

# Triviality problem and high-temperature expansions of higher susceptibilities for the Ising and scalar-field models in four-, five-, and six-dimensional lattices

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High-temperature expansions are presently the only viable approach to the numerical calculation of the higher susceptibilities for the spin and the scalar-field models on high-dimensional lattices. The critical amplitudes of these quantities enter into a sequence of universal amplitude ratios that determine the critical equation of state. We have obtained a substantial extension, through order 24, of the high-temperature expansions of the free energy (in presence of a magnetic field) for the Ising models with spin  $s \geq 1/2$  and for the lattice scalar-field theory with quartic self-interaction on the simple-cubic and the body-centered-cubic lattices in four, five, and six spatial dimensions. A numerical analysis of the higher susceptibilities obtained from these expansions yields results consistent with the widely accepted ideas, based on the renormalization group and the constructive approach to Euclidean quantum field theory, concerning the no-interaction (“triviality”) property of the continuum (scaling) limit of spin- $s$  Ising and lattice scalar-field models at and above the upper critical dimensionality.

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## I. INTRODUCTION

The renormalization group theory (RG) theory [1–3] predicts the value  $d_c = 4$  for the upper critical dimensionality of the  $N$ -component lattice scalar-field theory and of the short-range classical Heisenberg  $N$ -component spin systems with  $O(N)$ -symmetric interaction. When  $d \geq 4$ , the critical fluctuations become too weak to drive the leading critical exponents away from the “classical” values taken in the mean-field (MF) approximation and can induce only minor corrections to scaling. In particular, in 4D the simple MF asymptotic forms of the thermodynamical quantities at criticality should be corrected by logarithmic factors, whose precise structure is also predicted by the RG. In higher dimensions, the dominant singularities have purely MF forms and the fluctuations can only influence the critical amplitudes and the corrections to scaling. These RG predictions entail the “triviality” [1,4–7] of the quantum  $N$ -component scalar-field theories in  $d \geq d_c$  or, more precisely, the property that the continuum (scaling) limit of the lattice approximation of the theories (or of the spin models) describes fields whose connected fourth- and higher-order correlation-functions vanish and, therefore, are free or generalized free.

The main clues of this no-interaction property had been pointed out long ago [8], but more stringent arguments were produced only by modern developments of the RG theory [1–3]. At the same time, a rigorous constructive approach [4,7,9–13] based on the representation [5] of the lattice scalar field as a gas of polymers made it possible to prove conclusively that the continuum Euclidean quantum field theory, built as the scaling limit of a lattice theory (with the simplest nearest-neighbor discretization of the Laplacian)

in the symmetric phase, is “nontrivial” [14,15] in  $d \leq 3$  and “trivial” in  $d \geq 5$  dimensions.

The rigorous results that exist in 4D (and, in general, for  $N > 4$ ) are still incomplete, although they strongly suggest that, nevertheless, the triviality property still holds. Therefore, some room is left not only to numerical studies but also to a variety of efforts [16–18] (and the related controversies [19]) aimed either to exploit possible gaps in the arguments or to relax some of the hypotheses underlying the constructive approach in order to make the definition of a “nontrivial” continuum theory feasible.

For  $d \geq 4$ , the Monte Carlo (MC) simulation approach to the numerical verification of the RG predictions is not yet completely satisfactory. The detailed exploration of the near-critical behavior is hampered by the necessity of considering systems of very large sizes and, in particular, at  $d = 4$ , by the difficulty of an accurate characterization of the slowly varying logarithmic deviations from MF behavior. Moreover, for  $d \geq 4$ , the finite-size-scaling theory and the confluent corrections to scaling have been debated [20–25]. Thus, relatively few of the numerous available MC studies [17,22,26–35] are likely to be extensive enough to yield a satisfactory overall description of these systems at criticality, in spite of the remarkable progress in the simulation algorithms with reduced critical slowdown.

On the other hand, for these systems high-temperature (HT) expansions have been until now derived only for a small number of observables and are too short [36–38], or, perhaps, barely adequate [39–43], to extract reliable information in the critical region. We believe, however, that the HT series methods might bring further insight into this context, provided that, for a conveniently enlarged set of observables, the lengths of the expansions can be significantly extended. Recently, new stochastic algorithms [44–47] have shown promise of deriving extremely long, although approximate, HT expansions valid for finite-size lattice systems. The application

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[47] of these methods also to the triviality problem is particularly interesting.

The traditional graphical [36–39,48–54] or iterative [55–58] methods of calculation, although severely limited by the fast increase of their combinatorial complexity with the order of expansion, remain necessary to derive the exact HT series coefficients, valid in the thermodynamical limit, which are needed for a reliable use of the known analytic extrapolation tools [59–61], such as Padé approximants (PAs) or differential approximants (DAs). Finally, it is worth adding that, in the case of high-dimensional models, these exact HT expansions still seem to be the only practicable method to compute the higher-order field derivatives of the free energy at zero field, usually called *higher susceptibilities*.

In this paper, we focus on the HT series approach to provide further numerical evidence to support the RG predictions in the case of the  $N = 1$  lattice scalar-field models and of the Ising spin- $s$  systems. For this purpose, we have computed and analyzed exact HT expansions of the higher susceptibilities, through order 24, to study their critical behavior and an important class of universal combinations of critical amplitudes (UCCAs), whose properties might also be of interest.

The paper is organized as follows. In Sec. II we define the spin- $s$  Ising and the lattice scalar-field systems for which we have substantially extended the HT expansions of the specific free energy in the presence of a uniform magnetic field. We then make due reference to the few HT data already in the literature. In Sec. III, we introduce the higher susceptibilities and indicate how their expected critical behavior varies with the lattice dimensionality. In Sec. IV, we review the definition of the dimensionless  $2n$ -points renormalized coupling constants in terms of the higher susceptibilities and indicate their role in the discussion of the RG predictions. We then introduce several classes of UCCAs related to the latter quantities. Section V is devoted to a detailed numerical analysis of our HT expansions, including discussions of numerical estimates of the critical temperatures, exponents, and several UCCAs for the models under scrutiny. The final section contains our conclusions.

## II. ISING-TYPE MODELS: DEFINITIONS AND NOTATION

In what follows, we shall be concerned only with spin- $s$  Ising or one-component lattice scalar fields so, unless explicitly needed, it will be convenient to drop the dependence of the physical quantities on the number  $N$  of components of the spin or of the field.

In a bounded region  $\Lambda \subset \mathbb{Z}^d$  of the  $d$ -dimensional lattice  $\mathbb{Z}^d$ , the spin- $s$  Ising model interacting with an external uniform magnetic field  $H$  is described by the Hamiltonian [48–51]

$$\mathcal{H}_\Lambda\{s\} = -\frac{J}{s^2} \sum_{\langle ij \rangle} s_i s_j - \frac{mH}{s} \sum_i s_i, \quad (1)$$

where  $s_i = -s, -s+1, \dots, s$  is the spin variable at the lattice site  $i$ ,  $m$  is the magnetic moment of a spin, and  $J$  is the exchange coupling. Within region  $\Lambda$ , the first sum extends over all distinct nearest-neighbor pairs of sites and the second sum over all lattice sites. Clearly, the conventional Ising model is obtained simply by setting  $s = 1/2$ .

The self-interacting one-component scalar-field lattice model in a magnetic field is described by the Hamiltonian [38,39,51,54]

$$\mathcal{H}_\Lambda\{\phi\} = - \sum_{\langle ij \rangle} \phi_i \phi_j + \sum_i [V(\phi_i) + H\phi_i]. \quad (2)$$

Here  $-\infty < \phi_i < +\infty$  is a continuous variable associated to the site  $i$  and  $V(\phi_i)$  is an even polynomial in the variable  $\phi_i$ . In this study, for brevity, we have discussed only the particular model in which  $V(\phi_i) = \phi_i^2 + g(\phi_i^2 - 1)^2$ , but considering interactions of a more general form requires only simple changes in the computation.

The Gibbs specific free energy  $\mathcal{F}(K, h)$  is defined, as usual, by

$$\begin{aligned} \mathcal{F}(K, h) &= - \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda| k_B T} \ln Z_\Lambda(K, h) \\ &= - \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda| k_B T} \ln \sum_{\text{conf}} \exp[-\mathcal{H}_\Lambda / k_B T], \end{aligned} \quad (3)$$

where  $|\Lambda|$  is the volume of the region  $\Lambda$  and  $K = J/k_B T$  (or  $K = 1/k_B T$  in the case of the scalar-field models), called the *inverse temperature* for short, is the HT expansion variable, with  $k_B$  the Boltzmann constant and  $T$  the temperature, while  $h = mH/k_B T$  denotes the reduced magnetic field.

We have studied the models described by Eqs. (1) and (2) on the hyper-simple-cubic (hsc) and the hyper-body-centered (hbcc) lattices. Following Ref. [49], for  $d \geq 4$ , the hbcc lattices are defined as those in which the first neighbors  $\hat{j}$  of the site  $\hat{i}$  are such that  $\hat{i} - \hat{j} = (\pm 1, \pm 1, \dots, \pm 1)$ . This choice has the technical advantage, decisive for the computations on high-dimensional lattices, that the “lattice free-embedding numbers” that enter into the contribution of each graph to the HT expansion factorize so they can be expressed as powers of those referring to the 1D lattice. As a consequence of this drastic simplification, the computing time of the expansions is independent of the lattice dimensionality, whereas, in the case of the hsc lattices, it grows exponentially with the dimensionality. We should also notice that, for  $d > 2$ , the coordination number  $q = 2^d$  of the hbcc lattice is much larger than the coordination number  $q = 2d$  of the hsc lattice and, therefore, the hbcc lattice expansions share the advantage of being notably smoother and converge faster than the hsc ones.

The expansions presented here are based on a calculation of the HT and low-field expansion of the free energy of various models described by the Hamiltonians Eqs. (1) and (2), in the presence of an external uniform magnetic field, which we have extended through order 24. In the case of the conventional Ising model, i.e., the model with spin  $s = 1/2$ , such an expansion, through order 17, was already in the literature [62,63] in the case of the four-dimensional hsc lattice (h4sc). Our expansion agrees only up to order 16 with these data and, as a consequence, with the series coefficients of the ordinary susceptibility  $\chi_2(K)$  and of the fourth field derivative of the free energy  $\chi_4(K)$ , obtained from them and analyzed in Ref. [37], as well as in some successive studies. For the conventional Ising model, in addition to the expansion in the case of the h4sc lattice, we have also computed the analogous expansion in the case of the hbcc lattice in 4D (h4bcc). For both the h4sc and the h4bcc lattices, we have,

moreover, computed HT and low-field expansions in the case of the Ising models with spin  $s = 1, 3/2, \dots, 3$  and in the case of the Euclidean one-component scalar-field lattice models with an even-polynomial self-interaction. Finally, we repeated the series derivation for the same set of models in 5D and 6D but restricted ourselves to the five-dimensional hbcc (h5bcc) and the six-dimensional hbcc (h6bcc) lattices for the reasons of computational simplification indicated above. These expansions do not exist in the literature.

Altogether, we examined these Ising-type models in 28 cases, distinct by spatial dimensionality, value of the spin, and structure of the lattice or of the interaction. In a given dimension, all these models are expected to belong to the same critical universality class and, therefore, to be characterized by the same set of critical exponents and UCCAs. Finally, let us also mention that our HT expansions for the Ising models in a magnetic field can be readily transformed into low-temperature (LT) and high-field expansions, from which the spontaneous magnetization and the LT higher susceptibilities can be derived.

In our calculation of the HT expansions, we have employed the linked-cluster graphical method of Ref. [48]. We have used an algorithm of graph generation and series calculation already described in Ref. [51]. The details of the computer implementation of this procedure, its validation, and its performance are discussed in the same paper, which was devoted to a study of the higher susceptibilities and the scaling equation of state for the 3D Ising universality class. Our extensions of the HT and low-field expansions are summarized in Table I. The set of series coefficients cannot fit into this paper because of its large size and will be tabulated elsewhere.

#### A. Available series expansions in zero field

It is appropriate to list here the few HT expansions of the higher susceptibilities for high-dimensional models at zero field that can already be found in the literature. All of them are restricted to the conventional spin-1/2 Ising model on the hsc lattices in zero field. The ordinary susceptibility  $\chi_2(K)$  was derived [36] through order 11 in dimensions  $d = 2, \dots, 6$ . More recently, these calculations were extended [38] to include, through the the same order,  $\chi_4(K)$  and the second moment of the correlation function  $\mu_2(K)$  in  $d = 2, 3, 4$  dimensions and carried [39] up to order 14. The expansion

TABLE I. Maximal order in  $K$  of the HT and low-field expansions of the free energy for the Ising and scalar-field models on four-, five-, and six-dimensional lattices.

	Existing data [63]	This work
h4sc Ising $s = 1/2$	17	24
h4sc Ising $s > 1/2$	0	24
h4sc scalar field	0	24
h4bcc Ising $s \geq 1/2$	0	24
h4bcc scalar field	0	24
h5bcc Ising $s \geq 1/2$	0	24
h5bcc scalar field	0	24
h6bcc Ising $s \geq 1/2$	0	24
h6bcc scalar field	0	24

of the susceptibility  $\chi_2(K)$  has been recently pushed [43] to order 19 on the h4sc and h5sc lattices. An expansion of  $\chi_2(K)$  valid for any dimension  $d$  was computed [64] through order 15. For  $\chi_2(K)$ ,  $\chi_4(K)$ , and the sixth field derivative of the free energy  $\chi_6(K)$ , strong coupling expansions through order 11, i.e., expansions in powers of the second-moment correlation length  $\xi^2(K) = \mu_2(K)/2d\chi_2(K)$ , instead of  $K$ , valid for any  $d$ , have also been obtained [65]. Of course, the usual HT expansions in powers of  $K$  can be recovered simply by reverting the appropriate expansion of  $\xi^2(K)$ .

### III. THE HT EXPANSIONS OF THE HIGHER SUSCEPTIBILITIES

The assumption of asymptotic scaling [66–70] for the singular part  $\mathcal{F}_s(\tau, h)$  of the reduced specific free energy, valid for dimension  $d \neq d_c$ , when both  $h$  and  $\tau$  approach zero, is usually expressed in the form

$$\mathcal{F}_s(\tau, h) \approx |\tau|^{2-\alpha} Y_{\pm}(h/|\tau|^{\beta\delta}), \quad (4)$$

where  $\tau = (1 - K/K_c)$  is the reduced inverse temperature. The functions  $Y_{\pm}(w)$  are defined for  $0 \leq w \leq \infty$  and the  $+$  and  $-$  subscripts indicate that different functional forms are expected to occur for  $\tau < 0$  and  $\tau > 0$ . The exponent  $\alpha$  specifies the divergence of the specific heat,  $\beta$  describes the small  $\tau$  asymptotic form of the spontaneous specific magnetization  $M$  on the phase boundary ( $h \rightarrow 0^+, \tau < 0$ )

$$M \approx B(-\tau)^{\beta}, \quad (5)$$

and  $B$  denotes the critical amplitude of  $M$ . The exponent  $\delta$  characterizes the small  $h$  asymptotic behavior of the magnetization on the critical isotherm ( $h \neq 0, \tau = 0$ ),

$$|M| \approx B_c |h|^{1/\delta}, \quad (6)$$

and  $B_c$  is the corresponding critical amplitude. For  $d \geq d_c$ , the MF values expected for the exponents  $\alpha$ ,  $\beta$ , and  $\delta$  are  $\alpha = \alpha_{\text{MF}} = 0$ ,  $\beta = \beta_{\text{MF}} = 1/2$ , and  $\delta = \delta_{\text{MF}} = 3$ , while for the susceptibility exponent we have  $\gamma = \gamma_{\text{MF}} = 1$  and for the correlation-length exponent  $\nu = \nu_{\text{MF}} = 1/2$ . The usual scaling laws (but, of course, not the hyperscaling laws) follow from Eq. (4).

From our calculation of the magnetic-field-dependent free energy, we have gained extensions of the existing HT expansions in zero field and, in addition, made available a large body of data, in particular for the  $n$ -spin connected correlation functions at zero wave number and zero field (the “higher susceptibilities”), defined by the successive field derivatives of the specific free energy

$$\chi_n(K) = (\partial^n \mathcal{F}(K, h) / \partial h^n)_{h=0} = \sum_{s_2, s_3, \dots, s_n} \langle s_1 s_2 \dots s_n \rangle_c \quad (7)$$

in the Ising model case or by the analogous formula in the scalar-field case. For odd values of  $n$ , the quantities  $\chi_n(K)$  vanish in the symmetric HT phase, while they are nonvanishing for all  $n$  in the broken-symmetry LT phase.

For even values of  $n$  in the symmetric phase, the RG theory predicts that, for  $d > 4$ , we have

$$\chi_n(\tau) \approx C_n^+ |\tau|^{-\gamma_n} (1 + b_n^+ |\tau|^{\theta} + \dots) \quad (8)$$

as  $\tau \rightarrow 0^+$  along the critical isochore ( $h = 0, \tau > 0$ ). In Eq. (8),  $b_n^+$  and  $\theta$  denote, respectively, the amplitude and the exponent of the leading confluent correction to the asymptotic behavior. The explicit expressions obtained in the case of the spherical model [71–73] suggest that in 5D one should expect  $\theta = 1/2$ , whereas, in 6D,  $\theta = 1$ , with possible multiplicative logarithmic correction terms.

At the marginal dimension  $d_c$ , the homogeneity property described by Eq. (4) is not strictly true, because of the expected logarithmic corrections. In this case, for even values of  $n$ , in the symmetric phase, the RG theory predicts for the higher susceptibilities the following asymptotic behavior:

$$\chi_n(\tau) \approx C_n^+ \tau^{-\gamma_n} |\ln(\tau)|^{G_n(N)} (1 + O\{\ln[|\ln(\tau)|]/\ln(\tau)\}) \quad (9)$$

in the  $\tau \rightarrow 0^+$  limit. In both Eqs. (8) and (9), one has  $\gamma_n = \gamma_{MF} + (n-2)\Delta_{MF}$ , with the gap exponent  $\Delta_{MF} = \beta_{MF}\delta_{MF} = 3/2$ . The general expression for  $G_n(N)$  is

$$G_n(N) = \left(\frac{3}{2}n - 2\right) \frac{N+2}{N+8} - n/2 + 1 \quad (10)$$

so, in the  $N = 1$  case,  $G_n(1) = G = 1/3$ , independently of  $n$ . Clearly, the usual hyperscaling relation  $2\Delta = d\nu + \gamma$ , which is valid for  $d < d_c$ , fails by a power when  $d > 4$ , while it is only logarithmically violated in  $d = 4$ .

The simplest consequence of the usual scaling hypothesis Eq. (4), which will be tested using our HT expansions, is that the critical exponents of the successive derivatives of  $\mathcal{F}(\tau, h)$  with respect to  $h$  at zero field are evenly spaced by the gap exponent  $\Delta_{MF}$ . Also in 4D, this property can be simply and accurately checked by a HT analysis of the higher susceptibilities.

#### IV. RENORMALIZED COUPLINGS AND RELATED QUANTITIES

It is useful here to recall the definitions of the universal quantities  $g_{2n}^+$ , called *zero-momentum  $n$ -spin dimensionless renormalized coupling constants* (RCCs) in the symmetric phase. They enter into the approximate representations of the scaling equation of state and, moreover, play a key role in the discussion of the triviality properties of the  $d \geq 4$  systems. They are defined as the critical limit when  $K \rightarrow K_c^-$  of the expressions

$$g_4(K) = -\frac{v}{\xi^d(K)} \frac{\chi_4(K)}{\chi_2^2(K)} \quad (11)$$

$$g_6(K) = \frac{v^2}{\xi^{2d}(K)} \left\{ -\frac{\chi_6(K)}{\chi_2^3(K)} + 10 \left[ \frac{\chi_4(K)}{\chi_2^2(K)} \right]^2 \right\} \quad (12)$$

$$g_8(K) = \frac{v^3}{\xi^{3d}(K)} \left\{ -\frac{\chi_8(K)}{\chi_2^4(K)} + 56 \frac{\chi_6(K)\chi_4(K)}{\chi_2^5(K)} - 280 \left[ \frac{\chi_4(K)}{\chi_2^2(K)} \right]^3 \right\} \quad (13)$$

$$g_{10}(K) = \frac{v^4}{\xi^{4d}(K)} \left\{ -\frac{\chi_{10}(K)}{\chi_2^5(K)} + 120 \frac{\chi_8(K)\chi_4(K)}{\chi_2^6(K)} + 126 \frac{\chi_6^2(K)}{\chi_2^6(K)} - 4620 \frac{\chi_6(K)\chi_4^2(K)}{\chi_2^7(K)} + 15\,400 \left[ \frac{\chi_4(K)}{\chi_2^2(K)} \right]^4 \right\} \quad (14)$$

$$g_{12}(K) = \frac{v^5}{\xi^{5d}(K)} \left\{ -\frac{\chi_{12}(K)}{\chi_2^6(K)} + 220 \frac{\chi_{10}(K)\chi_4(K)}{\chi_2^7(K)} + 792 \frac{\chi_8(K)\chi_6(K)}{\chi_2^7(K)} - 17\,160 \frac{\chi_8(K)\chi_4^2(K)}{\chi_2^8(K)} - 36\,036 \frac{\chi_6^2(K)\chi_4(K)}{\chi_2^8(K)} + 560\,560 \frac{\chi_6(K)\chi_4^3(K)}{\chi_2^9(K)} - 1\,401\,400 \left[ \frac{\chi_4(K)}{\chi_2^2(K)} \right]^5 \right\} \quad (15)$$

and so on. The constant  $v$  is a lattice-dependent geometrical factor called the *volume per lattice site* [74]. A longer list of the RCCs appears in Ref. [51], where the equation of state is discussed only for the 3D case. For technical reasons, we have not yet extended the HT expansions of  $\mu_2(K)$  and, correspondingly, of the second-moment correlation length  $\xi(K)$ , so in this paper we shall study only the ratios of RCCs, for  $n > 2$ ,

$$r_{2n}(K) = \frac{g_{2n}(K)}{g_4(K)^{n-1}} \quad (16)$$

which share the computational advantage of being independent of  $\xi(K)$ . The critical limits of these ratios are universal quantities that will be denoted by  $r_{2n}^+$ .

At the upper critical dimension  $d_c$ , the quantities  $g_{2n}(K)$  are expected to vanish like powers of  $1/\ln(\tau)$ , when  $\tau \rightarrow 0^+$ . Therefore, the continuum limit theory is consistent only for vanishing renormalized coupling, i.e., it is trivial. We can check numerically that, in the same limit, the lowest ratios  $r_{2n}(K)$  remain finite in 4D. For  $d \geq 5$ , both the  $g_{2n}(K)$  and the  $r_{2n}(K)$  vanish in the critical limit like powers of  $\tau$ , so the mentioned property of triviality is also true for  $d > d_c$ .

Briefly recalling more detailed discussions [3,51,75], we can also observe that, for  $d > 4$ , in the small magnetization region, where the reduced magnetic field  $h(M, \tau)$  has a convergent expansion in odd powers of the magnetization  $M$ , the critical equation of state can be written in terms of an



appropriate variable  $z \propto M\tau^{-\beta}$  as

$$h(M, \tau) = \bar{h}|\tau|^{\beta\delta} F(z), \quad (17)$$

where  $\bar{h}$  is a constant and  $F(z)$  is normalized by the equation  $F'(0) = 1$ . In general, the small  $z$  expansion of  $F(z)$  can be written as

$$F(z) = z + \frac{1}{3!}z^3 + \frac{r_6^+}{5!}z^5 + \frac{r_8^+}{7!}z^7 + \dots \quad (18)$$

In the MF approximation, all  $r_{2n}^+$  vanish, and  $F(z)$  reduces to  $F_{\text{MF}}(z) = z + \frac{1}{3!}z^3$ .

At the upper critical dimension, the following form of the critical universal equation of state for an  $N$ -component system is obtained [2,3] from the RG

$$H \propto \left[ M\tau |\ln M|^{(N+2)/(N+8)} + \frac{1}{(N+8)} \frac{M^3}{|\ln M|} \right] \left( 1 + \frac{\text{const}}{|\ln M|} \right), \quad (19)$$

valid for small  $M$  and  $H$ . From Eq. (19) the general formula Eq. (9) and the expression (10) for  $G_n(N)$  can be deduced.

In terms of the higher susceptibilities, the simple sequence of quantities was defined [76] long ago

$$\mathcal{I}_{2r+4}(K) = \frac{\chi_2^r(K) \chi_{2r+4}(K)}{\chi_4^{r+1}(K)} \quad (20)$$

with  $r \geq 1$ . The finite and universal critical values

$$\mathcal{I}_{2r+4}^+ = \frac{(C_2^+)^r C_{2r+4}^+}{(C_4^+)^{r+1}} \quad (21)$$

of the functions  $\mathcal{I}_{2r+4}(K)$  in the limit  $K \rightarrow K_c^-$ , include some of the UCCAs first described in the literature.

Together with the sequence  $\mathcal{I}_{2r+4}^+$  of UCCAs, the sequences  $\mathcal{A}_{2r+4}^+$  and  $\mathcal{B}_{2r+8}^+$ , obtained as the critical limits of the functions

$$\mathcal{A}_{2r+4}(K) = \frac{\chi_{2r}(K) \chi_{2r+4}(K)}{[\chi_{2r+2}(K)]^2} \quad (22)$$

$$\mathcal{B}_{2r+8}(K) = \frac{\chi_{2r}(K) \chi_{2r+8}(K)}{[\chi_{2r+4}(K)]^2} \quad (23)$$

with  $r \geq 1$ , were also defined in Ref. [76].

In 4D the general formula Eq. (10) for  $G_n(N)$  implies that the powers of the logarithms that enter into the leading critical singularities Eq. (9) cancel in the quantities  $\mathcal{I}_{2r+4}(K)$ ,  $\mathcal{A}_{2r+4}(K)$ , and  $\mathcal{B}_{2r+8}(K)$  at the critical limit. Conversely, Eq. (10) can also be obtained recursively from the knowledge of only  $G_2(N)$  and  $G_4(N)$  by requesting that such a cancellation occurs.

The ratios  $r_{2n}(K)$  can be simply expressed in terms of the functions  $\mathcal{I}_{2r+4}(K)$ . For example,

$$r_6(K) = \frac{g_6(K)}{g_4(K)^2} = -\mathcal{I}_6(K) + 10, \quad (24)$$

$$r_8(K) = \frac{g_8(K)}{g_4(K)^3} = \mathcal{I}_8(K) - 56\mathcal{I}_6(K) + 280, \quad (25)$$

and so on. Taking the  $K \rightarrow K_c^-$  limit in Eqs. (24) and (25), and so on, and observing that the quantities  $r_{2n}(K)$  vanish as  $K \rightarrow K_c^-$  in the MF approximation [2], the corresponding critical values  $\hat{\mathcal{I}}_{2r+4}^+$  of the quantities in Eq. (20) can be simply evaluated, obtaining  $\hat{\mathcal{I}}_6^+ = 10$ ,  $\hat{\mathcal{I}}_8^+ = 280$ ,  $\hat{\mathcal{I}}_{10}^+ = 15400$ ,  $\hat{\mathcal{I}}_{12}^+ =$

1 401 400, and so on. It is also not difficult [71] to compute the MF values of the first few terms of the sequences  $\mathcal{A}_{2r+4}^+$  and  $\mathcal{B}_{2r+8}^+$ ; for example:  $\hat{\mathcal{A}}_8^+ = 14/5$ ,  $\hat{\mathcal{A}}_{10}^+ = 55/28$ ,  $\hat{\mathcal{B}}_{10}^+ = 154$ , and  $\hat{\mathcal{B}}_{12}^+ = 143/8$ .

In the next section, we shall study numerically the first few terms of the sequences  $\mathcal{I}_{2r+4}^+$ ,  $\mathcal{A}_{2r+4}^+$ , and  $\mathcal{B}_{2r+8}^+$  and observe that they share similar properties.

## V. RESULTS OF THE SERIES ANALYSIS

We address the reader to Refs. [50,51], for a more detailed description of the numerical approximation techniques necessary to estimate the critical parameters in the models under study, i.e., the locations of the critical points, their exponents of divergence, and the critical amplitudes for the various susceptibilities. We shall employ the DA method, a generalization [59–61] of the elementary PA method, to resum the HT expansions nearby the border of their convergence disks. This technique consists in estimating the values of the finite quantities or the parameters of the singularities of the expansions of the singular quantities from the solution, called the *differential approximant*, of an initial value problem for an ordinary linear inhomogeneous differential equation of first or higher order. The equation has polynomial coefficients defined in such a way that the series expansion of its solution equals, up to an appropriate order, the series under study. In addition to this technique, we shall also use a smooth and faster converging modification [61,77] of the traditional method of extrapolation of the coefficient-ratio sequence, sometimes called the *modified-ratio approximant* (MRA) method, to determine the location and the exponents of critical points.

Using PAs or DAs, one can achieve, in some cases, evaluations of the parameters that are unbiased, i.e., obtained without using independent estimates of the critical temperature in the construction of the approximants. In some cases, however, accurate estimates of the critical inverse temperatures are necessary to bias the determination of the critical exponents and amplitudes. As a general comment on the uncertainties of the estimates obtained by these methods, we have to observe that, inevitably, they are rather subjective. Therefore, we should be very cautious, compare the estimates obtained from the different approximation methods and also check how effectively our tools perform when applied to artificial model functions having the expected singularity structure. In our DA calculations, the uncertainties are taken as a small multiple of the spread among the estimates obtained from an appropriate sample of the highest-order approximants, i.e., those using most or all available expansion coefficients. Similarly, in the case of the MRAs, the error bars will be defined as a small multiple of the uncertainty of an appropriate extrapolation [50] of the highest-order approximants.

### A. The critical inverse temperatures of the models

If in 4D, as predicted by the RG theory, logarithmic factors modify the structure of the leading critical singularities and also appear in the corrections to scaling, as described by Eq. (9), we should expect that the numerical procedures mentioned above might suffer from a slower convergence than in the case of pure power-law scaling. For determination of the

TABLE II. Our estimates of the critical inverse temperatures of the spin- $s$  Ising models for various lattices and several values of the spin in 4D, 5D, and 6D.

Lattice	$s = 1/2$	$s = 1$	$s = 3/2$	$s = 2$	$s = 5/2$	$s = 3$
h4sc	0.149693(3)	0.215597(3)	0.255641(3)	0.282568(3)	0.301919(3)	0.316497(3)
h4bcc	0.0690114(8)	0.101165(2)	0.120592(2)	0.133605(2)	0.142930(2)	0.149942(2)
h5bcc	0.0326478(3)	0.0484554(3)	0.0579714(3)	0.0643290(3)	0.0688769(3)	0.0722915(3)
h6bcc	0.0159390(2)	0.0237914(2)	0.0285102(2)	0.0316592(2)	0.0339099(2)	0.0355989(2)

critical temperatures, different approximation methods such as PAs, DAs, and extrapolated MRAs have been used to study the expansions of the ordinary susceptibility  $\chi_2(K)$ , the quantity that generally shows the fastest convergence. Independently of the lattice type and dimensionality, our best estimates of the critical inverse temperatures for the systems under study are obtained extrapolating to large order  $r$  of expansion, a few (from four to seven) highest-order terms of the MRA sequences of estimates  $(K_c)_r$  of the critical inverse temperatures. To perform the extrapolation, we rely on the validity [50] of the simple asymptotic form

$$(K_c)_r = K_c - \frac{\Gamma(\gamma)}{\Gamma(\gamma - \theta)} \frac{\theta^2(1 - \theta)b_2}{r^{1+\theta}} + o\left(\frac{1}{r^{1+\theta}}\right). \quad (26)$$

In general,  $\theta$  and  $b_2$  indicate, respectively, the exponent and the amplitude of the leading confluent correction to scaling of  $\chi_2(K)$ .

In the 4D case, in which the asymptotic critical behavior of  $\chi_2(K)$  is described by Eq. (9), we can take  $\theta = 0$ . Therefore, the second term on the right-hand side of Eq. (26) vanishes and it must be replaced by a higher-order term depending on the exponent of the next-to-leading correction to scaling in Eq. (9). A similar argument applies in the 6D case in which we expect  $\theta = 1$ . In the 5D case, in which we expect  $\theta = 1/2$ , the coefficient of  $1/r^{1+\theta}$  in Eq. (26) also appears to be negligible, so the situation is similar to that of the 4D and the 6D cases. Since we do not know the values of the exponents of the next-to-leading correction to scaling, the simplest procedure of extrapolation might consist in assuming an asymptotic form  $(K_c)_r = K_c + w/r^{1+\epsilon}$  and in determining  $K_c$ ,  $w$ , and  $\epsilon$  by a best fit to our data. We obtain the values  $\epsilon = 0.6(2)$  in 4D,  $\epsilon = 1.1(2)$  in 5D, and  $\epsilon = 1.5(2)$  in 6D. These estimates are compatible with our previous remarks, indicating that the asymptotic behavior of Eq. (26) is determined by the next-to-leading rather than the leading correction to scaling. At the same time, as suggested by M. E. Fisher [78], the expectations [71–73] concerning the exponent of the leading corrections to scaling, whose amplitudes are

probably not negligible in spite of the fact that they are not seen by the MRAs, can be essentially confirmed studying by DAs the critical behavior of quantities like  $\mathcal{I}_{2r+4}(K)$ ,  $\mathcal{A}_{2r+4}(K)$ ,  $\mathcal{B}_{2r+8}(K)$ , and so on, and of their derivatives. As above remarked, in these quantities the dominant critical singularities cancel, while the leading corrections to scaling should survive and could be detected by DAs. In particular, a study of the derivatives of  $\mathcal{I}_6(K)$  and  $\mathcal{I}_8(K)$ , for the spin- $s$  Ising models, leads to the values  $\theta = 0.25(10)$  in 4D,  $\theta = 0.45(10)$  in 5D, and  $\theta = 0.95(10)$ , in very reasonable agreement with the predictions [71–73].

Our final results for the critical inverse temperatures of some spin- $s$  Ising and scalar-field models are collected in Tables II and III. In the 4D case, we have attached particularly generous error bars to our estimates. In  $d > 4$  dimensions, no logarithmic factors are expected to modify the leading MF behavior of the physical quantities, so our approximation tools are likely to yield estimates of a higher accuracy, which, moreover, appear to improve with increasing lattice dimensionality, both because of the decreasing size of the corrections to scaling and of the increasing lattice coordination number. All these results are confirmed also by the analyses employing DAs.

Only in the case of the Ising model with spin  $s = 1/2$  on the h4sc lattice, we can compare our estimates with those obtained in other studies by extrapolation of shorter HT series. In Ref. [42] the estimate  $K_c = 0.149696(4)$  was obtained from a series of order 17, while in Ref. [43] the result  $K_c = 0.149691(3)$  was derived from a series of order 19. As far as the most recent large-scale MC simulations are concerned, the estimate  $K_c = 0.149697(2)$  was obtained in Ref. [42], the value  $K_c = 0.149697(2)$  in Ref. [34], while the value  $K_c = 0.1496947(5)$  is reported in Ref. [35]. Our result in Table II is fully consistent with the older estimates. No comparison is possible either for higher values of the spin on the h4sc lattice or for any value of the spin on the h4bcc lattice, since no studies are available for these systems. In the case of the higher-dimensional lattices our analysis includes only the h5bcc and h6bcc lattices, which have not been studied elsewhere until now.

TABLE III. Our estimates of the critical inverse temperatures of the scalar-field models for various lattices and several values of the quartic self-coupling in 4D, 5D, and 6D.

Lattice	$g = 0.5$	$g = 0.6$	$g = 0.9$	$g = 1.1$	$g = 1.3$	$g = 1.5$
h4sc	0.283025(3)	0.280704(3)	0.270597(3)	0.262806(3)	0.254915(3)	0.247233(3)
h4bcc	0.136451(2)	0.134940(2)	0.129177(2)	0.124991(2)	0.120852(2)	0.116885(2)
h5bcc	0.0665777(2)	0.0657078(2)	0.0625961(2)	0.0604114(2)	0.0582806(2)	0.0562586(2)
h6bcc	0.0329566(1)	0.0324976(1)	0.0308921(1)	0.0297796(1)	0.0287008(1)	0.0276811(1)

### B. The logarithmic corrections in 4D

In the computation of the critical exponents, it is convenient also to distinguish the 4D case from the higher-dimensional ones. In 4D, when computing the exponent  $\gamma$  of  $\chi_2(K)$  by PAs or DAs, we obtain estimates very near to, but slightly larger than, unity. These estimates should then be regarded as the values of “effective exponents” that reflect the presence of a small correction to the leading classical behavior (and of subleading corrections). If we assume that the leading correction to MF behavior has the multiplicative logarithmic structure predicted by the RG, we can resort to a variety of procedures proposed [40,61,73,79] in the literature to isolate the logarithmic factor from the main power behavior and to measure its exponent. These prescriptions generally amount to cancel out the main power singularity in favor of the weak logarithmic one and, therefore, they need to be biased with an estimate of the inverse critical temperature, to which, in turn, the values obtained for the exponent of the logarithm are very sensitive.

For example, in the case of the ordinary susceptibility  $\chi_2(K)$ , one might study the auxiliary function  $l(K; \tilde{K}_c)$  defined by

$$l(K; \tilde{K}_c) = -(\tilde{K}_c - K) \ln(\tilde{K}_c - K) \frac{d}{dK} \ln[(\tilde{K}_c - K) \chi_2(K)], \quad (27)$$

where  $\tilde{K}_c$  is some accurate approximation of the true  $K_c$ . By Eq. (9),  $l(K; \tilde{K}_c) = G + O(\ln|\ln\tau|)$ , i.e., it yields the value of the exponent  $G$  when  $K \rightarrow K_c^-$  and  $\tilde{K}_c = K_c$ . Since  $\tilde{K}_c$  enters as a parameter into the definition of this biased indicator,

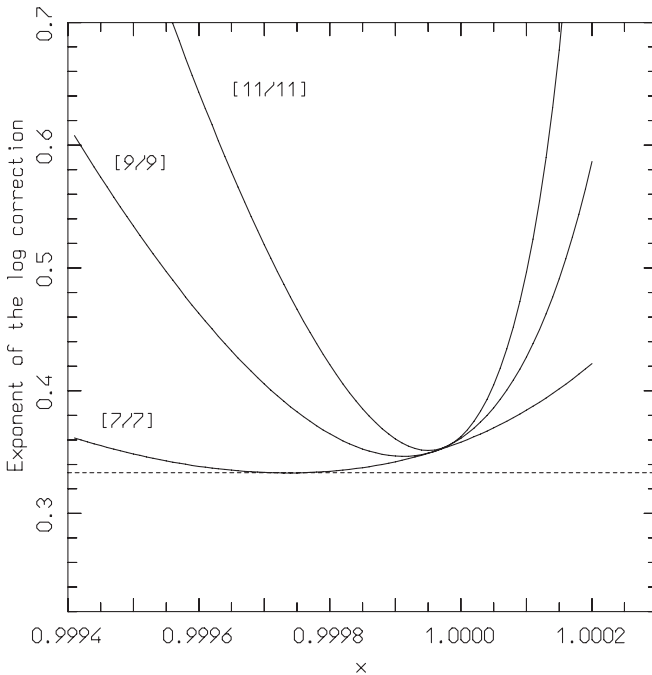


FIG. 1. Various PA approximants ([7/7], [9/9], and [11]) of the auxiliary function  $l(K; \tilde{K}_c)$  vs. the bias parameter  $x = \tilde{K}_c/0.101165$  normalized to our MRA estimate of  $K_c$  in case of the spin  $s = 1$  model on the h4bcc lattice. The dashed horizontal line represents the predicted value of the exponent of the logarithmic correction to the leading power behavior of the susceptibility.

we should consider how the estimate  $G(\tilde{K}_c)$  of the exponent depends on the choice of  $\tilde{K}_c$  in a small vicinity of our best estimate of the critical inverse temperature reported in Tables II or III. As a typical example, we show in Fig. 1 the plots of  $G(\tilde{K}_c)$  versus  $\tilde{K}_c$  (normalized to our MRA estimate of  $K_c$ ), computed by PAs of various orders, in the case of the Ising model with spin  $s = 1$  on the h4bcc lattice. It is reasonable to expect that the value of  $G(\tilde{K}_c)$  should depend slowly on  $\tilde{K}_c$  near the exact value of the critical inverse temperature so its best value might perhaps correspond to a stationary point. We observe that, for most PAs of  $G(\tilde{K}_c)$ , such a point does indeed exist and that the curves obtained by various PAs touch nearby this point, which generally does not differ much from our best estimate of  $K_c$  as reported in our tables. In the literature [40,61,73,79], the value of  $G(\tilde{K}_c)$  at the point where the various curves touch is generally taken as the most accurate estimate of the exponent  $G$ . However, this choice may be questioned, since the result appears to be insensitive to the order of approximation. As shown in Fig. 1, if we take the value of  $G(\tilde{K}_c)$  at the stationary point as the best approximation, the estimates are also close to the expected value  $G = 1/3$ .

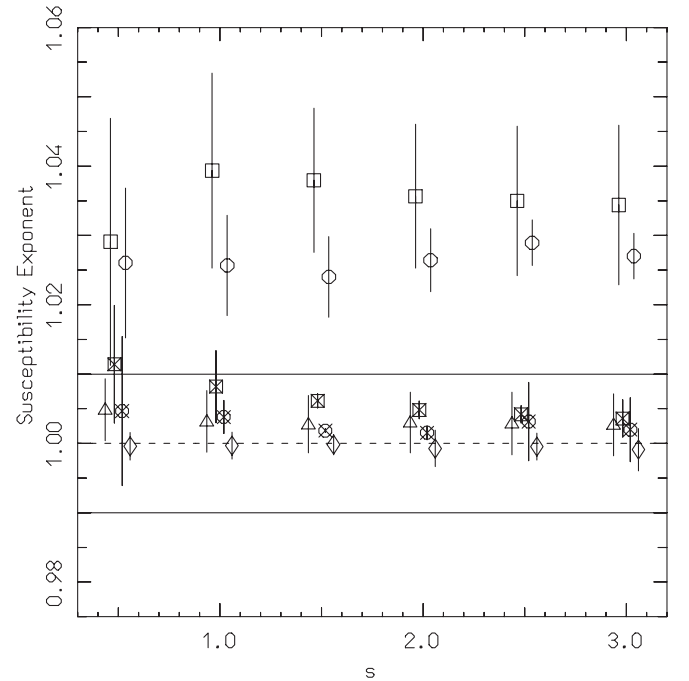


FIG. 2. The exponent  $\gamma$  of the susceptibility vs. the value  $s$  of the spin in the case of the spin- $s$  Ising models on the h4sc, h4bcc, h5bcc, and h6bcc lattices. For each value of the spin, the different estimates and their uncertainties are made more visible by slightly shifting their abscissas to avoid superpositions of the symbols. The exponents are denoted by open squares in the case of the h4sc lattice or by open circles in the case of the h4bcc lattice. If the logarithmic correction to the leading critical singularity expected in 4D is canceled out from the susceptibility expansion, we are led to the estimates represented by crossed open squares in the case of the h4sc lattice and by crossed open circles in the case of the h4bcc lattice. The estimates on the h5bcc and the h6bcc lattices are represented by open triangles and open rhombs, respectively. The dashed horizontal line represents the expected value  $\gamma_{MF} = 1$  of the exponent and the continuous lines indicate a 1% relative deviation from that value.

Unfortunately, also the choice of the stationary value as the best approximation is open to doubt, since in this case the successive approximations seem to worsen as the order of the series increases. We must, moreover, mention that, in the h4bcc Ising system, the values of  $G$  computed in this way range between  $\approx 0.4$  and  $\approx 0.3$ , as the spin varies from  $s = 1/2$  to  $s = 3$ . Finally, it is also unclear how to estimate the uncertainties involved in these procedures and, thus, how to interpret the spread of exponent estimates, which might be related to a strong spin dependence of the slowly decaying corrections appearing in Eq. (27). Other prescriptions to study the exponent of the logarithmic corrections do not lead to better results.

### C. The critical exponents of the higher susceptibilities

For each model under study, we have computed the exponent  $\gamma$  of the susceptibility by second- or third-order DAs biased with our estimate of the inverse critical temperature, namely by resorting to the standard prescription of imposing that the approximants are singular at the values of  $K_c$  reported in our Tables II and III and then computing the exponents. For  $d > 4$ , it is rigorously proved [9,10] that the systems must exhibit a MF critical behavior (with nontrivial subleading

corrections). Let us discuss, first, how our numerical tools perform in the 5D and 6D cases. We shall then argue that the differences between the features of this computation and those of the 4D case can be simply ascribed to the expected presence of a multiplicative logarithmic correction to the dominant MF power behavior. In Fig. 2, we have plotted our estimates of the exponent of the ordinary susceptibility versus the spin in the case of various spin- $s$  Ising models for  $d = 4, 5, 6$ . For  $d > d_c$ , our estimates reproduce the expected value  $\gamma_{MF} = 1$  very accurately, so the small deviations from this value can be safely viewed as only the residual effects of the confluent corrections to scaling. These deviations also show the expected decreasing size as the dimensionality of the system increases. Moreover, the critical universality, i.e., the independence of the exponents on the interaction structure, is well verified. On the contrary, at the upper critical dimension our calculations yield “effective” exponents larger than unity by  $\approx 3\%$ . We can interpret this result as an indication that the leading critical singularity of the susceptibility is slightly stronger than a pure MF singularity so it might indeed contain the logarithmic factor predicted by the RG, which is detected by the DAs as a powerlike factor with a very small exponent. This is confirmed by observing that, if the expected logarithmic singularity is canceled by dividing out the  $\ln(1 - K/K_c)^{1/3}$  correction factor from the susceptibility, the resulting estimate of the exponent  $\gamma$  generally gets within

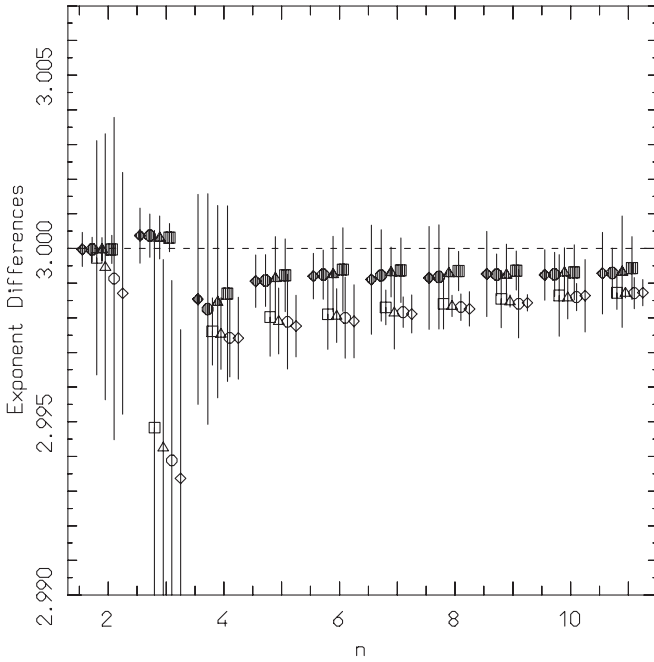


FIG. 3. The unbiased estimates of the exponent differences  $D_n = \gamma_{2n} - \gamma_{2n-2}$  for the susceptibilities  $\chi_{2n}$  and  $\chi_{2n-2}$ , for  $n = 2, 3, \dots, 11$  plotted vs.  $n$ , in the case of the scalar-field model on the h5bcc and h6bcc lattices. In the case of the h5bcc lattice, for each value of  $n$  we have reported a cluster of the four estimates of  $D_n$  corresponding to the values  $g = 0.9$  (open squares),  $g = 1.1$  (open triangles),  $g = 1.3$  (open circles), and  $g = 1.5$  (open rhombs) of the self-coupling of the field. The same quantities for the h6bcc lattice are represented by the corresponding black symbols. Although they correspond to the same value of  $n$ , the symbols within each cluster are slightly shifted apart to avoid cluttering and keep the spread of each estimate visible. The dashed horizontal line represents the expected value  $2\Delta_{MF} = 3$  of twice the gap exponent.

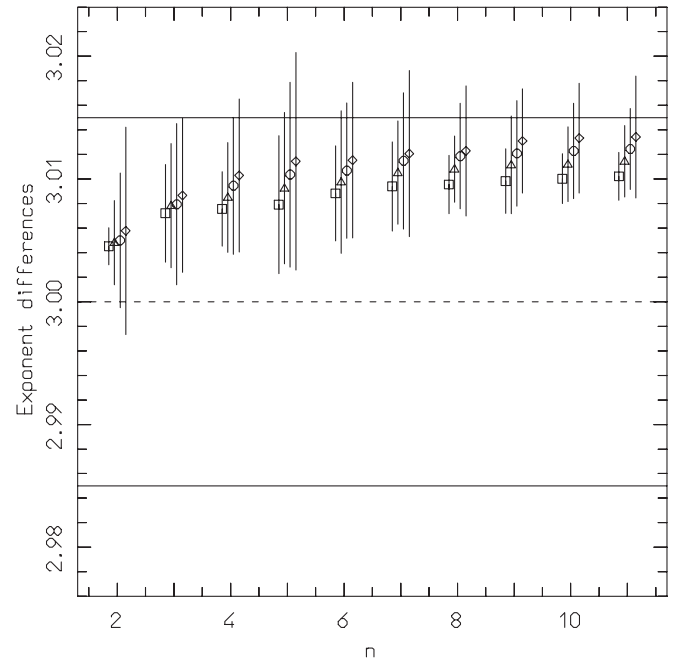


FIG. 4. The unbiased estimates of the exponent differences  $D_n = \gamma_{2n} - \gamma_{2n-2}$  of the susceptibilities  $\chi_{2n}$  and  $\chi_{2n-2}$ , for  $n = 2, 3, \dots, 11$  plotted vs.  $n$ , in the case of the scalar-field model on the h4bcc lattice. For each value of  $n$  we have reported a cluster of the four estimates of  $D_n$  corresponding to the values  $g = 0.9$  (squares),  $g = 1.1$  (triangles),  $g = 1.3$  (circles), and  $g = 1.5$  (rhombs) of the self-coupling of the field. Although they correspond to the same value of  $n$  the symbols within each cluster are slightly shifted apart to avoid cluttering and keep the spread of each estimate visible. The dashed horizontal line represents the expected value  $2\Delta_{MF} = 3$  of twice the gap exponent. The continuous horizontal lines indicate a relative deviation of 0.5% from the expected value.



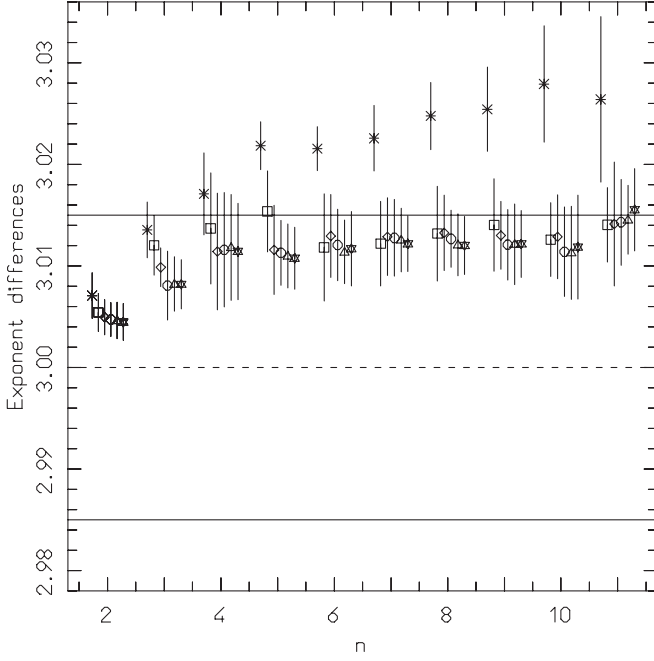


FIG. 5. The unbiased estimates of the exponent differences  $D_n$  plotted vs.  $n$  in the case of the Ising model on the h4bcc lattice for the following values of the spin:  $s = 1/2$  (asterisks),  $s = 1$  (open squares),  $s = 3/2$  (open rhombs),  $s = 2$  (open circles),  $s = 5/2$  (open triangles), and  $s = 3$  (open stars). The horizontal dashed line and the continuous lines have the same meaning as described in the caption to Fig. 4.

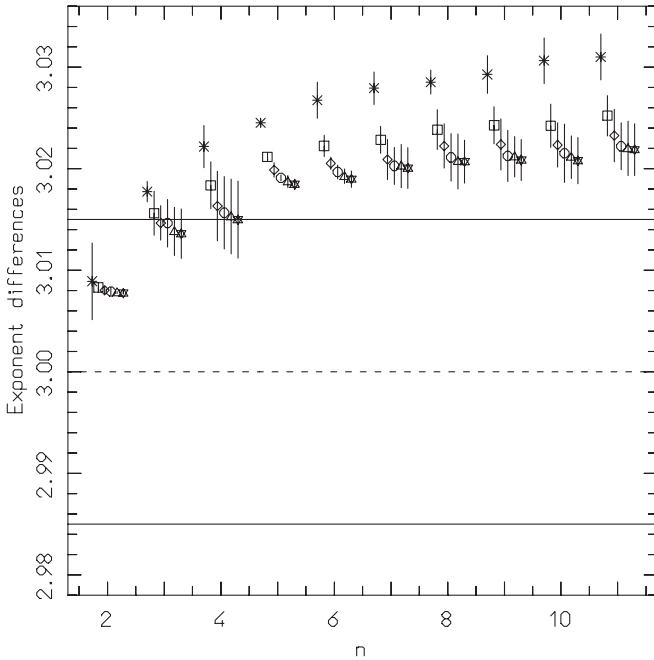


FIG. 6. Same as shown in Fig. 5 but for the Ising model on the h4sc lattice. The estimates of the exponent differences  $D_n$  are plotted vs.  $n$  for spin  $s = 1/2$  (asterisks),  $s = 1$  (open squares),  $s = 3/2$  (open rhombs),  $s = 2$  (open circles),  $s = 5/2$  (open triangles), and  $s = 3$  (open stars). The horizontal dashed line and the continuous lines have the same meaning as in Fig. 4.

$\approx 0.5\%$  of the MF value. Thus, the deviations are reduced to a smaller size and become compatible with the effects of the corrections to scaling.

Very accurate estimates can be obtained also for the differences  $D_n$  between the exponents of  $\chi_{2n}(K)$  and  $\chi_{2n-2}(K)$ ,

$$D_n = \gamma_{2n} - \gamma_{2n-2} = 2\Delta_{\text{MF}} = 3. \quad (28)$$

They can be computed from the ratios  $\chi_{2n}(K)/\chi_{2n-2}(K)$  by second- or third-order DAs biased with the critical inverse temperature. In the 4D case, we should not expect any effects from the logarithmic factors appearing in the leading singular behavior Eq. (9) of the higher susceptibilities, since these factors cancel in the above-indicated ratios. Instead of the results of the biased prescription, we prefer to show here the estimates from a computation by the unbiased “critical point renormalization” method [59]. This procedure consists in determining the difference  $D_n$  of Eq. (28) from the exponent of the singularity in  $x = 1$  of the series  $\sum a_r x^r$  with coefficients  $a_r = c_r^{2n}/c_r^{2n-2}$ , where  $c_r$  is the  $r$ -th coefficient of the expansion of  $\chi_s(K)$ . The biased DA calculation of the  $D_n$ , mentioned above, gives quite comparable results, so it is not necessary to report the corresponding figures.

The quantities  $D_n$  with  $n = 2, 3, \dots, 11$ , obtained by the unbiased method in the case of the scalar-field model on the h5bcc and h6bcc lattices, for several values of the coupling  $g$ , are plotted versus  $n$  in Fig. 3. The same computations

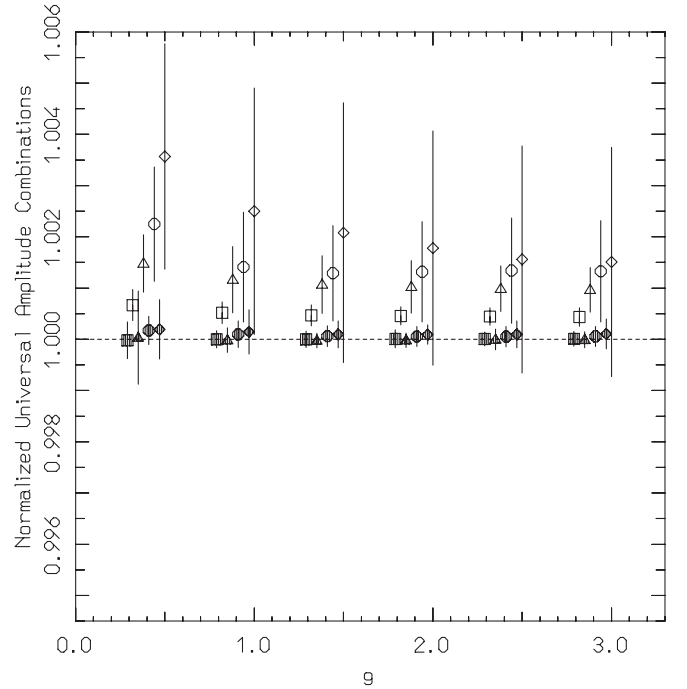


FIG. 7. The ratios  $Q_{2r+4} = \mathcal{I}_{2r+4}^+/\hat{\mathcal{I}}_{2r+4}^+$  with  $r = 1, 2, 3, 4$  vs. the value  $s$  of the spin for the Ising model on the h5bcc and h6bcc lattices. For each value of  $s$  the various symbols are slightly shifted apart to avoid superpositions and to keep the spread of each estimate visible. In the case of the h5bcc lattice we have represented  $Q_6$  with open squares,  $Q_8$  with open triangles,  $Q_{10}$  with open circles, and  $Q_{12}$  with open rhombs. The same ratios for the h6bcc lattice are represented by the corresponding black symbols. The horizontal dashed line represents the expected value of the ratios.

for the spin- $s$  Ising models with various values of the spin on the h5bcc and h6bcc lattices yield completely similar results and, therefore, we do not report the corresponding figure. Our estimates of  $D_n$  agree, generally within 0.1%, with the expected value  $2\Delta_{\text{MF}} = 3$ . Thus, the small size of these deviations from the MF value suggests that they can safely be related with the confluent corrections to scaling. The critical universality is also well verified. On the other hand, our results in the 4D case reported in Fig. 4 in the case of the scalar-field model on the h4bcc lattice, those reported in Fig. 5 for the Ising model on the h4bcc lattice, and those of Fig. 6 for the same system in the case of the h4sc lattice show relative deviations from the value of  $2\Delta_{\text{MF}}$ , 5 times larger than those in  $d > 4$  dimensions (i.e., of the order of 0.5%) but still sufficiently small to reflect only the residual influence of the expected subleading logarithmic corrections to the critical behavior.

#### D. Universal combinations of critical amplitudes

For  $d > 4$ , using second- or third-order DAs, the first few terms of the sequence of the UCCAs  $\mathcal{I}_{2r+4}^+$ ,  $\mathcal{A}_{2r+4}^+$ , and  $\mathcal{B}_{2r+8}^+$  can be evaluated to a good accuracy by extrapolating to  $K = K_c^-$  the estimates of the functions  $\mathcal{I}_{2r+4}(K)$ ,  $\mathcal{A}_{2r+4}(K)$ , and  $\mathcal{B}_{2r+8}(K)$ . For convenience, we have introduced the ratios of these quantities to their MF values and denoted them by  $\mathcal{Q}_{2r+4}$ ,  $\mathcal{R}_{2r+4}$ , and  $\mathcal{S}_{2r+8}$ , respectively. In Fig. 7, we have reported our estimates for the ratios  $\mathcal{Q}_6$ ,  $\mathcal{Q}_8$ ,  $\mathcal{Q}_{10}$ , and  $\mathcal{Q}_{12}$  versus the value  $s$  of the spin for Ising models on the h5bcc and h6bcc lattices. In complete agreement with the proven

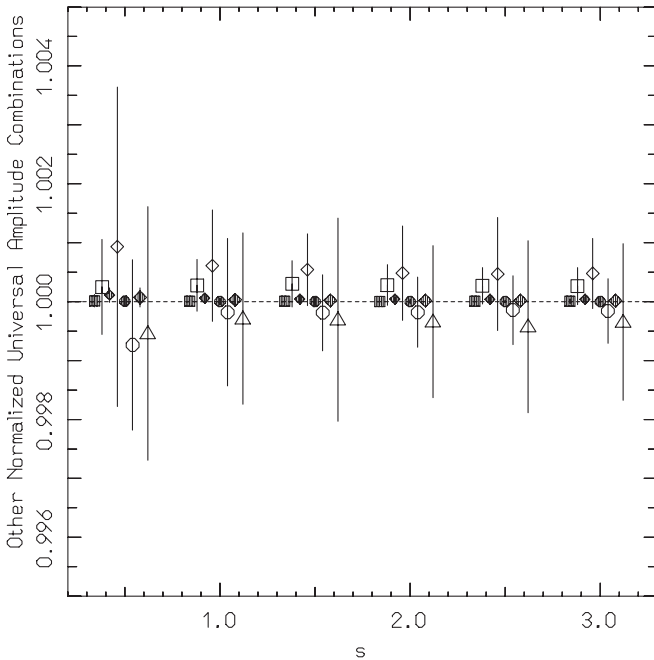


FIG. 8. The ratios  $\mathcal{R}_8^+$  (open squares),  $\mathcal{R}_{10}^+$  (open circles),  $\mathcal{S}_{10}^+$  (open rhombs), and  $\mathcal{S}_{12}^+$  (open triangles) vs. the spin  $s$  in the case of the Ising model with various values of the spin on the h5bcc lattice. The corresponding black symbols represent the estimates of the same quantities on the h6bcc lattice. For each value of  $s$  the various symbols are slightly shifted apart to avoid superpositions and to keep the spread of each estimate visible. The horizontal dashed line represents the expected value of the ratios.

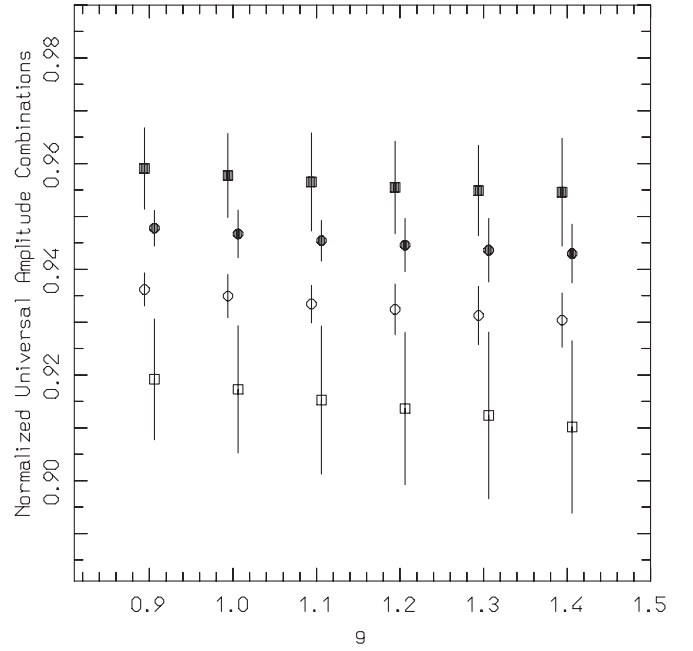


FIG. 9. The ratios  $\mathcal{Q}_6 = \mathcal{I}_6^+/\hat{\mathcal{I}}_6^+$  (black squares) and  $\mathcal{Q}_8 = \mathcal{I}_8^+/\hat{\mathcal{I}}_8^+$  (open squares) for the scalar-field model on the h4sc lattice vs. the coupling constant  $g$  of the field. The same quantities  $\mathcal{Q}_6$  (black circles) and  $\mathcal{Q}_8$  (open circles) vs. the coupling constant  $g$  for the scalar-field model on the h4bcc lattice. Like in the preceding figure, symbols associated to the same value of  $g$  are slightly shifted in order to avoid superpositions of the error bars.

MF nature of the critical behavior, these ratios generally equal unity, within the accuracy expected from our approximations that, in this case, allow not only for the influence of the confluent corrections to scaling but also for the uncertainties in the estimates of the critical temperatures needed to bias the calculations. Correspondingly, the critical limits of the ratios  $r_{2n}(K)$  vanish and the equation of state takes the MF form. Quite similar results are shown in Fig. 8 for the other normalized UCCAs  $\mathcal{R}_8^+$ ,  $\mathcal{R}_{10}^+$ ,  $\mathcal{S}_{10}^+$ , and  $\mathcal{S}_{12}^+$  that are plotted versus the spin  $s$  for Ising models with various values of the spin in the case of the h5bcc and h6bcc lattices.

Also in the 4D case, as shown in Fig. 9 for the first few UCCAs defined by Eq. (21) for the case of the scalar-field model, and in Fig. 10 for a few UCCAs defined by Eqs. (22) and (23) for the case of the spin- $s$  Ising model, the various quantities probably have been evaluated with reasonable accuracy, because the logarithmic factors, expected to appear in the leading critical behavior of the higher susceptibilities, cancel in the ratios defining the UCCAs. As a consequence, the uncertainties in the critical temperatures and the influence of the corrections to scaling should still be considered as the main sources of error. However, we observe that, generally, the first few ratios  $\mathcal{Q}_{2r+4}$ ,  $\mathcal{R}_{2r+4}$ , and  $\mathcal{S}_{2r+8}$  are slightly, but definitely, smaller than unity. We can imagine two possible explanations for this result: Either the deviations from unity have to be related only to (unlikely) residual effects of the logarithms in the leading and subleading behavior of the higher susceptibilities or the UCCAs are accurately estimated and they really do not take their MF values. Whatever the case, it is clear that also these data on the UCCAs confirm that,

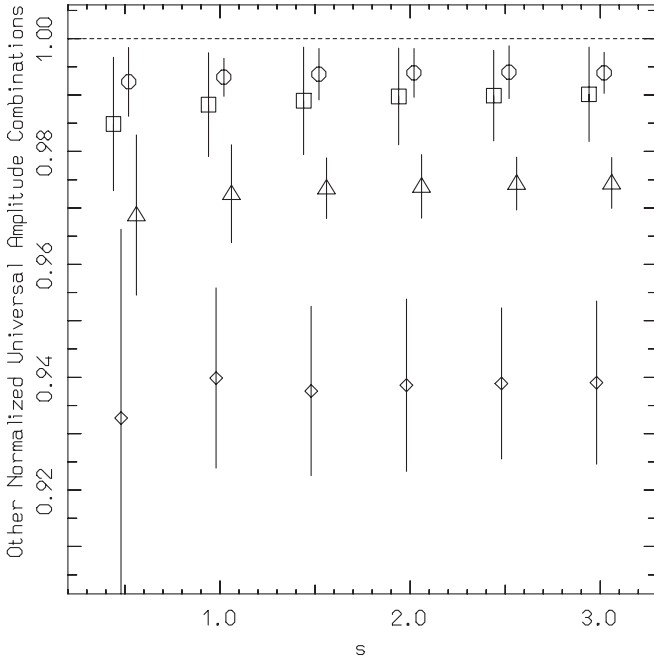


FIG. 10. The ratios  $\mathcal{R}_8^+$  (open squares),  $\mathcal{R}_{10}^+$  (open circles),  $\mathcal{S}_{10}^+$  (open rhombs), and  $\mathcal{S}_{12}^+$  (open triangles) plotted vs. the spin  $s$  in the case of the Ising model on the h4bcc lattice. Like in the preceding figure, symbols associated to the same value of the spin are slightly shifted in order to avoid superpositions of the error bars.

consistently with the RG predictions, the critical behavior in 4D is not MF like.

## VI. CONCLUSIONS

By analyzing our HT expansions of the zero-field higher susceptibilities, extended through order 24, in the case of the  $N = 1$  lattice scalar-field models and of the spin- $s$  Ising systems, we have provided further numerical evidence consistent with the critical behavior predicted by the RG in this class of models.

We have estimated the critical exponents of the ordinary and the higher susceptibilities and the values of a class of universal combinations of their critical amplitudes, which determine the form of the critical equation of state and are presently inaccessible by other computational methods. In 4D, the results of our analysis suggest that, within a good approximation, the critical exponents and this class of UCCAs, show small, but definitely nonvanishing, deviations from their values in the MF approximation. For the UCCAs, this fact had been already predicted long ago also within the RG formalism, by showing [80] that, at the upper critical dimension, at least one of the quantities in the above-mentioned class does not take the MF value. More generally, in 4D the deviations from the MF critical behavior are compatible with the small effects associated to the logarithmic corrections predicted by the RG. Our direct numerical checks concerning in particular the exponents of the logarithmic corrections to the dominant power behavior of the higher susceptibilities have only a rather limited accuracy, due to the modest sensitivity of the DAs to the logarithmic singularities, either in the leading behaviors and in the confluent corrections.

Quite to the contrary, the same kind of analysis performed on five- and six-dimensional lattices shows no numerical evidence of deviations from the leading classical behavior by an extent larger than the expected numerical uncertainties: both the exponents and the UCCAs appear to take the MF values within a high approximation, so the RG predictions concerning the triviality property are rather convincingly confirmed.

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