to the large stochastic error, because the degree of freedom of the ECP is much larger than that of a CP and the survival probability decreases more rapidly in an ECP than in a CP. If we assume that the critical point of the ECP for $N = 1$ and $K = 0$ coincides with p_{CP}, the dynamical exponent δ is estimated to be $\delta = 1.4$. We show the ratio between $P_{1,0}(t,p)$ and $t^{1.4}$ in the inset of Fig. 3. The local slope in the inset for large t is nearly zero at the critical point. Below p_c, the local slope for large t is negative, and it becomes positive above p_c.

We consider the survival probability in the limit $t \to \infty$, $P_{N,K}(p) \equiv \lim_{t\to\infty} P_{N,K}(t,p)$, and estimate the critical exponent β assuming that $p_c(1,0) = p_{\mathrm{CP}}$ and the power-law behavior $P_{N,K}(p) \approx (p - p_c(N,K))^{\beta}$ near the critical point. Figure 4 shows the survival probability as a function of $p - p_c(1,0)$ on a log-log scale. We find a good linear fit with slope $\beta = 2.1$. Since the critical exponent of survival probability of the interacting particle system belonging to

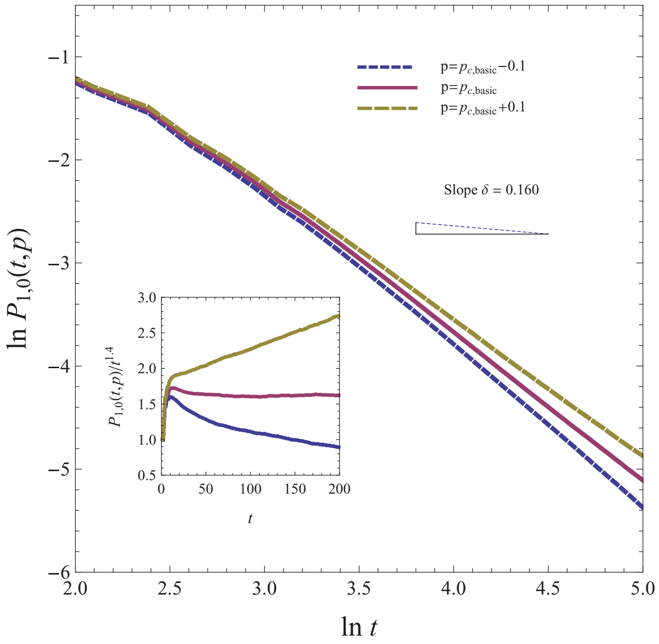

FIG. 3. (Color online) Plots of the survival probabilities for $N = 1$ and $K = 0$ as functions of t at $p = 0.7563, 0.7673$, and 0.7773. The slope corresponding to the dynamical exponent $-\delta$ is very different from the value of DP universality class. Inset: Survival probability normalized by $t^{1.4}$.

the DP universality class is estimated as $\beta = 0.277$, the ECP certainly does not belong to the DP universality class.

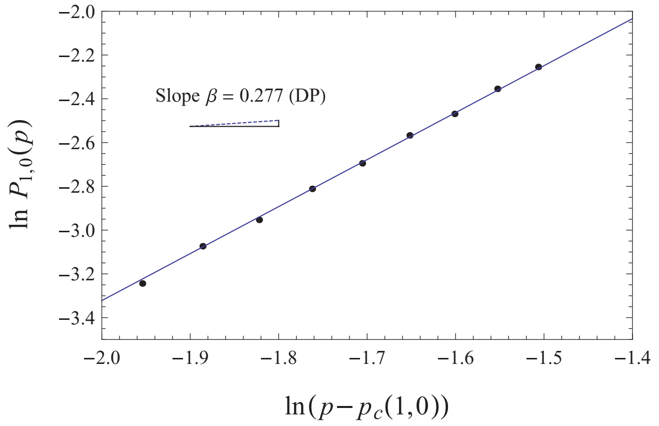

FIG. 4. (Color online) Plot of the survival probability for $N = 1$ and $K = 0$ as functions of $p - p_c(1,0)$. The slope corresponding to the critical exponent β is very different from the value of the DP universality class.

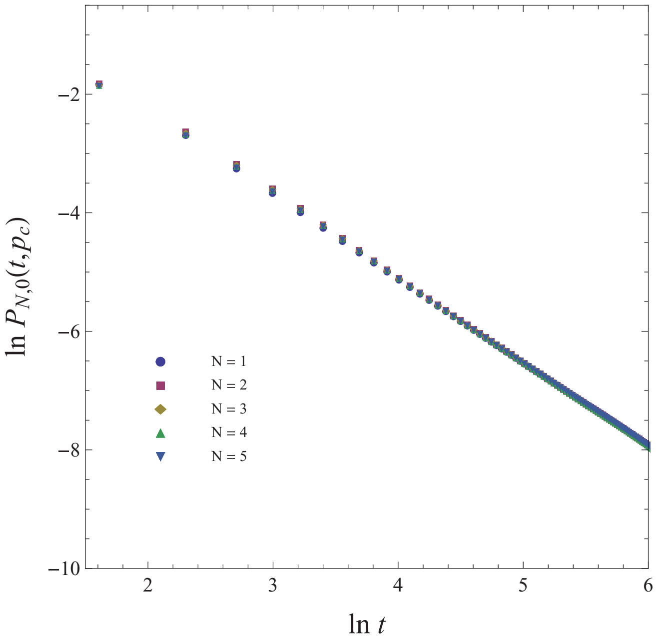

FIG. 5. (Color online) Plots of the survival probabilities obtained by fixing K to 0 and changing N from 1 to 5 at the corresponding critical points. Survival probabilities at critical points are accumulated on a single line.

We now consider the dependence of the critical behavior on N for fixed $K = 0$. Using the same method as described above, we estimate critical points for $N = 2$, 3, 4, and 5 from simulations over 10^8 realizations. The obtained critical point is the same as the critical point of the basic CP for all N, within the numerical errors. Figure 5 shows the survival probability as a function of t at the critical point on a log-log scale for $N =$ 1–5. Although the values for large t are slightly scattered, the slope (which is the dynamical exponent δ) is hardly dependent on N. We could not prove that the critical point of the ECP with $K = 0$ is equal to that of the CP independently of N. However, we briefly consider this problem for a special case. We assume that the fitness $f(\sigma)$ is an independent constant ζ of the state σ. Then, the survival probability $\tilde{P}_{N,0}(p,\zeta)$ is given by $P_{\text{CP}}(\zeta p)$. Because the survival probability of a CP near the critical point can be expressed as $C(p - p_{\text{CP}})^\beta$ with a constant C above the critical point, the survival probability of the above-mentioned ECP is given by

$$\tilde{P}_{N,0}(p,\zeta) = \begin{cases} C(p_c - \zeta p)^\beta & \zeta p > p_c, \\ 0 & \zeta p \leqslant p_c. \end{cases} \tag{12}$$

If ζ is assigned by selecting from a uniform probability distribution between 0 and 1, the survival averaged probability over ζ is given by

$$\tilde{P}_{N,0}(p) \equiv \int_0^1 \tilde{P}_{N,0}(p,\zeta) d\zeta, \tag{13}$$

$$= \begin{cases} C\frac{(p_c - p)^{1+\beta}}{(1+\beta)p} & p > p_c, \\ 0 & p \leqslant p_c. \end{cases} \tag{14}$$

Accordingly, the critical point in this case is equal to that of the CP, and the critical exponent is given by $1 + \beta$. This result suggests that the survival probability of the ECP may be expressed by a multiple integral of power functions with different zero points.

To examine the time dependence of the survival probability on the value of K, we fixed N to 5 and changed K from 0 to 4. Figure 6 shows the survival probabilities as a function of t at critical points $p_c(5,0) = 0.765$, $p_c(5,1) = 0.77$, $p_c(5,2) = 0.81$, $p_c(5,3) = 0.88$, and $p_c(5,4) = 0.93$ on a log-log scale. The dynamical exponent δ depends on K, and its values are estimated to be 1.4, 2.6, 3.1, 3.2, and 3.3 for $K = 0,1,\ldots,4$,

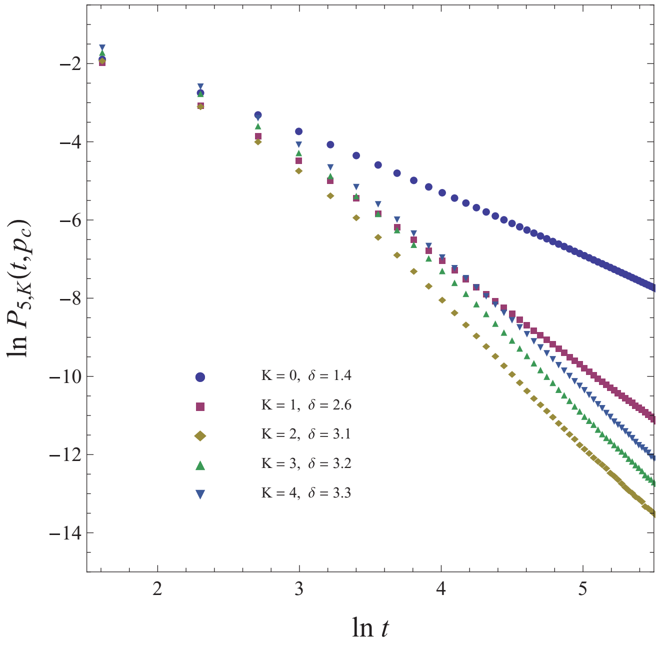

FIG. 6. (Color online) Plot of the survival probability by fixing N to 5 and changing K from 0 to 4 at the critical points. Slopes depend on K.

respectively. Interestingly, the survival probability decreases most rapidly at the middle value, i.e., for $K = 2$.

IV. CRITICAL BEHAVIOR OF FITNESS

The most significant feature of the ECP is that particles have fitness functions. We define a mean value of fitness per particle under the condition that the total number of events is n:

$$F_{N,K}(n,p) \equiv \frac{1}{2^N} \sum_{\sigma_0=\{0,1\}} \sum_{\sigma_1=\{0,1\}} \cdots \sum_{\sigma_{N-1}=\{0,1\}} \times \int_0^1 dx_0 \int_0^1 dx_1 \cdots \int_0^1 dx_{2^K} f_K \mathcal{P}(n, f_K). \quad (15)$$

For $N = 1$ and $K = 0$, the mean fitness values are given by

$$F(1,p) = \frac{7p}{12}, \quad (16)$$

$$F(2,p) = \frac{7p}{24} + \frac{7p^2}{24}, \quad (17)$$

$$F(3,p) = \frac{187p^2}{288} - \frac{29p^3}{576}, \quad (18)$$

$$F(4,p) = \frac{5p^2}{18} + \frac{12827p^3}{25920} - \frac{323p^4}{1920}, \quad (19)$$

$$F(5,p) = \frac{22127p^3}{25920} - \frac{37109p^4}{172800} - \frac{257737p^5}{9676800}. \quad (20)$$

Figure 7 shows the mean fitness (solid line) values calculated exactly up to $n = 15$ and the values obtained by the Monte Carlo simulation at $n = 10^5$, averaged over 10^6 trials. We find that the fitness exhibits a similar phase transition to the survival probability. Here, we focus on the average fitness over survival events $F^{sv}_{N,K}(n,p)$ for large n. Figure 8 shows $F^{sv}_{N,K}(n,p)$ at $n = 10^4$ for three different combinations of N and K: $(N,K) = (1,0)$, $(3,0)$, $(5,2)$. The fitness of surviving particles reaches maximum values near the critical point for all combinations; this feature is also observed for other combinations, and the value of p at which the fitness of survival particles is maximized approaches the critical point as the

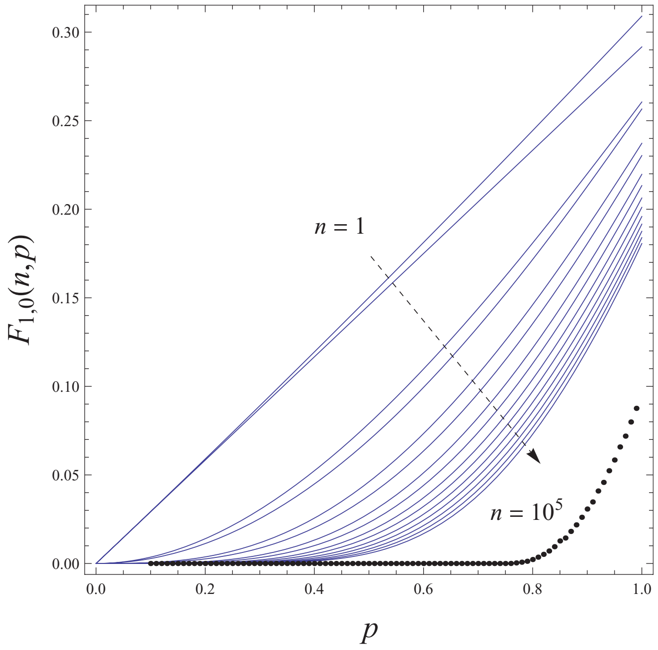

FIG. 7. (Color online) Plot of the fitness for $N = 1$ and $K = 0$ as functions of p. Solid lines are survival probabilities from $n = 1$ to $n = 15$. Solid circles denote the survival probabilities at $n = 10^5$ obtained by Monte Carlo simulations.

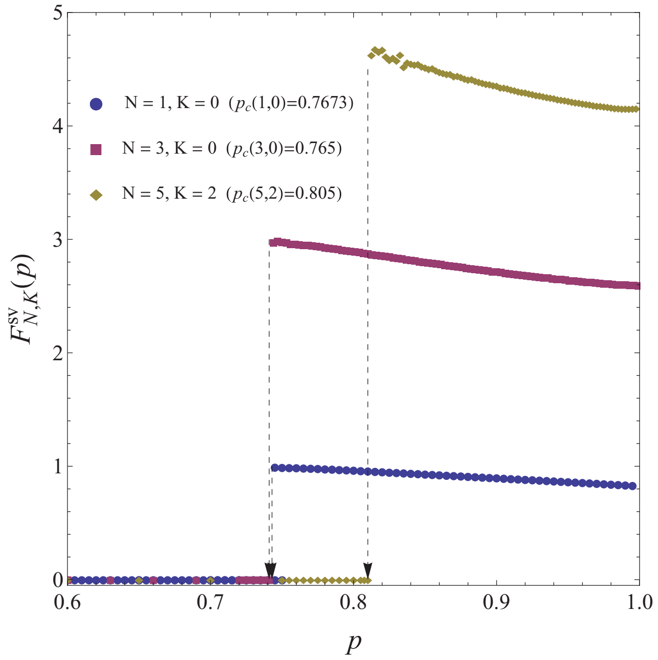

FIG. 8. (Color online) Fitness per particle averaged over the condition that the particle survives at $n = 10^4$ for three combinations $(N,K) = (1,0)$, $(3,0)$, and $(5,2)$. Conditional fitness values reach a maximum near the critical point for all combinations.

time increases. Thus, we presume that the fitness of survival particles reaches a maximum value at the critical point in the limit of $n \to \infty$.

The best way to increase the fitness is to increase the number of particles with large fitness. The creation rate of the particle is increased by increasing the value of p, and this serves to increase the number of particles with large fitness. However, this does not always increase the number of mutations, because mutations occur only when a new particle is created in our model. Creating a new particle requires a vacant site. For large p, the particles are crowded, and the opportunity for creating a new particle decreases. Accordingly, the particles with low fitness continue to exist for a long duration. On the other hand, for small p, the particles become extinct. Thus, the number of mutations probably maximizes at the critical point. It is well known that the shape of the cluster of DP is fractal at the critical point. If the cluster shape, which consists of particles in space-time, is fractal at the critical point even in an ECP, we would expect the length of the cluster surface, where new particles are created, to reach its maximum value at the critical point. Thus, it is very natural that the fitness of survival particles also reaches its maximum value at the critical point.

V. CONCLUSIONS

We have shown that an ECP exhibits rich nonequilibrium phase transitions. In comparison with a basic CP, our model is characterized by two additional parameters, N and K. The critical point depends on N and K. However, for $K = 0$, the critical point is very close to that of a basic CP. In the case when $K = 0$, the local fitness is determined only by a single bit of the state. We speculate that this particular property is the reason for the critical point to be the same as that of a basic CP within numerical errors, but we could not prove it.

The estimated exponents of an ECP are clearly different from those of the basic CP. We attempted to estimate the dynamical exponent of the number of particles, but we could not obtain accurate values owing to the very slow convergence of the local slope. However, we conjecture that it is also different from the value of the DP universality class.

By introducing the concept of fitness into the CP, we can apply the CP to wider variety of fields. In particular, since the fitness of the surviving particle reaches a maximum value at the critical point, we may apply the ECP to optimize the NK model. The NK model has been often used in the study of social sciences. In particular, the impact of the copy operation of optimizing a social system such as a corporation using the NK model was discussed [15–18]. The copying of the methods of agents who are highly adapted to an environment is very effective, and this process is regarded as the creation of a particle in an ECP. However, the fitness cannot be improved only by copying. The modification of the method is essential to improve the fitness and it is introduced as a mutation in an ECP. The fitness is quickly improved by increasing the number of agents because many different methods can be examined in parallel. However, the number of agents is limited in a real social system. Therefore, we must choose one strategy, either copy or mutation, during one step. The best choice depends on the initial condition and on time. We cannot claim that an ECP at the critical point realizes the best choice; however, the ECP may provide us with a simple way to optimize the social system using copy and trial methods.

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