

Continuous universality in nonequilibrium relaxational dynamics of O(2) symmetric systemsNiladri Sarkar^{*} and Abhik Basu[†]*Theoretical Condensed Matter Physics Division, Saha Institute of Nuclear Physics, Calcutta 700064, India*

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We elucidate a nonconserved relaxational nonequilibrium dynamics of a O(2) symmetric model. We drive the system out of equilibrium by introducing a nonzero noise cross correlation of amplitude D_\times in a stochastic Langevin description of the system, while maintaining the O(2) symmetry of the order parameter space. By performing dynamic renormalization group calculations in a field-theoretic set up, we analyze the ensuing nonequilibrium steady states and evaluate the scaling exponents near the critical point, which now depend explicitly on D_\times . Since the latter remains unrenormalized, we obtain universality classes varying continuously with D_\times . More interestingly, by changing D_\times continuously from zero, we can make our system move away from its equilibrium behavior (i.e., when $D_\times = 0$) continuously and incrementally.

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I. INTRODUCTION

The concept of universality near critical points in equilibrium systems has a long history and is theoretically well developed [1]. When equilibrium systems undergo second order phase transition at a critical point, they display universal scaling properties for thermodynamic quantities and correlation functions. These are characterized by a set of scaling exponents, which are *universal* in the sense that they depend only on the spatial dimension d and the symmetry of the order parameter (e.g., Ising, XY, etc.) [1], but not on the parameters that specify the (bare) Hamiltonian. Notable exceptions are the $2d$ XY model and the related models, where the renormalization group flow is characterized by a fixed line and consequently the scaling exponents exhibit a continuous dependence on the value of the bare stiffness parameter that appears in the model Hamiltonian. The idea of universality may be readily extended to equilibrium dynamics close to critical points, where the systems exhibit universality through the dynamic scaling exponents, which characterize the time-dependent unequal-time correlation functions. Their universality classes depend upon the presence or absence of conservation laws and the nondissipative (reactive) terms in the underlying dynamical equations [2]. For driven dissipative out of equilibrium systems with nonequilibrium steady states (NESSs), the general picture about universality is still wide open. In the recent past, attempts with significant success have been made in classifying the physics of nonequilibrium systems at long time and large length scales into universality classes. For example, the robustness of the standard universality classes in critical dynamics to detailed-balance violating perturbations are shown in Ref. [3]. In addition, only models having conserved order parameters and spatially anisotropic noise correlations exhibit novel features. In contrast, recent works demonstrate that truly nonequilibrium dynamic phenomena, whose steady states cannot be described in terms of Gibbsian distributions, are rather sensitive to all kinds of perturbations. Well-known examples include driven diffusive models [4], and fluid- and magnetohydrodynamic

turbulence [5–7]. Overall, in contrast to equilibrium systems, how one may classify the universality classes for systems out of equilibrium remains an unresolved issue to date. It is well known that for models driven out of equilibrium, not only dynamical properties but even the static properties (e.g., the static correlation functions) depend crucially on the distributions of noises which appear in a Langevin description of the model. In light of this a useful strategy to investigate nonequilibrium universality classes is to construct simple models with nonthermal noises, whose dynamics will reveal this sensitive dependence of universal properties on noise distributions in a systematic manner.

In this paper we examine the particular issue of universality in nonequilibrium for a simple relaxational dynamics (model A in the terminology of Ref. [2]) of an O(2)-symmetric system (equivalently, the *classical XY model*) in dimension $d = d_c - \epsilon$ where the upper critical dimension of the model $d_c = 4$. The O(2) symmetric model is a special case with $N = 2$ for the more general O(N) model. The equilibrium critical dynamics of these models are discussed in detail in Ref. [2]. The model is forced out of equilibrium by specific choices of the variances of the additive noises in the Langevin equations for the dynamical variables (see below). Phase transitions and associated universal properties at the critical point in systems with relaxational (model-A type in the language of Ref. [2]) dynamics have been shown to be remarkably robust against the introduction of various competing dynamics which are local and do not conserve the order parameter [8], including those which break the discrete symmetry of the system [9]. We show that its NESS depend sensitively on the parameters of the model. We use field-theoretic renormalization group calculation [10–14] using dimensional regularization [14] based on an ϵ -expansion [11] scheme.

Our principal results are as follows: (i) As a temperaturelike variable in the model (see below) is lowered, our model undergoes a phase transition from a *high temperature* paramagnetic disordered phase to a *low temperature* ferromagnetic ordered phase undergoing a second order phase transition at a nonequilibrium critical point. (ii) Universal scaling behavior near the critical point determined by a set of standard scaling exponents characterizing the correlation and the response functions that depend explicitly on the magnitude of the noise cross correlations; in effect we obtain a continuous

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universality parametrized by the noise cross correlations, the latter being a marginal operator in the model. The remainder of the paper is organized as follows: In Sec. II we set up our continuum O(2) symmetric dynamical model for a nonconserved order parameter to study its universal properties near the critical point. We introduce noises which break the fluctuation-dissipation-theorem (FDT) [15], and thus drive the system out of equilibrium, but keep the rotational invariance in the order parameter space unbroken. In the next section, Sec. III, we set up the field-theoretic formulation for our model in terms of a path integral description. We use a diagrammatic perturbation theory and calculate fluctuation corrections to different vertex functions up to the two-loop order. We then use a minimal subtraction scheme to calculate different critical exponents within an ϵ expansion. In Sec. IV we summarize and discuss the implications of our results.

II. MODEL EQUATIONS

In this section we set up our model equations to describe a simple nonequilibrium generalization of the relaxational (model A) dynamics in the overdamped limit of a nonconserved O(2) symmetric order parameter. The equilibrium characteristics of this dynamical model have been extensively discussed in the literature; see, e.g., Ref. [2]. We consider a second order phase transition described by a vector order parameter ϕ_i , $i = 1, 2$. As we furthermore assume isotropy in order parameter space, the static critical properties are described by an O(2)-symmetric ϕ^4 -type Landau-Ginzburg-Wilson free energy functional in d space dimensions,

$$F[\phi_i] = \int d^d x \left[\frac{\tau}{2} (\phi_1^2 + \phi_2^2) + \frac{1}{2} \{ (\nabla \phi_1)^2 + (\nabla \phi_2)^2 \} + \frac{u}{4!} (\phi_1^2 + \phi_2^2)^2 \right], \quad (1)$$

where $\tau = (T - T_c)/T_c$ is the bare relative distance from the mean-field critical temperature T_c and $u > 0$ is a (bare) coupling constant. The free energy functional F is manifestly rotation invariant in the order parameter space. This F determines the equilibrium probability distribution for ϕ_i . Free energy functional F allows us to compute any of the two (independent) critical exponents, e.g., the anomalous dimension η and the correlation length exponent ν , by means of renormalization group procedure, based on diagrammatic perturbation theory with respect to the static nonlinear coupling u within a systematic expansion in terms of $\epsilon = 4 - d$ about the static upper critical dimension $d_c = 4$. These exponents have well-defined physical meaning. For example, the exponent η characterizes how the order parameter correlation function at criticality decays in a spatial power-law fashion, $\langle \phi_i(\mathbf{r}) \phi_j(\mathbf{r}') \rangle \propto 1/|\mathbf{r} - \mathbf{r}'|^{d-2+\eta} \delta_{ij}$, or equivalently of the static susceptibility $\chi(\mathbf{q}) \propto 1/q^{2-\eta}$ where \mathbf{q} is a wave vector, and the exponent ν describes how the correlation length ξ diverges as the renormalized critical temperature T_c is approached, $\xi \propto |T - T_c|^{-\nu}$. Further, the fluctuation-corrected true transition temperature T_c is smaller as compared to the mean-field critical temperature T_{0c} , i.e., $\tau_{0c} = T_c - T_{0c} < 0$.

In contrast to equilibrium systems, for systems out of equilibrium, there is no detailed balance and even the static

quantities must be calculated from the underlying dynamics directly. Dynamics of descriptions of such systems in terms of continuum degrees of freedom are often based on stochastically driven Langevin equations of motion for the relevant dynamical degrees of freedom. For Langevin equations describing processes relaxing toward a thermal equilibrium state the correlation functions of a given degree of freedom and the corresponding susceptibility are connected through the FDT, which in turn fixes specific relations between the variances of the noises and the diffusivities. For example, the nonconserved relaxational (model A) dynamics for a vector order parameter ϕ_i is given by

$$\frac{\partial \phi_i}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi_i} + g_i, \quad (2)$$

where $i = 1, 2$; F is given by Eq. (1), Γ is a kinetic coefficient, and g_i are temporally δ -correlated zero-mean Gaussian stochastic noises with specified variances. Assuming spatial translational invariance we can write generally

$$\langle g_i(\mathbf{q}, t) g_j(-\mathbf{q}, 0) \rangle = 2D_{ij}(\mathbf{q}) \delta(t). \quad (3)$$

If we now set $D_{ij}(\mathbf{k}, t) = 2K_B T \Gamma \delta_{ij}$, where K_B is the Boltzmann constant and T is the temperature, then the FDT is obeyed and the corresponding Fokker-Planck equation admits a steady-state equilibrium solution $P_{eq} \sim \exp[-F/K_B T]$. In contrast, in nonequilibrium situations there are no general relations linking the noise variance and the kinetic coefficients and the FDT is broken. Since noises in a Langevin description describe the effects of the environment (e.g., thermal baths), such nonequilibrium noises reflect external drives. What are the simplest choices of the noise variances which explicitly break the FDT, without having to break the O(2) symmetry? One possible way to do that is to introduce two different noise strengths in the noise correlation matrix and break the FDT. This can be realized by the choice

$$\begin{aligned} \langle g_1(\mathbf{q}, t) g_1(-\mathbf{q}, 0) \rangle &= 2\Gamma D_1 \delta(t), \\ \langle g_2(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle &= 2\Gamma D_2 \delta(t), \\ \langle g_1(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle &= 0. \end{aligned} \quad (4)$$

Such a choice as above will certainly break the FDT but unfortunately will break the O(2) symmetry of the ensuing dynamics as well. Reference [16] investigated nonequilibrium critical properties of O(n)-symmetric models with reversible mode-coupling terms. Specifically, a variant of the model of Sasvári, Schwabl, and Szépfalussy (SSS) is studied, where violation of detailed balance is incorporated by allowing the order parameter and the dynamically coupled conserved quantities to be governed by heat baths of different temperatures. They, however, find that upon approaching the critical point detailed balance is restored, and the equilibrium static and dynamic critical properties are recovered. Yet another option is to couple the system with the corresponding conserved angular momentum and introduce dynamical anisotropy in the noise for the conserved quantities, i.e., by constraining their diffusive dynamics to be at different temperatures T_{\parallel} and T_{\perp} in d_{\parallel} - and d_{\perp} -dimensional subspaces, respectively; see Ref. [17] for detailed calculation for the SSS model for planar ferro- and isotropic antiferromagnets. Reference [17] showed the equilibrium fixed point (with isotropic noise) to

be stable with respect to these nonequilibrium perturbations, and the familiar equilibrium exponents therefore describe the asymptotic static and dynamic critical behavior. Novel critical features are only found in extreme limits, where the ratio of the effective noise temperatures T_{\parallel}/T_{\perp} is either zero or infinite. In a similar study, Ref. [18] discussed nonequilibrium dynamics in a liquid-gas model with reversible mode couplings. The model is driven out of equilibrium by introducing different temperatures for different dynamical variables, or, by having anisotropic noises. However, no new genuine nonequilibrium stable fixed point is found (within one-loop calculations). Similar approaches to nonequilibrium critical dynamics of the relaxational models C and D (in the language of Ref. [2]) are discussed in Ref. [19] and involve coupling a nonconserved and conserved order parameter, respectively, with a conserved density, where the order parameter and density fields are in contact with heat baths at different temperatures. Within a one-loop calculation it finds, in certain cases, *continuously varying* static and dynamic critical exponents, as a function of a dimensionless nonequilibrium parameter in the model. An alternative route to violation of detailed balance in the simple model A-type relaxational dynamics for the order parameter ϕ_i is to introduce nonzero noise cross correlations which will make the noise matrix off-diagonal. This will break the FDT as the noise matrix is then not proportional to the kinetic coefficient matrix (which is proportional to the unit matrix in the present case). We take cross noise strengths as $\hat{D}(\mathbf{q})$. We write

$$\begin{aligned} \langle g_1(\mathbf{q}, t) g_1(-\mathbf{q}, 0) \rangle &= \langle g_2(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle = 2D\Gamma\delta(t), \\ \langle g_1(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle &= 2\hat{D}(\mathbf{q})\Gamma\delta(t). \end{aligned} \quad (5)$$

In general the function $\hat{D}(\mathbf{q})$ is a complex function of wave vector \mathbf{q} .

The form of the function $\hat{D}(\mathbf{q})$ may be restricted by demanding rotational invariance of the noise variance matrix [equivalently by demanding O(2) symmetry of the dynamics]. Under a rotation by an arbitrary angle θ in the order parameter space the noise variance matrix transforms to

$$N' = \Gamma \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} D & \hat{D} \\ \hat{D}^* & D \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad (6)$$

where the noise variance matrix before rotation is

$$N = \Gamma \begin{pmatrix} D & \hat{D} \\ \hat{D}^* & D \end{pmatrix}. \quad (7)$$

Now we demand $N = N'$ due to rotational invariance. This after a simple algebra then yields that the noise cross correlation amplitude should be fully imaginary or $\hat{D}(\mathbf{q}) = -\hat{D}^*(\mathbf{q})$. Since in the real space $\hat{D}(\mathbf{r})$ must be a real function, we find $\hat{D}(\mathbf{q})$ must be an odd function of \mathbf{q} . In order for the noise cross correlation to have the same naïve dimension as D [so that both D and \hat{D} are equally *relevant* in a renormalization group (RG) sense], we set $\hat{D}(\mathbf{q})\hat{D}(\mathbf{q}) = D_{\times}^2$ where D_{\times}^2 is a constant (and has the same dimension as D^2). We henceforth replace $\hat{D}(\mathbf{q})$ by $i\hat{D}(\mathbf{q})$ where $\hat{D}(\mathbf{q})$ is now completely real. This reflects the fully imaginary nature of

the cross correlation explicitly. Thus the explicit forms of the two equations of motion for ϕ_1 and ϕ_2 are

$$\begin{aligned} \frac{1}{\Gamma} \frac{\partial \phi_1}{\partial t} &= -\tau\phi_1 + c\nabla^2\phi_1 - \frac{u}{3!}\phi_1^3 - \frac{u}{3!}\phi_1\phi_2^2 + \frac{g_1}{\Gamma}, \\ \frac{1}{\Gamma} \frac{\partial \phi_2}{\partial t} &= -\tau\phi_2 + c\nabla^2\phi_2 - \frac{u}{3!}\phi_2^3 - \frac{u}{3!}\phi_2\phi_1^2 + \frac{g_2}{\Gamma}, \end{aligned} \quad (8)$$

complemented by the noise variances as below:

$$\begin{aligned} \langle g_1(\mathbf{q}, t) g_1(-\mathbf{q}, 0) \rangle &= \langle g_2(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle = 2D\Gamma\delta(t), \\ \langle g_1(\mathbf{q}, t) g_2(-\mathbf{q}, 0) \rangle &= 2i\hat{D}(\mathbf{q})\Gamma\delta(t). \end{aligned} \quad (9)$$

One may in addition consider including a conserved angular momentum as a slow variable in the problem (see, e.g., model E in Ref. [2]). We do not do that here for simplicity. Model equations (8) suffice for our purposes of exploring nonuniversal features in a simple setting. Are there limits on the value of D_{\times} in this model? To obtain that we demand the noise variance matrix to have eigenvalues which are real positive or zero. The eigenvalues concerned are $D \pm D_{\times}$. Thus $|D_{\times}| \leq D$, or in terms of a dimensionless number $N_{\times} = (D_{\times}/D)^2$, $N_{\times} \leq 1$. In the subsequent calculations we will find that N_{\times} enters into the expressions of different scaling exponents explicitly.

Equations of motion (8) are written in an O(2) invariant representation. Using the equivalence between O(2) and U(1) representations, one may write an equivalent U(1) representation of the dynamics. The free energy in the U(1) representation takes the form

$$F_U[\psi\psi^*] = \int d^d x \left[\tau\psi\psi^* + (\nabla\psi)(\nabla\psi^*) + \frac{u}{3!}(\psi\psi^*)^2 \right], \quad (10)$$

where complex fields $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$; ψ^* is the complex conjugate of ψ . The corresponding Langevin equations of motion in the overdamped limit are given by

$$\frac{\partial \psi}{\partial t} = -\Gamma \frac{\delta F_U}{\delta \psi^*} + \xi, \quad (11)$$

where zero-mean Gaussian distributed complex noise ξ has the following correlations in the Fourier space:

$$\begin{aligned} \langle \xi(\mathbf{q}, t) \xi(-\mathbf{q}, 0) \rangle &= 0 = \langle \xi^*(\mathbf{q}, t) \xi^*(-\mathbf{q}, 0) \rangle, \\ \langle \xi(\mathbf{q}, t) \xi^*(-\mathbf{q}, 0) \rangle &= 2D\Gamma\delta(t) + 2i\hat{D}(\mathbf{q})\Gamma\delta(t). \end{aligned} \quad (12)$$

Thus introduction of noise cross correlations in the O(2) description is equivalent to adding an imaginary and odd function of \mathbf{q} in the variance $\langle \xi\xi^* \rangle$. Before we embark upon detailed calculation let us consider possible physical (microscopic) realizations of our continuum model in terms of stochastic lattice-gas models. However, what we discuss below does not fully and precisely define a microscopic model, but rather outlines broad features that an eventual appropriate microscopic lattice-gas type realization should possess. Consider a system of XY [O(2) spins] either on a (hypercubic) lattice or a continuum in d dimensions, interacting with an additional mobile species in the system, which diffuses randomly, undergoing symmetric exclusion process (SEP) to any of the nearest sites, if vacant. A simple model of interaction between these two species could be where each diffusing particle carries an XY spin attached to it, and

the nearest-neighbor exchange coupling J_{ij} that defines the XY model is related to the local particle density $n_i(t)$ at site i via J_{ij} being a function of $n_i(t)n_j(t)$. Next, noting that the microscopic dynamics of both the spins and particles are stochastic, characterized by two sets of random numbers $\tilde{g}_{1i}(t)$ and $\tilde{g}_{2i}(t)$, respectively, we impose that \tilde{g}_{1i} and \tilde{g}_{2i} are cross-correlated, with the cross-correlation function being of the form (in the continuum limit) $A\delta(\mathbf{x}_1 - \mathbf{x}_2) + B(\mathbf{x}_1 - \mathbf{x}_2)$, where \mathbf{x}_1 and \mathbf{x}_2 are two points in the lattice, A is a numerical constant and $B(\mathbf{x})$ is an odd function of position \mathbf{x} having the same dimension as $\delta(\mathbf{x})$. The presence of the odd function $B(\mathbf{x}_1 - \mathbf{x}_2)$ in the joint probability ensures a lack of reflection invariance of the underlying stochastic microscopic dynamics. Thus the measured quantities (e.g., correlation functions of appropriate densities) should reflect this lack of reflection invariance. At this level, the dynamics for the additional species is clearly conserving. This implies there will be no time scale present on which the additional variables of the mobile species can be treated as fast and eliminated to yield an effective equation of motion for the XY spins alone with the effects of the diffusing species buried in the additive noises in the effective spin equations. However, if particle nonconservation is introduced, e.g., via evaporation-deposition effects, the local particle density dynamics will be fast and then may be eliminated to produce an effective spin dynamics. Since the noises in such effective theories contain information about the already eliminated fast degrees of freedom (in this case the local diffusing particle density), there will be nonzero noise cross correlations of specified structures as above, due to the particular chosen structure of the underlying reflection invariance breaking microscopic dynamics. Alternatively, one may introduce the driving as a temporally δ -correlated fluctuating magnetic field $\mathbf{h}(\mathbf{x}, t) = (h_x, h_y)$ with h_x and h_y having short ranged spatial correlations as in Eq. (9). In both cases, noise cross correlations of appropriate structures will be generated in the effective Langevin description. With this short background in mind, let us now investigate the universal scaling properties of the model described by the Langevin equations (8) together with the noise variances (9). The presence of nonlinear terms in Eqs. (8) rules out exact solutions, and we resort to perturbative calculations that we discuss below.

III. NONEQUILIBRIUM STEADY STATES

Let us first consider the high temperature phase of the system. At high temperature with $\tau > 0$, the system is in the paramagnetic phase, i.e., $\langle \phi_i(\mathbf{x}, t) \rangle = 0$, where $\langle \dots \rangle$ means averaging over the noise distributions. The correlation length ξ remains finite for all $\tau > 0$. The only effect of the noise cross correlations is to make the cross-correlation function $\langle \phi_1(\mathbf{x}, t)\phi_2(0, 0) \rangle$ nonzero with a finite correlation length ξ . Further, as in equilibrium critical dynamics, the paramagnetic phase is linearly stable and the fluctuations have a finite lifetime for all wave vectors. Nevertheless, the FDT is violated for all $\tau > 0$ due to the noise cross correlations.

Near the critical point, the system becomes scale invariant and the correlation length ξ diverges, leading to the emerging macroscopic physics near the critical point being vastly different from the paramagnetic phase. A quantitative description

of the nature of correlations near the critical point requires the principles and formalisms of the dynamic renormalization group (DRG), which we execute here by using a field-theoretic framework. Detailed discussions of the technical aspect of field-theoretic DRG calculations are well documented in the literature; see, e.g., Refs. [10,20]. In order to set up the background let us examine the linearized version of the model equations (8) together with the noise correlations (5) at $\tau = 0$ by dropping all the nonlinear terms ($u = 0$). The system remains $O(2)$ invariant, but the FDT is already broken at this linear level due to the noise cross correlations. Obviously, the field correlations from the linearized model equations can be exactly calculated. This linear theory, in the critical region, defined by $\tau = 0$, is massless resulting in divergent long wavelength fluctuations, as can be seen by explicit calculations of the correlation functions $C_{ij} = \langle \phi_i(\mathbf{x}, t)\phi_j(0, 0) \rangle$, $i = 1, 2$, which may be written down in a scaling form at the critical point $\tau = 0$:

$$C_{11}(\mathbf{x}, t) = C_{22}(\mathbf{x}, t) = x^{2-d} f(t/x^z), \quad (13)$$

where f is an analytic function of its argument. The cross-correlation function C_{12} (and by symmetry C_{21}) displays the same scaling form, but with a different amplitude, and it is an odd function of \mathbf{x} .

What is the nature of these diverging fluctuations when the nonlinear terms are present ($u > 0$)? The presence of nonlinear terms no longer allows for exact solutions, in contrast to the linearized theory. However, this question may be systematically addressed via standard implementation of DRG procedure, based on a perturbative expansion in the *small* coupling u about the linear theory. The perturbative corrections to the correlation function may be equivalently viewed as arising from modifications (renormalization) of the parameters τ, u, Γ, D and the dynamical fields ϕ_1 and ϕ_2 . Renormalizability of the theory ensures that correlator C_{ij} will retain scaling forms similar to Eq. (13) with exponents different from those appearing in Eq. (13) and new scaling functions at (renormalized) $\tau = 0$:

$$C_{11}(\mathbf{x}, t) = C_{22}(\mathbf{x}, t) = x^{2-d-\eta} f_s(t/x^z), \quad (14)$$

where η and z are the anomalous dimension and dynamic exponents, respectively, and f_s is a new scaling function [21]. For the linear theory described above, $\eta = 0$ and $z = 2$. The nonlinear coupling u is expected to change these exponents for the linear theory. The equilibrium critical dynamics of several nonlinear problems have been described in Ref. [2]. In our subsequent analysis below we assume renormalizability and justify it *postfacto* by a low order (up to two-loop) perturbation theory.

Operationally, the DRG procedure is conveniently performed in terms of a path integral description based on a dynamic generating functional, which is to be constructed out of the Langevin equations (8) and the corresponding noise variances given by Eq. (9) following standard procedures [13,22]. The dynamic generating functional for the present model is given by

$$\mathcal{Z} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}\hat{\phi}_1 \mathcal{D}\hat{\phi}_2 \exp \left[-\frac{D}{\Gamma} \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_1 \hat{\phi}_1 - \frac{D}{\Gamma} \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_2 \hat{\phi}_2 - i \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_1 \frac{\hat{D}_k}{\Gamma} \hat{\phi}_2 \right]$$

$$\begin{aligned} & \times \exp \left[i \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_1 \left\{ \frac{1}{\Gamma} \partial_t \phi_1 + \frac{\delta F}{\delta \phi_1} \right\} \right] \\ & \times \exp \left[i \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_2 \left\{ \frac{1}{\Gamma} \partial_t \phi_2 + \frac{\delta F}{\delta \phi_2} \right\} \right], \end{aligned} \quad (15)$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are conjugate response fields corresponding to the order parameter fields ϕ_1 and ϕ_2 used to obtain the noise-averaged generating functional \mathcal{Z} . Clearly, with our choice of the noise cross correlations, the generating functional respects the $O(2)$ symmetry of the underlying dynamics, but explicitly breaks the FDT. It now remains to be seen whether this breakdown of the FDT remains valid even for the *effective, renormalized* version of this theory or it is restored in the long wavelength limit. In the above we have used the Ito prescription [23] while writing down the generating functional (15).

The perturbative calculational framework begins with the construction of the perturbation expansion of one-particle irreducible Feynman diagrams for all possible correlations and the corresponding vertex functions constructed out of the fields ϕ_1 , $\hat{\phi}_1$, ϕ_2 , and $\hat{\phi}_2$. Such a perturbation expansion is meaningful only when the coupling u is small. This is generally accomplished using the standard DRG procedure, which involves an order by order expansion about $d_c - d$, where d_c is the upper critical dimension and d is the physical space dimension. As a result, the renormalized coupling u_R flows to small values, of order $\epsilon = d_c - d$. As for equilibrium field theories, a straightforward scaling analysis yields that $d_c = 4$ for this model: We scale $x \rightarrow b_x x$ and $t \rightarrow b_t t$ in the action, where $b_x > 1$ and $b_t > 1$ are arbitrary parameters, and find out how the other quantities scale to maintain scale invariance. It is seen from the bare equation, leaving aside the nonlinear terms, that we must have $b_t = b_x^2 = b^2$ in order to have dynamical scaling, where we have taken $b_x = b$. For the action to remain invariant, the fields ϕ_1 and ϕ_2 pick up a canonical dimension $d/2 - 1$ and the coupling constant u a dimension $4 - d$. Hence the critical dimension at which the coupling constant u becomes dimensionless is $d_c = 4$. This yields that the perturbation theory will be infrared (IR)

singular for $d \leq 4$, and consequently the system will show nontrivial critical behavior in that regime, while for $d \geq 4$ the perturbation theory contains ultraviolet (UV) divergences, and the (static) mean-field exponents together with dynamic exponent $z = 2$ (dynamic exponent of the linearized theory) will describe the system at the critical point. To make the field theory UV renormalized it is needed to introduce the multiplicative renormalization constants in order to render all the nonvanishing two- and four-point functions finite. Within the DRG procedure this is achieved by demanding the renormalized vertex functions in the theory, or their appropriate momentum and frequency derivatives, to be finite when the corresponding loop integrals representing fluctuation corrections are considered as functions of conveniently chosen frequency and momentum, well outside the IR regime. In order to compute the associated one- and two-loop momentum integrals we employ the dimensional regularization scheme and choose $\tau = \mu^2$ as our normalization point, where μ is an intrinsic momentum scale of the renormalized theory. From the renormalization constants (Z factors) that render the underlying field theory finite in the ultraviolet (UV), one may then derive the RG flow equation, which describes how correlation functions change under scale transformations. Since the theory becomes scale invariant in the vicinity of a critical point (or an RG fixed point), one may employ the information previously gained about the UV behavior to access the physically interesting power laws governing the infrared (IR) regime at the critical point ($\tau \propto T - T_c \rightarrow 0$) for long wavelengths (wave vector $\mathbf{q} \rightarrow 0$) and low frequencies ($\omega \rightarrow 0$). The scaling behavior of the correlation or vertex functions may be extracted by finding their dependence on μ by using the RG equation.

Ward identities due to the rotational invariance of the model (in the order parameter space) ensures the following exact relations between different vertex functions:

$$\Gamma_{\hat{\phi}_1 \phi_1}(\mathbf{k}, \omega) = \Gamma_{\hat{\phi}_2 \phi_2}(\mathbf{k}, \omega) = \Gamma_{11}(\mathbf{k}, \omega), \quad (16)$$

$$\Gamma_{\hat{\phi}_1 \hat{\phi}_1}(\mathbf{k}, \omega) = \Gamma_{\hat{\phi}_2 \hat{\phi}_2}(\mathbf{k}, \omega) = \Gamma_{20}(\mathbf{k}, \omega) \quad (17)$$

and

$$\begin{aligned} & \Gamma_{\hat{\phi}_1 \phi_1 \phi_1 \phi_1}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3) \\ & = \Gamma_{\hat{\phi}_2 \phi_2 \phi_2 \phi_2}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3) \\ & = 2\Gamma_{\hat{\phi}_1 \phi_1 \phi_1 \phi_2}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3) \\ & = 2\Gamma_{\hat{\phi}_2 \phi_2 \phi_2 \phi_1}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3) \\ & = \Gamma_{13}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3). \end{aligned} \quad (18)$$

Thus in the present model, the only UV-divergent two- and four-point vertex functions which require multiplicative renormalization are (i) $\partial_\omega \Gamma_{11}(\mathbf{k}, \omega)$, (ii) $\partial_{k^2} \Gamma_{11}(\mathbf{k}, \omega)$, (iii) $\Gamma_{11}(\mathbf{k}, \omega)$, (iv) $\Gamma_{20}(\mathbf{k}, \omega)$, and (v) $\Gamma_{13}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, -\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3)$. Each of them is to be rendered finite through multiplicative

renormalization by means of introducing a Z factor. Thus there are five Z factors in total. However, there are four parameters (Γ, D, τ, u) and two fields ($\hat{\phi}_i, \phi_i, i = 1 \text{ or } 2$), thus six altogether, available for renormalization; thus this leaves us at liberty to choose one of the renormalization constants in a convenient manner.

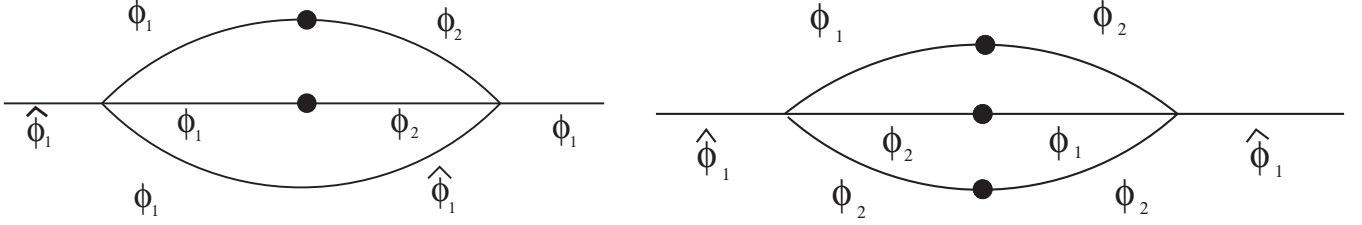


FIG. 1. Representative two-loop diagrams coming from nonzero noise cross correlations contributing to $\Sigma_G(k, \omega)$ (left) and $\Sigma_D(k, \omega)$ (right). A line with a filled circle represents a correlation function, a line without any filled circle represents a propagator. We do not show all the diagrams here.

The renormalized kinetic coefficient Γ_R , noise strength D_R , mass τ_R , and coupling constant u_R are defined in terms of the above vertex functions as

$$\begin{aligned} \partial_\omega \Gamma_{11}(0,0) &\equiv \frac{i}{\Gamma_R}, & \partial_{k^2} \Gamma_{11}(0,0) &\equiv 1, & \Gamma_{11}(0,0) &\equiv \tau_R, \\ \Gamma_{20}(0,0) &\equiv -\frac{2D_R}{\Gamma_R}, & \Gamma_{13}(\mathbf{k}_i = 0, \omega_i = 0) &\equiv u_R. \end{aligned} \quad (19)$$

The above definitions of the renormalized parameters allow us to calculate different renormalization Z factors in the problem.

The perturbation theory here is constructed out of the bare propagator and correlation functions, which are to be read off from the harmonic part of the action functional. From the generating functional we get the bare propagators as

$$\langle \phi_1(\mathbf{k}, \omega) \hat{\phi}_1(-\mathbf{k}, -\omega) \rangle_0 \equiv G_1^0(k, \omega) = \frac{i}{-\frac{i\omega}{\Gamma} + \tau + k^2} = \langle \phi_2(\mathbf{k}, \omega) \hat{\phi}_2(-\mathbf{k}, -\omega) \rangle_0 = G_2^0(k, \omega) \quad (20)$$

and bare correlators as

$$\begin{aligned} \langle \phi_1(\mathbf{k}, \omega) \phi_1(-\mathbf{k}, -\omega) \rangle_0 &\equiv C_1^0(k, \omega) = \frac{2D}{\Gamma \left[\frac{\omega^2}{\Gamma^2} + (\tau + k^2)^2 \right]} = \langle \phi_2(\mathbf{k}, \omega) \phi_2(-\mathbf{k}, -\omega) \rangle_0 \equiv C_2^0(k, \omega), \\ \langle \phi_1(k, \omega) \phi_2(-k, -\omega) \rangle_0 &\equiv C_x^0(k, \omega) = \frac{2i \hat{D}(k)}{\Gamma \left[\frac{\omega^2}{\Gamma^2} + (\tau + k^2)^2 \right]}. \end{aligned} \quad (21)$$

The self-energy $\Sigma_G(\mathbf{k}, \omega)$ is formally given by the Dyson equation:

$$G_1^{-1}(k, \omega) = -\frac{i\omega}{\Gamma} + \tau + k^2 - \Sigma_G(k, \omega) = G_2^{-1}(k, \omega) = \Gamma_{11}(k, \omega). \quad (22)$$

In the same way one may define $\Sigma_D(k, \omega)$ through the relation

$$\Gamma_{20}(k, \omega) = 2D + \Sigma_D(k, \omega). \quad (23)$$

We now calculate fluctuation corrections to the relevant vertex functions. One-loop diagrammatic contributions to Γ_{13} do not receive any contribution from D_\times and are structurally identical to their equilibrium counterparts. Similarly one-loop corrections to $\Sigma_G(\mathbf{k}, \omega)$ are independent of D_\times and are identical to the corresponding equilibrium contributions. In contrast, there are additional two-loop D_\times -dependent diagrammatic corrections to $\Sigma_G(k, \omega)$ and $\Sigma_D(k, \omega)$ whose evaluations require careful consideration (see Fig. 1).

We separately consider the following contributions $\partial_\omega \Sigma_G(0,0)$, $\partial_{k^2} \Sigma_G(0,0)$, and $\Sigma_D(0,0)$. The cross-correlation contributions $\Sigma_G^\times(k, \omega)$ and $\Sigma_D^\times(k, \omega)$ to $\Sigma_G(k, \omega)$ and $\Sigma_D(k, \omega)$, respectively, which do not arise in equilibrium, are all even in $\hat{D}(\mathbf{q})$ (or in D_\times), since the model must be invariant under $D_\times \leftrightarrow -D_\times$. Such contributions to $\Sigma_G(\mathbf{k}, \omega)$ are of the form (up to numerical factor)

$$\Sigma_G^\times(k, \omega) = u^2 \left(\frac{2}{6} - \frac{1}{9} \right) \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{q}_2)}{(\tau + q_2^2)} \frac{\Gamma}{[-i\omega + \Gamma q_1^2 + \Gamma q_2^2 + \Gamma \{3\tau + (\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)^2\}]}, \quad (24)$$

and

$$\Sigma_D^\times(k=0, \omega=0) = u^2 \left(\frac{1}{3} - \frac{1}{18} \right) \frac{1}{\Gamma} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{q}_2)}{(\tau + q_2^2)} \frac{D}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2}, \quad (25)$$

where \mathbf{k} and ω are the external wave vector and frequencies, respectively. We separately need to find out the $k^0\omega$ and $k^2\omega^0$ parts of of the integral (24). We need to calculate $\frac{\partial}{\partial a}\Sigma_G^\times(k,\omega)|_{k=0,\omega=0}$ where $a = k^2, \omega$. Let us consider $a = \omega$:

$$\frac{\partial}{\partial \omega} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{q}_2)}{(\tau + q_2^2)} \frac{1}{-i\omega + \Gamma[3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2]}. \quad (26)$$

Since we are using an ϵ expansion based on a minimal subtraction scheme, we need to extract the diverging parts of Eq. (26), to be given by poles in ϵ . It is noteworthy that integral (26) has a structure very similar to and has the same logarithmic divergence (by simple power counting) as its equilibrium counterpart [i.e., when $\hat{D}(\mathbf{q})$ is replaced by D in Eq. (26)]. Clearly, the dominant contribution to it comes from $\mathbf{p} = \mathbf{q}_1 + \mathbf{q}_2 \sim 0$, which controls the critical behavior of this integral. We write (retaining only the small- \mathbf{p} contribution), up to constants and numerical factors,

$$\begin{aligned} \frac{\partial}{\partial \omega} \Sigma_G^\times(k,\omega) \Big|_{k=0,\omega=0} &\sim \frac{\partial}{\partial \omega} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{p} - \mathbf{q}_1)}{(\tau + q_2^2)} \frac{1}{-i\omega + \Gamma[3\tau + q_1^2 + q_2^2 + p^2]} \\ &\sim \frac{\partial}{\partial \omega} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_2^2)} \frac{1}{-i\omega + \Gamma[3\tau + q_1^2 + q_2^2 + p^2]} \\ &= D_\times^2 \frac{\partial}{\partial \omega} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(\tau + q_1^2)} \frac{1}{(\tau + q_2^2)} \frac{1}{-i\omega + \Gamma[3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2]} \\ &= -i \frac{1}{3\Gamma^2} D_\times^2 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(\tau + q_1^2)} \frac{1}{(\tau + q_2^2)} \frac{1}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2}, \end{aligned} \quad (27)$$

where we have replaced integral (26) by its dominant contribution coming from $\mathbf{p} \sim 0$. The last line of Eq. (27) is obtained by symmetrizing the previous line. The reduced integral (27) may now be evaluated exactly in the same way just as its equilibrium counterpart (i.e., when D_\times is replaced by D). In a similar way, the cross-correlation contribution to the anomalous dimension (again logarithmically divergent on a naive power counting basis) may be written as (up to constants and numerical factors)

$$\frac{\partial}{\partial k^2} \Sigma_G^\times(k,\omega) \Big|_{k=0,\omega=0} \sim D_\times^2 \frac{\partial}{\partial k^2} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(\tau + q_1^2)} \frac{1}{(\tau + q_2^2)} \frac{1}{\tau + (\mathbf{k} + \mathbf{q}_1 + \mathbf{q}_2)^2}, \quad (28)$$

and the cross-correlation contributions to noise strengths (25) become (up to constants and numerical factors)

$$\Sigma_D^\times(0,0) \sim DD_\times^2 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(\tau + q_1^2)} \frac{1}{(\tau + q_2^2)} \frac{1}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2}. \quad (29)$$

In the above, in our evaluations of the cross-correlation contributions $\frac{\partial}{\partial a}\Sigma_G^\times(k,\omega)|_{k=0,\omega=0}$, $a = k^2, \omega$ and $\Sigma_D^\times(0,0)$, we have picked up the dominant contribution given by $\mathbf{p} = \mathbf{q}_1 + \mathbf{q}_2 \sim 0$. Subdominant contributions are neglected and are expected to be *small* as we heuristically justify: For example, for $\mathbf{p} \gg \mathbf{q}_1$, the integrand in $\Sigma_D^\times(0,0)$ [see Eq. (29) above] is

$$\frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{q}_2)}{(\tau + q_2^2)} \frac{D}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2} \sim \frac{\hat{D}(\mathbf{q}_1)}{(\tau + q_1^2)} \frac{\hat{D}(\mathbf{p})}{\tau + p^2} \frac{D}{\tau + p^2} \frac{1}{3\tau + 2p^2}, \quad (30)$$

which can be both positive and negative since $\hat{D}(\mathbf{q}_1)$ and $\hat{D}(\mathbf{p})$ are odd functions of their arguments, and hence contributions from outside the dominant region $\mathbf{p} \sim 0$ will be small due to mutual cancellations. Our results, although backed up by heuristic arguments, nevertheless bring out remarkable features, as we shall see below. Thus after putting every diagrammatic contribution (up to two-loop order) together we obtain for $\Sigma_G(\mathbf{k},\omega)$

$$\begin{aligned} \Sigma_G(k,\omega) &= \frac{2uD}{3} \int \frac{d^d q}{(2\pi)^2} \frac{1}{\tau + q^2} + \Gamma u^2 D^2 \left(\frac{1}{2} + \frac{1}{18} + \frac{1}{9} \right) \int \frac{d^2 q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{\tau + q_1^2} \\ &\times \frac{1}{\tau + q_2^2} \frac{1}{-i\omega + \Gamma\{3\tau + q_1^2 + q_2^2 + (\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)^2\}} \\ &+ \Gamma u^2 D_\times^2 \left(\frac{2}{6} - \frac{1}{9} \right) \int \frac{d^2 q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{\tau + q_1^2} \frac{1}{\tau + q_2^2} \frac{1}{-i\omega + \Gamma\{3\tau + q_1^2 + q_2^2 + (\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)^2\}}. \end{aligned} \quad (31)$$

Similarly the two-loop contributions to $\Sigma_D(0,0)$ come out to be

$$\begin{aligned} \Sigma_D(k=0,\omega=0) &= \frac{u^2 D^3}{\Gamma} \left(\frac{1}{6} + \frac{1}{18} \right) \int \frac{d^2 q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{\tau + q_1^2} \frac{1}{\tau + q_2^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \\ &+ \frac{u^2 DD_\times^2}{\Gamma} \left(\frac{1}{3} - \frac{1}{18} \right) \int \frac{d^2 q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{\tau + q_1^2} \frac{1}{\tau + q_2^2} \frac{1}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2}. \end{aligned} \quad (32)$$

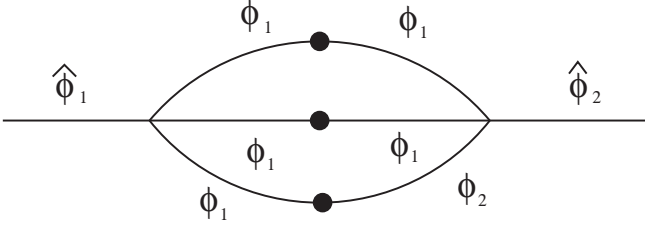


FIG. 2. A representative two-loop diagram contributing to $\Sigma_{\times}(k, \omega)$. Symbols have meanings as before.

Although formally there exists two-loop diagrammatic corrections (there are no one-loop corrections) to $\hat{D}(\mathbf{k})$, all of these vanish in the long wavelength limit due to the fact that $\hat{D}(\mathbf{k})$ is an odd function of wave vector \mathbf{k} . In Fig. 2 we consider one such two-loop diagram:

The corresponding expression $\Sigma_{\times}(k, \omega)$ is (up to constants and numerical factors)

$$\Sigma_{\times}(k, \omega) \sim \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{\hat{D}(\mathbf{q}_1)}{\tau + q_1^2} \frac{1}{\tau + q_2^2} \frac{1}{\tau + (\mathbf{q}_1 + \mathbf{q}_2)^2} \times \frac{1}{3\tau + q_1^2 + q_2^2 + (\mathbf{q}_1 + \mathbf{q}_2)^2}. \quad (33)$$

Clearly, the integral in Eq. (33) vanishes. We shall come back to this issue of nonrenormalization of $D_{\times}(\mathbf{k})$ again at the end.

Finally, to complete evaluating diagrammatic corrections we now evaluate the Γ_{13} up to one-loop order at zero external wave vector and frequency. There are no contributions from D_{\times} to the four point vertex function. We obtain

$$\Gamma_{(3,1)} = u^2 D \frac{\mu^{-\epsilon}}{2(4\pi)^d} \Gamma(\epsilon/2) \left[\frac{1}{4} + \frac{1}{36} \right]. \quad (34)$$

After evaluating all the two point and four point vertex functions we can now renormalize vertex functions $\Gamma_{(11)}(0,0)$, $\frac{\partial}{\partial \omega} \Gamma_{(11)}(\omega=0,0)$, $\frac{\partial}{\partial k^2} \Gamma_{(11)}(0,k=0)$, and $\Gamma_{(3,1)}(0,0)$. We now define renormalization Z factors for the present model. We begin by introducing the renormalized fields ϕ_i^R and $\hat{\phi}_i^R$:

$$\phi_i^R = Z^{1/2} \phi_i, \quad \hat{\phi}_i^R = \hat{Z}^{1/2} \hat{\phi}_i. \quad (35)$$

This implies that the renormalized vertex functions become

$$\Gamma_{(N,\hat{N})}^R = Z^{-N/2} \hat{Z}^{-\hat{N}/2} \Gamma_{(N,\hat{N})}. \quad (36)$$

We further define the renormalized parameters as

$$D^R = Z_D D, \quad \tau^R = Z_{\tau} \tau \mu^{-2}, \quad (37)$$

$$u^R = Z_u u A_d \mu^{d-4}, \quad \Gamma^R = Z_{\Gamma} \Gamma,$$

where μ is the scale factor introduced to make the renormalized parameters dimensionless. Here $A_d = \frac{1}{2^{d-1} \pi^{d/2}}$. Thus we get

$$\frac{\partial}{\partial \omega} \Gamma_{(11)}^R(\omega=0,0) = Z^{-1/2} \hat{Z}^{-1/2} \frac{i}{\Gamma} \left[1 + \frac{2D^2 u^2 \mu^{-2\epsilon}}{3(4\pi)^d \epsilon} \ln\left(\frac{4}{3}\right) + \frac{2D_{\times}^2 u^2 \mu^{-2\epsilon}}{9(4\pi)^d \epsilon} \ln\left(\frac{4}{3}\right) \right] \equiv \frac{i}{\Gamma^R} = \frac{i}{Z_{\Gamma} \Gamma}, \quad (38)$$

$$\frac{\partial}{\partial k^2} \Gamma_{(11)}^R(0,k=0) = i Z^{-1/2} \hat{Z}^{-1/2} \left[1 + \frac{D^2 u^2 \mu^{-2\epsilon}}{9(4\pi)^d \epsilon} + \frac{D_{\times}^2 u^2 \mu^{-2\epsilon}}{27(4\pi)^d \epsilon} \right] = i, \quad (39)$$

$$\Gamma_{(20)}^R(0,0) = -\hat{Z}^{-1} \frac{2D}{\Gamma} \left[1 + \frac{2D^2 u^2 \mu^{-2\epsilon}}{3(4\pi)^d \epsilon} \ln\left(\frac{4}{3}\right) + \frac{5D_{\times}^2 u^2 \mu^{-2\epsilon}}{6(4\pi)^d \epsilon} \ln\left(\frac{4}{3}\right) \right] = -\frac{2D^R}{\Gamma^R}, \quad (40)$$

$$\Gamma_{(3,1)}^R(0,0) = \hat{Z}^{-1/2} Z^{-3/2} \frac{u}{6} \left[1 - \frac{10}{6\epsilon} D u \mu^{-\epsilon} \right], \quad (41)$$

$$\Gamma_{(11)}^R(0,0) = Z^{-1/2} \hat{Z}^{-1/2} \tau \left[1 - \frac{4u D \mu^{-\epsilon}}{3(4\pi)^{d/2} \epsilon} - \frac{2D^2 u^2 \mu^{-2\epsilon}}{3(4\pi)^d \epsilon} \left(\frac{2}{\epsilon} + 1 \right) - \frac{2D_{\times}^2 u^2 \mu^{-2\epsilon}}{9(4\pi)^d \epsilon} \left(\frac{2}{\epsilon} + 1 \right) \right] = \tau^R, \quad (42)$$

from which we can calculate all the Z factors. We use the freedom to choose one of the Z factors freely to set $Z_D = 1$. Henceforth we set $D = 1$ for simplicity without any loss of generality. Assuming $D_{\times}^2 = N_{\times} D^2$, where N_{\times} is any dimensionless parameter, the other Z factors are obtained up to two-loop order as follows:

$$Z_{\Gamma} = 1 - \frac{1}{36} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} \left(6 \ln \frac{4}{3} - 1 \right) - \frac{1}{108} N_{\times} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} \left(6 \ln \frac{4}{3} - 1 \right), \quad (43)$$

$$\hat{Z} = 1 + \frac{1}{36} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} + \frac{11}{72} N_{\times} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} \ln \frac{4}{3} + \frac{1}{108} N_{\times} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon}, \quad (44)$$

$$Z = 1 + \frac{1}{36} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} - \frac{11}{72} N_{\times} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} \ln \frac{4}{3} + \frac{1}{108} N_{\times} \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon}, \quad (45)$$

$$Z_{\tau} = 1 - \frac{2}{3} \frac{(u A_d \mu^{-\epsilon})}{\epsilon} - \frac{1}{18} \left(\frac{7}{2} + \frac{6}{\epsilon} \right) \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon} - \frac{1}{54} N_{\times} \left(\frac{7}{2} + \frac{6}{\epsilon} \right) \frac{(u A_d \mu^{-\epsilon})^2}{\epsilon}. \quad (46)$$

Defining the Wilson's flow functions as

$$\zeta_\phi = \mu \frac{\partial}{\partial \mu} \ln Z, \quad \zeta_{\hat{Z}} = \mu \frac{\partial}{\partial \mu} \ln \hat{Z}, \quad (47)$$

$$\zeta_\Gamma = \mu \frac{\partial}{\partial \mu} \ln Z_\Gamma, \quad (48)$$

$$\zeta_D = \mu \frac{\partial}{\partial \mu} \ln Z_D, \quad (49)$$

$$\zeta_\tau = \mu \frac{\partial}{\partial \mu} \ln \frac{\tau^R}{\tau} = -2 + \mu \frac{\partial}{\partial \mu} \ln Z_\tau, \quad (50)$$

and the β function for the nonlinear coupling as

$$\beta_u = \mu \frac{\partial}{\partial \mu} u^R = u \left(-\epsilon + \frac{10}{6} u \right), \quad (51)$$

we get a stable nontrivial fixed point at $u^* = \frac{6}{10}\epsilon$ and we can evaluate the critical exponents from these flow functions at the fixed point. The flow functions at the fixed point pick up values up to order ϵ^2 as follows:

$$\zeta_\phi = -\frac{1}{50}\epsilon^2 - \frac{1}{150}N_\times \epsilon^2 + \frac{11}{100}N_\times \epsilon^2 \ln \frac{4}{3}, \quad (52)$$

$$\zeta_{\hat{Z}} = -\frac{1}{50}\epsilon^2 - \frac{1}{150}N_\times \epsilon^2 - \frac{11}{100}N_\times \epsilon^2 \ln \frac{4}{3}, \quad (53)$$

$$\zeta_\Gamma = \frac{1}{50}\epsilon^2 (6 \ln \frac{4}{3} - 1) + \frac{1}{150}N_\times \epsilon^2 (6 \ln \frac{4}{3} - 1), \quad (54)$$

$$\zeta_\tau = -2 + \frac{2}{3}\epsilon + O(\epsilon^2). \quad (55)$$

The basic renormalization group (RG) equation is derived on the basis that the unrenormalized correlation and vertex functions do not depend on the arbitrary scale μ . Hence if we hold the bare parameters D , τ , and μ fixed, we must have

$$\begin{aligned} 0 &= \mu \frac{d}{d\mu} \Big|_{D,\tau,\mu} \Gamma_{N,\hat{N}} \\ &= \mu \frac{d}{d\mu} \left[\hat{Z}^{\hat{N}/2} Z^{N/2} \Gamma_{N,\hat{N}}^R(\mu, \Gamma^R, \tau^R, u^R) \right]. \end{aligned} \quad (56)$$

As Z factors also depend on μ , the RG equation finally takes the form

$$\begin{aligned} \hat{Z}^{\hat{N}/2} Z^{N/2} \mu \left[\frac{\partial}{\partial \mu} + \hat{N}/2 \frac{\partial}{\partial \mu} \ln \hat{Z} + \frac{N}{2} \frac{\partial}{\partial \mu} \ln Z + \frac{\partial \Gamma^R}{\partial \mu} \frac{\partial}{\partial \Gamma^R} \right. \\ \left. + \frac{\partial \tau^R}{\partial \mu} \frac{\partial}{\partial \tau^R} + \frac{\partial u^R}{\partial \mu} \frac{\partial}{\partial u^R} \right] \Gamma_{N,\hat{N}}^R = 0. \end{aligned} \quad (57)$$

At the critical point we have scale invariance separately under scaling of space, time, fields, and parameters. These are determined by the momentum and frequency canonical dimensions of the fields and parameters. After proper scaling as described above in this section we have canonical dimensions of fields and parameters as

$$\begin{aligned} d_{\phi_1, \phi_2}^k &= d/2 - 1, & d_{\phi_1, \phi_2}^\omega &= 0, \\ d_{\hat{\phi}_1, \hat{\phi}_2}^k &= d/2 - 1, & d_{\hat{\phi}_1, \hat{\phi}_2}^\omega &= 1, \\ d_\Gamma^k &= -2, & d_\Gamma^\omega &= 1, \\ d_\tau^k &= 0, & d_\tau^\omega &= 0. \end{aligned} \quad (58)$$

Canonical scale invariance at the fixed point ($\beta_u^* = 0$) for the correlation function implies

$$\left[\mu \frac{\partial}{\partial \mu} - x \frac{\partial}{\partial x} - 2\Gamma^R \frac{\partial}{\partial \Gamma^R} - d_C^k \right] C^R(x, t, \mu, \Gamma^R, \tau^R) = 0$$

and

$$\left[\Gamma^R \frac{\partial}{\partial \Gamma^R} - t \frac{\partial}{\partial t} - d_C^\omega \right] C^R(x, t, \mu, \Gamma^R, \tau^R) = 0, \quad (59)$$

where d_C^k and d_C^ω are the momentum and frequency dimension of the correlation function $C(x, t)$, respectively. In this case $d_C^k = d - 2$ and $d_C^\omega = 0$. Therefore from Eq. (59) we have

$$\Gamma^R \frac{\partial}{\partial \Gamma^R} C^R(bx, b^2t, \mu, \Gamma^R, \tau^R) = t \frac{\partial}{\partial t} C^R(x, t, \mu, \Gamma^R, \tau^R). \quad (60)$$

The RG equation for the correlation function at the fixed point can be written as

$$\begin{aligned} 0 &= \mu \frac{\partial}{\partial \mu} C = \mu \frac{\partial}{\partial \mu} [Z^{-1} C^R] \\ &= \left[\mu \frac{\partial}{\partial \mu} - \zeta_\phi + \zeta_\Gamma \Gamma^R \frac{\partial}{\partial \Gamma^R} + \zeta_\tau \tau^R \frac{\partial}{\partial \tau^R} \right] \\ &\quad \times C^R(x, t, \mu, \Gamma^R, \tau^R). \end{aligned} \quad (61)$$

Combining the two separate spacial and temporal scale invariant equations (59) and using Eqs. (60) and (61) we get at the fixed point

$$\begin{aligned} \left[-x \frac{\partial}{\partial x} - \zeta_\tau \tau^R \frac{\partial}{\partial \tau^R} - (2 + \zeta_\Gamma) t \frac{\partial}{\partial t} + 2 - d - \eta \right] \\ \times C^R(x, t, \Gamma^R, \tau^R) = 0, \end{aligned} \quad (62)$$

where we have used $\eta = -\zeta_\phi$. From Eq. (62) it can be seen that at the critical point ($\tau = 0$) and equal time ($t = 0$) the correlation function should take the form

$$C(x) \sim x^{2-d-\eta}, \quad (63)$$

which gives the spatial scaling of the equal-time correlation function at the critical point. In the case of time-dependent correlation function at the critical point the scale invariant equation takes the form

$$\left[-x \frac{\partial}{\partial x} - (2 + \zeta_\Gamma) t \frac{\partial}{\partial t} + 2 - d - \eta \right] C^R(x, t, \Gamma^R) = 0. \quad (64)$$

Assuming dynamical scaling, the solution of $C(x, t)$ should be of the form $C(x, t) \sim x^{2-d-\eta} g(x^{2+\zeta_\Gamma}/t)$, which implies that

$$2 + \zeta_\Gamma = z \quad (65)$$

should be the dynamic exponent. At equal time ($t = 0$), near the critical point ($\tau^R \neq 0$), the equation for $C(x, t)$ can be written as

$$\left[-x \frac{\partial}{\partial x} - \zeta_\tau \tau^R \frac{\partial}{\partial \tau^R} + 2 - d - \eta \right] C^R(x, t, \tau^R) = 0. \quad (66)$$

This implies that the correlation function should be of the form $C(x, \tau) \sim x^{2-d-\eta} f(x^{\zeta_\tau}/\tau)$, and the correlation length exponent

$$\nu = -\frac{1}{\zeta_\tau}. \quad (67)$$

For the propagator $G = \langle \phi \hat{\phi} \rangle$, the scale invariant equations are given by

$$\left[\mu \frac{\partial}{\partial \mu} - x \frac{\partial}{\partial x} - 2\Gamma^R \frac{\partial}{\partial \Gamma^R} - d_G^k \right] G^R(x, t, \mu, \Gamma^R, \tau^R) = 0$$

and

$$\left[\Gamma^R \frac{\partial}{\partial \Gamma^R} - t \frac{\partial}{\partial t} - d_G^\omega \right] G^R(x, t, \mu, \Gamma^R, \tau^R) = 0, \quad (68)$$

where $d_G^k = d - 2$ and $d_G^\omega = 1$. From the second of Eqs. (68) it is obvious that

$$\Gamma^R \frac{\partial}{\partial \Gamma^R} G^R(x, t, \mu, \Gamma^R, \tau^R) = \left[1 + t \frac{\partial}{\partial t} \right] G^R(x, t, \mu, \Gamma^R, \tau^R). \quad (69)$$

The RG equation for the propagator at the fixed point can be written as

$$\begin{aligned} 0 &= \mu \frac{\partial}{\partial \mu} G = \mu \frac{\partial}{\partial \mu} [Z^{-1/2} \hat{Z}^{-1/2} G^R] \\ &= \left[\mu \frac{\partial}{\partial \mu} - \frac{1}{2} \zeta_\phi - \frac{1}{2} \zeta_{\hat{\phi}} + \zeta_\Gamma \Gamma^R \frac{\partial}{\partial \Gamma^R} + \zeta_\tau \tau^R \frac{\partial}{\partial \tau^R} \right] \\ &\quad \times G^R(x, t, \mu, \Gamma^R, \tau^R). \end{aligned} \quad (70)$$

Using Eqs. (70) and (69) in Eq. (68) we get

$$\begin{aligned} &\left[-x \frac{\partial}{\partial x} - \zeta_\tau \tau^R \frac{\partial}{\partial \tau^R} - (2 + \zeta_\Gamma) \left\{ 1 + t \frac{\partial}{\partial t} \right\} \right. \\ &\quad \left. + 2 - d - \frac{1}{2} \eta - \frac{1}{2} \hat{\eta} \right] G^R(x, t, \Gamma^R, \tau^R) = 0, \end{aligned} \quad (71)$$

where $\hat{\eta} = -\zeta_{\hat{\phi}}$.

From Eq. (71) the time dependent propagator at the critical point ($\tau = 0$) can be written as

$$\begin{aligned} &\left[-x \frac{\partial}{\partial x} - (2 + \zeta_\Gamma) \left\{ 1 + t \frac{\partial}{\partial t} \right\} + 2 - d - \frac{1}{2} \eta - \frac{1}{2} \hat{\eta} \right] \\ &\quad \times G^R(x, t, \Gamma^R) = 0. \end{aligned} \quad (72)$$

Assuming dynamical scaling $G(x, t)$ should be of the form $G \sim x^{2-d-\eta/2-\hat{\eta}/2-2-\zeta_\Gamma} f(\frac{x^{2+\zeta_\Gamma}}{t})$. The static susceptibility $\chi(x)$ is proportional to $\int_0^\infty dt G(x, t)$, which on integrating over time gives us

$$\chi(x) \sim x^{2-d-\eta/2-\hat{\eta}/2}. \quad (73)$$

From Eqs. (65), (63), and (67) and using Eqs. (54), (52), and (55) we get the dynamic exponent, anomalous dimension and correlation length exponent of the model to the leading order in ϵ :

$$z = 2 + \frac{1}{50} \epsilon^2 (6 \ln \frac{4}{3} - 1) + \frac{1}{150} N_\times \epsilon^2 (6 \ln \frac{4}{3} - 1), \quad (74)$$

$$\eta = \frac{1}{50} \epsilon^2 + \frac{1}{150} N_\times \epsilon^2 - \frac{11}{100} N_\times \epsilon^2 \ln \frac{4}{3}, \quad (75)$$

$$\hat{\eta} = \frac{1}{50} \epsilon^2 + \frac{1}{150} N_\times \epsilon^2 + \frac{11}{100} N_\times \epsilon^2 \ln \frac{4}{3}, \quad (76)$$

$$\frac{1}{\nu} = 2 - \frac{2}{3} \epsilon + O(\epsilon^2). \quad (77)$$

If $N_\times = 0$ we get back the equilibrium exponents as expected. Their contribution from the nonequilibrium part depends on the value of N_\times . Let us now consider the consequences of a nonzero N_\times . First of all, as is evident from the results presented above, the static susceptibility and the equal-time

correlation function *do not* display the same spatial scaling, since $\eta \neq \hat{\eta}$. This is an important evidence of breakdown of the FDT in the renormalized theory. Note that in the linearized theory breakdown of the FDT is manifested in the existence of nonzero cross-correlation functions: The correlation matrix picks up nonzero off-diagonal elements, whereas the dynamic susceptibility matrix remains diagonal. Nevertheless, all elements of the correlation and the dynamic susceptibility matrix exhibit the same scaling properties at the critical point. In contrast, in the renormalized nonlinear theory, not only is the correlation matrix off-diagonal whereas the susceptibility matrix remains diagonal, the elements of the correlation matrix scale differently from those of the susceptibility matrix. The latter result is purely a nonlinear effect. Further, the dynamic exponent for finite N_\times , $z(N_\times)$ is larger than $z(N_\times = 0)$, its value for equilibrium dynamics. Thus relaxation for finite N_\times is *slower* than for the corresponding equilibrium dynamics. The correlation length exponent ν has been calculated only up to $O(\epsilon)$ and is equal to its equilibrium value. However, as there are N_\times -dependent corrections to $\Sigma_G(0, 0)$ at the two-loop order, the value of ν is likely to be different from its equilibrium value at the two-loop order. Last, the N_\times dependence of all the scaling exponents are *continuous*—all of them vary continuously with N_\times and go over to their equilibrium values when N_\times is set to zero. Our claim of the scaling exponents varying continuously with N_\times rests on the marginality of N_\times . We have shown explicitly up to two-loop order that $\hat{D}(\mathbf{k})$ does not renormalize. Hence N_\times does not renormalize up to two-loop order and depends on the bare value of D_\times^2 . Any nonzero fluctuation corrections to $\hat{D}(\mathbf{q})$ must be an odd function of its wave vector argument. In order to have that one must have an odd number of internal cross-correlation line. Since all internal wave vectors are integrated over, such a contribution will vanish in the limit of vanishing external wave vector and frequency. Thus $\hat{D}(\mathbf{q})$ and remains unrenormalized and hence N_\times appears as a dimensionless marginal operator to any order in perturbation.

IV. SUMMARY

To summarize, we have analyzed the universal scaling properties of a nonequilibrium version of the O(2)-symmetric dynamical model near the critical point. We write down a nonconserved relaxational dynamics for the order parameter field. We have introduced cross correlations between the two additive noises in the Langevin equations, so that the FDT is immediately broken. We then show that if the cross correlation is imaginary and odd in wave vector, the underlying O(2) symmetry is still maintained. We calculate the scaling exponents of the model in a DRG framework using an ϵ -expansion scheme, where $\epsilon = 4 - d$ with 4 being the upper critical dimension of the model. We show that at the two-loop order there are diagrammatic corrections to the various two-point vertex functions in the model arising from the cross correlations. We have used heuristic arguments to extract the dominant contributions to the two-loop diagrams involving cross correlations, which have allowed us to evaluate the respective cross-correlation contributions in a simple and controlled manner. We finally argue that the cross-correlation amplitude appears as a marginal operator in the problem.

Since this amplitude appears in the expressions of the scaling exponents we have an example of a continuously varying universality class. Technically speaking we obtain a *fixed line*, parametrized by the value of the parameter N_{\times} introduced above, instead of a single or isolated fixed points. Every point on the fixed line characterizes a universality class, parametrized again by N_{\times} . The fixed line begins from $N_{\times} = 0$, which is the equilibrium fixed point. This stands in contrast to, e.g., Ref. [3], where nonequilibrium noises lead to additional fixed points, but not a fixed line as here. There are other dynamical models where cross-correlated noises lead to universal properties varying continuously with the amplitude of the noise cross correlations. Notable examples are the stochastically driven generalized Burgers model (GBM) [24] and magnetohydrodynamic turbulence (MHD) [25]. However, the d -dimensional GBM and MHD models are intrinsically nonequilibrium and do not generally have an equilibrium limit: Switching off the noise cross correlation does not make these models equilibrium in general d dimensions. In contrast, the present model has a well-defined equilibrium limit given by $N_{\times} = 0$ for any dimension d . Thus not only does our model here exhibit continuously varying universal properties, it can be driven away from equilibrium continuously and incrementally by tuning N_{\times} . Continuously varying universality has been found in Ref. [19] as well. However, Ref. [19] required coupling of the order parameter field with a conserved density. In contrast in our work we have the order parameter field only as the relevant dynamical field. Quantitative accuracy of our results is limited by the heuristic arguments we resorted to while evaluating the diagrams arising from noise cross correlations. In order to verify this, direct numerical simulations of the model Langevin equations, or simulations of appropriately defined lattice-gas models, should be performed. In the present paper we have discussed the universal scaling

properties at the critical point only. Numerical simulations of a driven O(3) model [26] display existence of spatiotemporal chaotic low-temperature regime below its critical point in the absence of stochastic noises. This chaos, when *controlled*, is replaced by spatially periodic steady helical states, which are robust against noise. In view of these results in Ref. [26], it would be interesting to examine the properties of the ordered phase below T_c , and their dependences on the parameter N_{\times} introduced above. In the above we have confined ourselves in discussing a usual order-disorder transition and the associated universality at the critical point. For any model, such a scenario holds as long as the physical dimension is greater than the lower critical dimension d_L of the model. For equilibrium models with continuous symmetries, e.g., the O(2)-symmetric model in equilibrium, the Mermin-Wagner theorem tells us that $d_L = 2$. For models out of equilibrium, there are no such general results. It would be interesting to examine the present model, perhaps through nonperturbative methods, at $d = 2$, in particular the role and dynamics of topological defects in the presence of noise cross correlations. We hope our theoretical results will inspire more detailed theoretical studies on more realistic models or experimental work on driven systems with coupled variables, where the role of noise cross correlations in determining the universal properties may be explicitly tested.

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