PHYSICAL REVIEW E **85**, 016706 (2012)

Taming hypersingular integrals using dimensional continuation

Zehao Li[1] and L. R. Ram-Mohan[2]

[1]*Department of Physics, Worcester Polytechnic Institute, Worcester, Massachusetts 01609, USA*

[2]*Departments of Physics, Electrical and Computer Engineering, Worcester Polytechnic Institute, Worcester, Massachusetts 01609, USA*

(Received 25 March 2011; revised manuscript received 13 December 2011; published 9 January 2012)

We use the method of dimensional continuation to isolate singularities in integrals containing products of Green's functions or their derivatives. Rules for the extraction of the finite part of so-called hypersingular integrals are developed, which should be useful in methods based on boundary integral techniques in science and engineering. In applications to potential theory, electromagnetic scattering, and crack dynamics in continuum mechanics, boundary integrals now can be readily evaluated using computational techniques without recourse to complex analysis or contour distortions since the hypersingularities occurring in intermediate steps of the computations can be isolated and ignored while taking the finite parts of the integrals into account in a consistent manner. We have also identified new forms of the Dirac δ function in D dimensions, which are useful and convenient in the calculations. A summary of the integrable singular integrals is given in tabular form. We extend the considerations to a wider class of Green's functions and present a theorem, with additional results arising from it, that shows that hypersingular integrals associated with three-dimensional potential problems can be reduced to one-dimensional finite integrals rather than two-dimensional integrals, again leading to direct evaluations in such cases. These calculations are compared with existing results to show the efficacy of the approach.

DOI: 10.1103/PhysRevE.85.016706 PACS number(s): 02.60.−x, 02.40.Xx, 02.30.Em, 41.20.Cv

I. INTRODUCTION

The properties of Green's functions and other generalized functions are defined [1] by the "company they keep," in the sense that their behavior is determined by an integration of such functions multiplied by well-behaved functions [2]. However, frequently in physical calculations in science and engineering we encounter derivatives of Green's functions as in the boundary integral method or its numerical implementation in the boundary element method (BEM). This leads to nonintegrable singularities that require careful attention in treating them.

In quantum field theory (QFT), we have an analogous situation in which products of Green's functions appearing in loop diagrams lead to infinities. Particularly lucid comments on this issue of the need to define new rules for the evaluation of products of singular functions have been given by Bogoliubov and Shirkov [3]. The method of analytic continuation in spatial dimension D of the integrals, to isolate the singular part and to identify the relevant finite values of the integrals, is used in relativistic field theory in perturbative evaluations of physically relevant quantities. In QFT the nature of the divergences requires "dimensional regularization" by which the infinities are absorbed into physically observable parameters through the process of renormalization [4].

Fortunately, in potential theory, electromagnetic field computations, and the theory of crack dynamics and continuum mechanics, the singularities occurring in intermediate stages of the calculations can be shown to cancel out. Thus, while renormalization is not an issue in this case, managing the infinities in the theory and performing numerical analysis is an issue, and it can be troublesome, as evinced by the focus of attention on this in the literature. Several investigations in the literature refer to the integrals appearing in the integral representation of potentials and fields and in their evaluation by the BEM as hypersingular integrals [5–8].

In all the reports in the literature dealing with hypersingular integrals, the approach for calculating them is to use either a distorted surface (two-dimensional [2D]) or contour (one-dimensional [1D]) to directly address the issue of the singularity. The presence of the hypersingularities typically reduces the numerical accuracy attained in the integrals, and the separation of the finite and infinite parts is a particularly lengthy procedure. Transformation of variables and complex analysis to evaluate the integrals are also employed in these papers. We cite a recent cross section of typical articles in this area in Refs. [9–16].

There are a few analytic methods to solve the problem of singular integrals, such as the Galerkin approximation using local polynomials (as in the finite element method) [9–11], the Cauchy principle value technique to obtain the finite part in integration [12–14], or complex analysis through contour deformation [15]. These techniques usually consider a local coordinate system around the singularity. The approaches differ only in the details of the evaluation of the singular integrals to separate the finite and infinite parts. However, all these earlier methods require very lengthy procedures due to the arbitrary shape of the discretized elements, such as triangles, and generally such discretized elements lack symmetry needed to simplify the integrals. The same complicated procedure has to be applied to each new type of Green's function that is appropriate to the problem, such as for Laplace problems [9], elasticity problems [10,11], or in fracture analysis [13].

Here we wish to present a new, independent method for the evaluation of the hypersingular integrals, whereby a more universal approach can be implemented. We propose the use of *dimensional continuation* in the evaluation of integrals of the well-known Green's functions. Since the singularities of Green's functions and their derivatives can be tamed by the radial part of the Jacobian, r^{D-1}, arising in D dimensions, we arrive at a closed-form expression for the integrals at high enough values of D. On returning to the dimension of interest by analytic continuation, the singularity there can be explicitly isolated and shown to cancel out in all applications. This is the essence of the method of dimensional continuation.

TABLE I. Comparison between the 2D calculations based on the final analytic expressions given by Fata *et al.* [18] and the results from our 1D reduction of the ISI using the theorem in Sec. IV.

Singular integrals	2D analytic results [18]	Present 1D calculations
I_1	7.161 515 826 913 852	(Nonsingular)
I_1^1	1.373 374 685 494 244	1.373 374 685 494 246
I_3^1	1.549 306 788 877 796	1.549 306 788 877 799
$I_1 - I_3^{11}$	3.484 787 720 187 224	3.484 787 720 187 223
I_3^{12}	0.125 979 758 603 789	0.125 979 758 603 789
$I_3 - 3\, I_5^{11}$	3.412 616 456 100 593	3.412 616 456 100 595
I_5^{12}	0.978 715 205 225 059	0.978 715 205 225 061

We identify the rules for obtaining consistent results through the use of such methods for the hypersingular integrals occurring in the BEM [17]. We provide a systematic approach to the treatment of the singularities in typical integrals using the standard example of the Poisson equation in Sec. II and show in Sec. II B how to isolate them using the dimensional continuation method. Further, we define integrable singular integrals that typically occur in physical applications in Sec. II C. These results are then used in deriving new expressions for the Dirac δ functions in D dimensions in Sec. II D; such expressions help resolve the singularity in the derivative of the solution of the Poisson equation over the infinite domain. In Sec. III we derive further results for a general class of Green's functions on the integrability of their derivatives and show how integrals can be performed over a general shape in the discretized region around the singularity. In Sec. IV we prove that the hypersingular integrals appearing in potential theory can be reduced by one more dimension; in other words, the boundary integral method that uses Green's theorem and reduces the dimensionality of the problem from three dimensions to two can have its singular integrals reduced by yet one more dimension. This is demonstrated for the Poisson's equation, and the results are compared in Table I with extant 2D evaluations in the literature [18,19]. Examples of such integrals occurring in electromagnetic scattering are considered in Sec. V, with the example of fracture analysis presented in Appendix B. A summary of the integrable singular integrals is reported in tabular form in Tables II, III, and IV in Appendix C. Concluding remarks are given in Sec. VI.

It is hoped that the present approach will provide an effective, powerful, and practical method of evaluating the so-called hypersingular integrals in computational science and engineering applications, with an automated approach to accounting for these issues in a direct manner.

II. POISSON'S EQUATION IN AN INFINITE DOMAIN

We consider the usual three-dimensional (3D) Laplace's equation with an inhomogeneous term in order to identify the problem of singular integrals and illustrate our method in resolving this problem. We also obtain a new generalized expression for the Dirac δ function in arbitrary dimensions.

A. Poisson's equation in three dimensions

In the infinite domain, the solution of the Poisson equation, $\nabla^2\varphi(\mathbf{r}) = -4\pi\rho(\mathbf{r})$, is given by [20] $\varphi(\mathbf{r}) = \int \rho(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|\, d^3\mathbf{r}'$. Here the potential is represented by $\varphi(\mathbf{r})$, and $\rho(\mathbf{r})$ is the charge density. The Green's function for the Poisson problem is $G(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$. The potential's first- and second-order derivatives are

$$\partial_i\varphi = -\int \frac{(r_i - r_i')\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}', \tag{1}$$

$$\partial_i\partial_j\varphi = \int \left[-\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{3(r_i-r_i')(r_j-r_j')}{|\mathbf{r}-\mathbf{r}'|^5}\right]\rho(\mathbf{r}')\, d^3\mathbf{r}'. \tag{2}$$

When $i = j$, we should have the result

$$\sum_{i=1}^{3}\partial_i\partial_i\varphi(\mathbf{r}) = \nabla^2\varphi(\mathbf{r}) = -4\pi\rho(\mathbf{r}), \tag{3}$$

from the standard identity $\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$. Thus we should be able to carry out the above integral explicitly, and we expect to have

$$\sum_{i=1}^{3}\int \left[-\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{3(r_i-r_i')^2}{|\mathbf{r}-\mathbf{r}'|^5}\right]\rho(\mathbf{r}')\, d^3\mathbf{r}' = -4\pi\rho(\mathbf{r}). \tag{4}$$

However, the individual integrals are singular as $\mathbf{r}\to\mathbf{r}'$ because of the factors of $1/|\mathbf{r}-\mathbf{r}'|^3$ and $1/|\mathbf{r}-\mathbf{r}'|^5$ in the integrand. In the following we consider the singularities in detail. Equation (4) also suggests that we can define a new form of the Dirac δ function, and we will consider this issue rigorously in Sec. II D.

B. Singular integrals

We classify a set of singular integrals that frequently occur in integral equations. Consider singular integrals of the form

$$\int_{|\mathbf{r}|<R} \frac{f(\mathbf{r})}{|\mathbf{r}|^d}\, d^D\mathbf{r}, \tag{5}$$

where $f(\mathbf{r})$ has a Taylor series expansion around the origin, and D is the dimension of space that we will take to be continuous. We introduce a shift in the denominator [8,21] by substituting $|\mathbf{r}| \Rightarrow \sqrt{r^2+\epsilon^2}$ in order to easily isolate the infinite part of the singular integral. At the end of the calculation, the limit $\epsilon \to 0$ will be imposed.

1. Basic singular integrals

We first consider a basic singular integral defined as

$$I_0(R;d,\delta) = \int_{|\mathbf{r}|<R} \frac{1}{|\mathbf{r}|^d}\, d^D\mathbf{r}. \tag{6}$$

Doing the "angular" integrations in D dimensions, we note that $d^D\mathbf{r} = A_D\, r^{D-1}\, dr$, where $A_D = 2\pi^{D/2}/\,\Gamma(D/2)$ is the surface area of the D-dimensional unit hypersphere. We change $|\mathbf{r}|$ in the denominator to $\rho = \sqrt{r^2+\epsilon^2}$ to write

$$I_0(R;d,\delta) \Rightarrow A_D \int_0^R \rho^{-d}\, r^{D-1}\, dr.$$

To leading order in ϵ the integral then becomes

$$\frac{I_0(R;d,\delta)}{A_D} = \epsilon^{\delta}\, \frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)} + \frac{R^\delta}{\delta}, \tag{7}$$

where $\delta = D - d$. The details of evaluating this integral are given in Appendix A. We now consider three limits for the integral I_0:

1. When $\delta > 0$, the first term vanishes when $\epsilon \to 0$. In this case I_0 is not a singular integral.
2. When $\delta \to 0$, we have

$$\frac{I_0(R;d,\delta)}{A_D} = -\frac{\Gamma'(d/2)}{2\,\Gamma(d/2)} - \frac{\gamma}{2} + \ln\frac{R}{\epsilon} + O(\delta), \tag{8}$$

where γ is Euler's constant $\gamma = 0.5772\ldots$. In this case the integral has a logarithmic singularity as $\epsilon \to 0$.

3. When $\delta < 0$, we have

$$\frac{I_0(R;d,\delta)}{A_D} = \frac{R^\delta}{\delta} + \epsilon^{-|\delta|}\left[\frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)}\right]. \tag{9}$$

In this case we see that the singular integral has an $\epsilon^{-|\delta|}$-type infinity as $\epsilon \to 0$.

Therefore, we separate the infinite part of the singular integral $I_0(R;d,\delta)$ as follows:

$$\frac{I_0(R;d,\delta)}{A_D} = \begin{cases} \frac{R^\delta}{\delta}, & \text{for } \delta > 0, \text{ no infinity},\\ -\frac{\Gamma'(d/2)}{2\,\Gamma(d/2)} - \frac{\gamma}{2} + \ln\frac{R}{\epsilon}, & \text{for } \delta = 0, \text{ log infinity},\\ \frac{R^\delta}{\delta} + \epsilon^{-|\delta|}\left[\frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)}\right], & \text{for } \delta < 0,\ \epsilon^{-|\delta|} \text{ infinity}. \end{cases} \tag{10}$$

Notice that the nature of the infinite part is determined only by $\delta = D - d$.

2. General singular integrals

The general singular integrals are defined as

$$I_k\big(R;d,\delta,\{n_i\}_{i=1}^k\big) = \int_{|\mathbf{r}|<R} \frac{x_1^{n_1} x_2^{n_2}\cdots x_k^{n_k}}{|\mathbf{r}|^{d+N}}\, d^D\mathbf{r}, \tag{11}$$

where $N = \sum_{i=1}^k n_i$. Let us first focus on the form

$$I_1(R;d,\delta,n) = \int_{|\mathbf{r}|<R} \frac{x^n}{|\mathbf{r}|^{d+n}}\, d^D\mathbf{r}. \tag{12}$$

The details of the evaluation of these integrals are given in Appendix A, and we provide only the results here. When n is odd, this integral vanishes, and when n is even, I_1 is given by

$$I_1(R;d,\delta,n) = \frac{(D-2)!!(n-1)!!}{(D+n-2)!!}\, I_0(R;d+n,\delta), \tag{13}$$

where $(n_i-1)!! = (n_i-1)(n_i-3)\cdots 1$, and $(-1)!! = 1$. The most commonly occurring nonzero case in typical applications is when $n = 2$ [see, for example, Eq. (4)], for which we obtain

$$I_1(R;d,\delta,2) = \int_{|\mathbf{r}|<R} \frac{x^2}{|\mathbf{r}|^{d+2}}\, d^D\mathbf{r} = \frac{1}{D}\, I_0(R;d+2,\delta). \tag{14}$$

Since the type of infinity just depends on δ, $I_1(R;d,\delta,n)$ has the same singular behavior as $I_0(R;d,\delta)$. In fact, the general singular integrals I_k are always multiples of the above basic singular integral I_0. When all n_i are even numbers, I_k is given by

$$I_k\big(R;d,\delta,\{n_i\}_{i=1}^k\big) = \frac{(D-2)!!\prod_{i=1}^k(n_i-1)!!}{(D+n-2)!!}\, I_0(R;d+N,\delta), \tag{15}$$

and I_k vanishes otherwise. We note that these integrals all have the same singular behavior as I_0.

3. ϵ-singular integrals

We can have a singular integral that has ϵ in the numerator. We assume that ϵ is a constant when performing the integration. Hence we will have

$$I_k^\epsilon\big(R;d,\delta,\{n_i\}_{i=0}^k\big) = \int_{|\mathbf{r}|<R} \frac{\epsilon^{n_0} x_1^{n_1} x_2^{n_2}\cdots x_k^{n_k}}{|\mathbf{r}|^{d+n_0+N}}\, d^D\mathbf{r} = \epsilon^{n_0} I_k\big(R;d+n_0,\delta-n_0,\{n_i\}_{i=1}^k\big), \tag{16}$$

where $N = \sum_{i=1}^k n_i$. We need consider only the case when all n_i are even, since the integral vanishes otherwise. For nonzero cases, we have

$$I_k^\epsilon\big(R;d,\delta,\{n_i\}_{i=0}^k\big) = \frac{(D-2)!!\prod_{i=1}^k(n_i-1)!!}{(D+N-2)!!} \times \epsilon^{n_0} I_0(R;d+N+n_0,\delta-n_0). \tag{17}$$

To simplify the notation we define $I_0^\epsilon(R;d,\delta,n_0) = \epsilon^{n_0} I_0(R;d+n_0,\delta-n_0)$, so that

$$I_k^\epsilon\big(R;d,\delta,\{n_i\}_{i=0}^k\big) = \frac{(D-2)!!\prod_{i=1}^k (n_i-1)!!}{(D+N-2)!!} \times I_0^\epsilon(R;d+N,\delta,n_0). \tag{18}$$

Therefore, I_k^ϵ is transformed to I_0^ϵ, and hence we need to discuss the properties of I_0^ϵ. This is done in the following.

4. Basic ϵ-singular integrals

With the result for I_0 derived above in Eq. (10), we have

$$I_0^\epsilon(R;d,\delta,n_0) = \epsilon^\delta \left[\frac{\Gamma\left(\frac{n_0-\delta}{2}\right)\Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d+n_0}{2}\right)}\right] + \epsilon^{n_0}\left(\frac{R^{\delta-n_0}}{\delta-n_0}\right). \tag{19}$$

The earlier definitions for I_k^ϵ and I_0^ϵ are convenient because a factor of ϵ^δ can be pulled out. Because n_0 is always greater than 0, the second term vanishes in the limit $\epsilon \to 0$ if $\delta \neq n_0$. When $\delta = n_0$, recall that I_0^ϵ is a multiple of I_0, and by Eq. (10) we have

$$\begin{aligned} I_0^\epsilon(R;d,\delta,n_0) &= \epsilon^{n_0} I_0(R;d+n_0,\delta-n_0) \\ &= \epsilon^{n_0}\left\{-\frac{\Gamma'[(d+n_0)/2]}{2\,\Gamma[(d+n_0)/2]} - \frac{\gamma}{2} + \ln\frac{R}{\epsilon}\right\} \\ &\to 0. \end{aligned} \tag{20}$$

Therefore, $I_0^\epsilon \to 0$ as $\epsilon \to 0$ for all $\delta > 0$, even when $\delta = n_0$. In summary, we obtain the following:

1. When $\delta > 0$, we simply get $I_0^\epsilon(R;d,\delta,n_0) = 0$.
2. When $\delta \to 0$, we have

$$I_0^\epsilon(R;d,\delta,n_0) = \frac{\Gamma\left(\frac{n_0}{2}\right)\Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d+n_0}{2}\right)} + O(d), \tag{21}$$

which is finite.

3. When $\delta < 0$, we have

$$I_0^\epsilon(R;d,\delta,n_0) = \epsilon^{-|\delta|}\left[\frac{\Gamma\left(-\frac{\delta-n_0}{2}\right)\Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{d+n_0}{2}\right)}\right], \tag{22}$$

which has a singularity arising from the $\epsilon^{-|\delta|}$ factor.

C. Integrable singular integrals

If two singular integrals have the same infinite part their difference is a finite number. More generally, a linear combination of singular integrals may sum to a finite number when their infinite parts cancel. We call such combinations as *integrable singular integrals* (ISIs) [22]. As will be shown below, most of the singular integrals in physics applications of potential theory and engineering analysis using Green's functions are ISIs [23].

For the nonzero cases, the integrals $I_k(R;d,\delta,\{n_i\}_{i=1}^k)$ are always a multiple of $I_0(R;d+N,\delta)$, so that both classes of integrals have the same type of infinity: logarithmic infinity when $\delta = 0$ and $\epsilon^{-|\delta|}$-type infinity when $\delta < 0$. Therefore, we can take the linear combination of $I_k(R;d,\delta,\{n_i\}_{i=1}^k)$ and $I_0(R;d,\delta)$ to cancel the singular parts and obtain ISIs [24]. Such ISIs are given by

$$I_0(R;d,\delta) - \frac{(d+N-2)!!}{(d-2)!!\prod_{i=1}^k (n_i-1)!!} I_k\big(R;d,\delta,\{n_i\}_{i=1}^k\big)$$

$$= \begin{cases} \left[1 - \frac{(d+N-2)!!(D-2)!!}{(D+N-2)!!(d-2)!!}\right]\frac{A_D}{\delta} R^\delta, & \text{for } \delta < 0, \\ \frac{1}{2}\left[\Psi\left(\frac{d+N}{2}\right) - \Psi\left(\frac{d}{2}\right)\right] A_D, & \text{for } \delta = 0, \end{cases} \tag{23}$$

where all n_i are even, and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. $\Psi[(d+N)/2] - \Psi(d/2)$ can be written as

$$\Psi\left(\frac{d+N}{2}\right) - \Psi\left(\frac{d}{2}\right) = \frac{2}{d} + \frac{2}{d+2} + \cdots + \frac{2}{d+N-2}. \tag{24}$$

Another type of ISI includes I_k^ϵ. Because I_k^ϵ can always be transformed to I_0^ϵ, we just need to consider I_0^ϵ. We noted earlier that I_0^ϵ is finite when $\delta = 0$, and is integrable. When $\delta < 0$, we have

$$I_0(R;d,\delta) - \left[\frac{\Gamma\left(-\frac{\delta}{2}\right)}{\Gamma\left(-\frac{\delta}{2}+\frac{n_0}{2}\right)}\frac{\Gamma\left(\frac{d}{2}+\frac{n_0}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right] I_0^\epsilon(R;d,\delta,n_0)$$

$$= \frac{A_D}{\delta} R^\delta. \tag{25}$$

We call the above the *fundamental ISIs* because all the other ISIs can be written as linear combinations of them. Some simple examples of fundamental ISIs are given as follows:

1. By setting $k=1, n_1=2$ in Eq. (23), we obtain *the simplest ISI,* which takes the form

$$\int_{|\mathbf{r}|<R}\left(\frac{1}{|\mathbf{r}|^d} - d\,\frac{x^2}{|\mathbf{r}|^{d+2}}\right) d^D\mathbf{r} = \frac{A_D}{D} R^\delta. \tag{26}$$

It can be checked that this formula holds for all $\delta \geqslant 0$ or $\delta < 0$.

2. By setting $n_0 = 2$ in Eq. (25), we have another ISI obtained from I_0 and I_0^ϵ, which we call ϵ^2-ISI, for which

$$\int_{|\mathbf{r}|<R}\left(\frac{1}{|\mathbf{r}|^d} - \frac{d}{d-D}\,\frac{\epsilon^2}{|\mathbf{r}|^{d+2}}\right) d^D\mathbf{r} = \frac{A_D}{\delta} R^\delta. \tag{27}$$

D. The Dirac δ function in ISI

We note that

$$\delta^{(D)}(\mathbf{r}) = \frac{1}{A_D}\sum_{i=1}^D\left(\frac{1}{|\mathbf{r}|^D} - \frac{D\,r_i^2}{|\mathbf{r}|^{D+2}}\right) \tag{28}$$

is a Dirac δ function in the D dimension in the sense that [25]

$$\frac{1}{A_D}\sum_{i=1}^D \int \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^D} - \frac{D(r_i - r_i')^2}{|\mathbf{r}-\mathbf{r}'|^{D+2}}\right]\rho(\mathbf{r}')\,d^D\mathbf{r}' = \rho(\mathbf{r}). \tag{29}$$

To verify the above we take a series expansion of $\rho(\mathbf{r}')$:

$$\rho(\mathbf{r}') = \rho(\mathbf{r}) + (\mathbf{r}'-\mathbf{r})\cdot\nabla\rho(\mathbf{r}) + O[(\mathbf{r}'-\mathbf{r})^2]. \tag{30}$$

The leading term of the expansion gives

$$\sum_{i=1}^{D}\int\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D}}-\frac{D(r_i-r_i')^2}{|\mathbf{r}-\mathbf{r}'|^{D+2}}\right]\rho(\mathbf{r})\,d^D\mathbf{r}' = \rho(\mathbf{r})D\int\left(\frac{1}{|\mathbf{s}|^D}-\frac{Ds_i^2}{|\mathbf{s}|^{D+2}}\right)d^D\mathbf{s}, \quad (31)$$

where $\mathbf{s}=\mathbf{r}'-\mathbf{r}$. This is the simplest ISI with $d=D$. So from Eq. (26) we have

$$\sum_{i=1}^{D}\int\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D}}-\frac{D(r_i-r_i')^2}{|\mathbf{r}-\mathbf{r}'|^{5}}\right]\rho(\mathbf{r})\,d^D\mathbf{r}' = A_D\rho(\mathbf{r}). \quad (32)$$

We can show that the further terms in the series expansion are zero. Actually, any integral of the following form can be expressed as

$$\sum_{i=1}^{D}\int_{|\mathbf{s}|<R}\left(\frac{1}{|\mathbf{s}|^D}-\frac{Ds_i^2}{|\mathbf{s}|^{D+2}}\right)\prod_{i=1}^{D}s_i^{a_i}\,d^D\mathbf{s}=\lambda\, I_0(R;d,\delta), \quad (33)$$

where $d=D-\sum a_i$, with a_i being the power of s_i in the Taylor expansion of $\rho(\mathbf{r}')$, $\delta=D-d=\sum a_i>0$, and λ is a constant that is obtained by doing the angular integration and is given by Eq. (15). Because $\delta>0$, we know this is a regular integral with no singularity, and from Eq. (10) we obtain

$$\sum_{i=1}^{D}\int_{|\mathbf{s}|<R}\left(\frac{1}{|\mathbf{s}|^D}-\frac{Ds_i^2}{|\mathbf{s}|^{D+2}}\right)\prod_{i=1}^{D}s_i^{a_i}\,d^D\mathbf{s}=A_D\lambda\,\frac{R^\delta}{\delta}. \quad (34)$$

On the other hand, by evaluating the difference between two such integrals over the ranges $|\mathbf{s}|<R_1$ and $|\mathbf{s}|<R_2$ with $R_2>R_1$ we have

$$\sum_{i=1}^{D}\int_{R_1}^{R_2}\left(\frac{1}{|\mathbf{s}|^D}-\frac{Ds_i^2}{|\mathbf{s}|^{D+2}}\right)\prod_{i=1}^{D}s_i^{a_i}\,d^D\mathbf{s}=A_D\lambda\,\frac{R_2^\delta-R_1^\delta}{\delta}. \quad (35)$$

We note that for $\mathbf{s}\neq 0$

$$\sum_{i=1}^{D}\left(\frac{1}{|\mathbf{s}|^D}-\frac{Ds_i^2}{|\mathbf{s}|^{D+2}}\right)=\frac{D}{|\mathbf{s}|^D}-\frac{Ds^2}{|\mathbf{s}|^{D+2}}=0. \quad (36)$$

Hence the left-hand side of Eq. (35) is zero, so that $\lambda=0$. Therefore the integral in Eq. (34) vanishes. Combined with Eq. (32), we reconstruct the relation Eq. (29).

Also, we can write the δ function as a limit:

$$\delta^{(D)}(\mathbf{r})=\frac{D}{A_D}\lim_{\epsilon\to 0}\left[\frac{1}{(\sqrt{r^2+\epsilon^2})^D}-\frac{r^2}{(\sqrt{r^2+\epsilon^2})^{D+2}}\right] = \frac{D}{A_D}\lim_{\rho\to r^+}\left(\frac{1}{\rho^D}-\frac{r^2}{\rho^{D+2}}\right). \quad (37)$$

This new representation of the Dirac δ function can be used to directly prove Eq. (4).

III. FURTHER RESULTS ON INTEGRABLE SINGULAR INTEGRALS

We present here two important additional results that can further substantially simplify the evaluation of integrable singular integrals.

A. Singular integrals arising from the derivatives of Green's functions

With the formulas obtained from dimensional continuation above we can prove a general theorem that shows that *the singular integrals coming from the derivative of Green's functions must actually be finite.*

Theorem: If the Green's function $G(\mathbf{r})$ can be expressed as the following series expansion:

$$G(\mathbf{r})=\sum_M\sum_{\{n_i\}}a_{M,\{n_i\}}\frac{r_1^{n_1}r_2^{n_2}\cdots r_D^{n_D}}{r^M}, \quad (38)$$

then we have the following equality:

$$\int_{r\leqslant R}\frac{\partial G(\mathbf{r})}{\partial r_i}\,d^D\mathbf{r}=\frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}r_i\,G(\mathbf{r})\,d^D\mathbf{r}, \quad (39)$$

which is always a finite number. We have relegated the proof of this theorem to Appendix A. With this theorem we can show that the integral of the double derivatives of the Green's function is also integrable, as shown below. In fact, this implies that any finite-order derivatives of Green's functions can be integrated.

Assume the conditions of the above theorem hold for $G(\mathbf{r})$. Then the integral of the double derivative of G is also finite, where we have

$$\begin{aligned}\int_{r\leqslant R}\partial_i\partial_j G\,d^D\mathbf{r} &= \frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}r_i\,\partial_j G\,d^D\mathbf{r}\\ &=\frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}[\partial_j(r_i\,G)-\delta_{ij}\,G]\,d^D\mathbf{r}\\ &=\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}r_ir_j\,G\,d^D\mathbf{r}\right)\\ &\quad-\delta_{ij}\,\frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}G\,d^D\mathbf{r}.\end{aligned} \quad (40)$$

B. Singular integrals over a general shape

In practical applications, we usually have singular integrals over volumes V of any general shape rather than necessarily spherically symmetric regions. We now generalize our method to account for this in the following. Assume the conditions in the above theorem hold for $G(\mathbf{r})$. For an analytic function $u(\mathbf{r})$, we have

$$\begin{aligned}\int_{r\leqslant R}u\,\partial_i G\,d^D\mathbf{r} &= \int_{r\leqslant R}[\partial_i(u\,G)-G\,\partial_i u]\,d^D\mathbf{r}\\ &=\frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R}r_i\,u\,G\,d^D\mathbf{r}-\int_{r\leqslant R}G\,\partial_i u\,d^D\mathbf{r}.\end{aligned} \quad (41)$$

We employ the characteristic function (Ref. [26], p. 313) defined by

$$\chi_V(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \in V, \\ 0 & \mathbf{r} \notin V. \end{cases} \tag{42}$$

[This is a generalization of the usual step function $\theta(x)$, which is zero for $x < 0$ and unity for $x > 0$.] Then if we take a large enough spherical integration range such that it contains V, we have

$$\int_{r \leqslant R} \chi_V\, \partial_i G\, d^D\mathbf{r} = \int_V \partial_i G\, d^D\mathbf{r}. \tag{43}$$

By the Stone-Weierstrass theorem (Ref. [26], p. 159), the characteristic function χ_V can be approximated closely by polynomial functions. We can therefore apply this result to obtain

$$\begin{aligned} \int_V \partial_i G\, d^D\mathbf{r} &= \frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R} r_i\, \chi_V\, G\, d^D\mathbf{r} - \int_{r\leqslant R} G\, \partial_i \chi_V\, d^D\mathbf{r} \\ &= \hat{\mathbf{e}}_i \cdot \int_{\partial V} \mathbf{n}(\mathbf{r})\, G(\mathbf{r})\, d^{D-1}\mathbf{r}, \end{aligned} \tag{44}$$

where ∂V is the boundary of V, $\mathbf{n}(\mathbf{r})$ is the (outward directed) unit normal vector on ∂V at $\mathbf{r}$, and $\hat{\mathbf{e}}_i$ is the ith unit basis vector. The first term in Eq. (44), having $\partial/\partial R$, vanishes because χ_V is zero outside V, so the integral does not depend on R. The derivative of χ_V will become (the negative of) the δ function on ∂V, as the derivative of the step function $\theta(x)$ is the delta function $\delta(x)$, so the second term becomes an integration on the boundary.

The above discussion is a physical explanation rather than a strict mathematical proof. In order to provide a rigorous proof, we need to have several conditions for V, such as the requirement for compactness and convexity, and by saying that χ_V can be approximated by polynomials we actually mean a uniform convergence as stated by the Stone-Weierstrass theorem. We also note that Eq. (44) can be applied when G has 0^{-n} singularities inside V; however, it cannot be applied when G has logarithmic singularities.

The formula in Eq. (44) is especially useful in the boundary integral equation (BIE) method, where we break up the surface into discrete triangles and have singular integrals of the forms $\int_\Delta \rho\, \partial_i G\, dS$ and $\int_\Delta \rho\, \partial_i \partial_j G\, dS$. In a small triangle Δ enclosing the singularity we can assume ρ to be a constant ρ_0. With the above formula, we have

$$\int_\Delta \partial_i G\, dS = \sum_{k=1}^{3} \left(\hat{\mathbf{e}}_i \cdot \mathbf{n}_k \int_{l_k} G\, dl \right), \tag{45}$$

where the sum over k means that we evaluate the integral over the three sides of the triangle, thereby reducing the BIE integral to a sum of one-dimensional integrals.

IV. POISSON PROBLEM IN A FINITE DOMAIN

A. Poisson's equation in three dimensions

Now we consider the boundary integral approach to a typical 3D potential problem for which the Poisson Green's function is $G(\mathbf{x},\mathbf{y}) = 1/|\mathbf{x}-\mathbf{y}|$. The boundary integral equation is given by

$$\int_\Gamma G(\mathbf{x},\mathbf{y}) \frac{\partial \varphi(\mathbf{y})}{\partial n}\, dS_{\mathbf{y}} - \int_\Gamma \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) \varphi(\mathbf{y})\, dS_{\mathbf{y}} = 4\pi \varphi(\mathbf{x}), \tag{46}$$

where $\varphi(\mathbf{x})$ is the potential needed to be solved on the boundary Γ, and $\mathbf{n}(\mathbf{y})$ is the unit normal vector on Γ at $\mathbf{y}$. In a numerical approximation of the boundary integral equation we divide the surface Γ into nonoverlapping contiguous triangles, so the original integration can be expressed by a sum of 2D integrations over flat triangles, and we express $\varphi(\mathbf{y})$ by polynomial functions in each triangle. In essence this is the boundary element method. One problem arising in this calculation is that if $\mathbf{x}$ and $\mathbf{y}$ lie in the same triangle, the integration over this triangle will become singular because the singularity appearing in $G(\mathbf{x},\mathbf{y})$ and its derivative (and in other examples its higher derivatives). We note that this problem does not occur if $\mathbf{x}$ and $\mathbf{y}$ lie in different triangles because they are separated and their distance will have a lower bound, and $G(\mathbf{x},\mathbf{y})$ will be a finite number.

In the following, we restrict our attention to the case when $\mathbf{x}$ and $\mathbf{y}$ are in the same triangle Δ, and we define $\mathbf{r} = \mathbf{x} - \mathbf{y}$. The class of singular integrals that appear in such calculations are

$$\begin{aligned} I_1 &= \int_\Delta \frac{1}{r}\, dS, \quad I_1^i = \int_\Delta \frac{r_i}{r}\, dS, \\ I_3^i &= \int_\Delta \frac{r_i}{r^3}\, dS, \quad I_3^{ij} = \int_\Delta \frac{r_i\, r_j}{r^3}\, dS, \\ \delta_{ij}\, I_3 - 3\, I_5^{ij} &= \int_\Delta \left(\frac{\delta_{ij}}{r^3} - \frac{3\, r_i\, r_j}{r^5} \right) dS. \end{aligned} \tag{47}$$

The analytic results for these integrals have been evaluated previously by accounting for the singularities through lengthy procedures in two dimensions [18,19,27]. Here we are using the notation in Ref. [18] for the integrals. Within our framework, as developed in this report, the analytic expressions can be obtained more easily because, with the exception of I_1 which is a finite integral, the 2D integrals can be transformed to (1D) line integrals, by employing the result in Sec. III B, as follows:

$$\begin{aligned} I_1^i &= \sum_{k=1}^{3} \left(\hat{\mathbf{e}}_i \cdot \mathbf{n}_k \int_{l_k} r\, dS \right), \quad I_3^i = -\sum_{k=1}^{3} \left(\hat{\mathbf{e}}_i \cdot \mathbf{n}_k \int_{l_k} \frac{1}{r}\, dS \right), \\ I_3^{ij} &= -\int_\Delta r_j\, \partial_i \frac{1}{r}\, dS = -\sum_{k=1}^{3} \left(\hat{\mathbf{e}}_i \cdot \mathbf{n}_k \int_{l_k} \frac{r_j}{r}\, dl \right) + \delta_{ij}\, I_1, \\ \delta_{ij}\, I_3 - 3\, I_5^{ij} &= \int_\Delta \partial_i \partial_j \frac{1}{r}\, dS = \sum_{k=1}^{3} \left(\hat{\mathbf{e}}_i \cdot \mathbf{n}_k \int_{l_k} \partial_j \frac{1}{r}\, dl \right). \end{aligned} \tag{48}$$

For the Poisson Green's function, it can be verified that the numerical results from the line integration perfectly match the analytic expressions of the earlier work when they are evaluated numerically [18]. We have verified that our line integrals can also be evaluated analytically to obtain the same expressions. In Table I we give the numerical comparison between the two methods. We have taken the three vertices of the triangle, shown in Fig. 3, to be (−2, − 1), (2, − 2), and (1,1) as a general example for obtaining concrete numerical results.

For more complex Green's functions, evaluating the analytic expressions on arbitrary 2D surfaces with singularities appearing inside them will be very lengthy procedures within the framework of the methods used in the literature. Analytic approaches in two dimensions would be more complex than the 1D analytic approach presented here. Second, if numerical integrations are performed over the 2D region, the presence of the singularities reduce accuracy of the integrals in the intermediate steps of the analysis. The 1D numerical integrals that one would encounter correspondingly in our method will not have any reduction in accuracy due to the singularities since the integrations are on the boundary.

B. Poisson's equation in two dimensions

For completeness we present a short elaboration of singular integrals appearing in the 2D boundary integral method, even though the following integrals are not ISIs. In the 2D Poisson problem, cast in terms of the boundary integral method, we have [17]

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}\oint dl' \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\varphi(\mathbf{r}')}{\partial n'} - \varphi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right], \quad (49)$$

where $G(\mathbf{r},\mathbf{r}') = -2\ln|\mathbf{r}-\mathbf{r}'|$. We can assume φ and $\partial_{n'}\varphi$ to be constants, as a worst case scenario, over a small line element from ℓ_a to ℓ_b, so that we need to evaluate the singular integrals:

$$B_1 = \int_{\ell_a}^{\ell_b} \ln s\, dl', \quad B_2 = \int_{\ell_a}^{\ell_b} \frac{\mathbf{s}\cdot\mathbf{n}'}{s^2}\, dl', \quad (50)$$

where $\mathbf{s} = \mathbf{r} - \mathbf{r}'$. B_1 is a well-defined integrable end-point singular integral typified by

$$\int_0^R \ln x\, dx = \lim_{\epsilon\to 0}(x\ln x - x)\big|_\epsilon^R = R\ln R - R. \quad (51)$$

The integral B_2 can be evaluated using the point-shifting technique used earlier. We make use of the geometry displayed in Fig. 1 and write $dl' = s\,d\theta/\cos\alpha = s\,d\theta/\hat{\mathbf{s}}\cdot\mathbf{n}'$. Then,

$$B_2 = \int_{\ell_a}^{\ell_b} \frac{\mathbf{s}\cdot\mathbf{n}'}{s^2}\frac{s\,d\theta}{\hat{\mathbf{s}}\cdot\mathbf{n}'} = \int_{\ell_a}^{\ell_b} d\theta. \quad (52)$$

This integral in the limit $\epsilon \to 0$ corresponds to an angle subtended by the contour at the singular point, so that for a straight contour [see Fig. 2(a)] we have $B_2 = \pi$, while for a corner, as shown in Fig. 2(b), we have $B_2 = 3\pi/2$, as an exterior angle. Further details can be obtained for 2D treatments of the boundary integral method in Ref. [17].

V. ISI IN ELECTROMAGNETIC SCATTERING

We illustrate the above considerations with a brief application to the evaluation of electromagnetic fields emitted by a conducting surface [28,29], where again integrable singularities occur. (An additional example from the field of fracture dynamics is described briefly in Appendix B.)

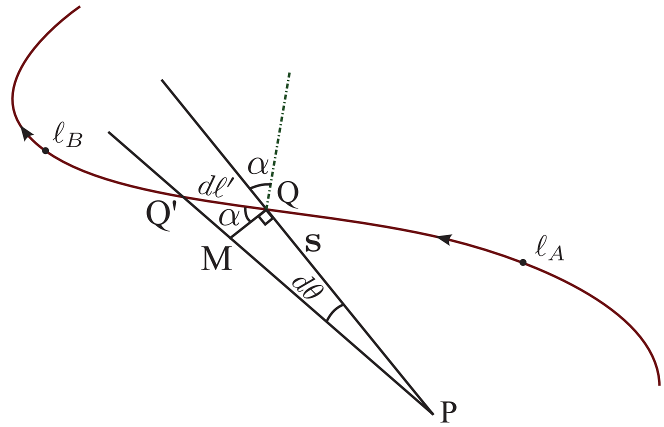

FIG. 1. (Color online) The contour used to identify the terms in the integrand of the boundary integral approach for evaluating the 2D Poisson potential.

A. 3D scattering

In 3D, the electric field radiated by a conducting surface takes the form [21,30]

$$\mathbf{E} = -ikZ_0\int_S\left[G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}') + \frac{1}{k^2}\nabla\nabla G(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\right]dS', \quad (53)$$

where the Green's function is given by $G(\mathbf{r},\mathbf{r}') = e^{ik\varrho}/4\pi\varrho$, with $\varrho = |\mathbf{r}-\mathbf{r}'|$, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space [28]. The second term in the integral involves a second derivative of the Green's function, and therefore the corresponding integral is a hypersingular integral. It is usual to discretize the surface into small elements. As a simple approximation, we may assume that the current $\mathbf{J}(\mathbf{r}')$ is a constant $\mathbf{J_0}$ over a suitably small element (in general it can be taken to be a simple polynomial). We can then write the second term of the integral explicitly as

$$\int_{\Delta S}\nabla\nabla G(\mathbf{r},\mathbf{r}')\cdot\mathbf{J_0}\,dS = \mathbf{J_0}\cdot\sum_{i,j}\hat{\varrho}_i\hat{\varrho}_j\int_{\Delta S}G_{ij}\,dS, \quad (54)$$

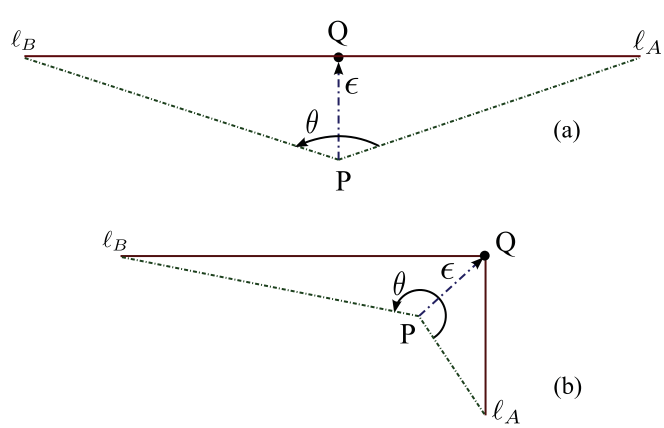

FIG. 2. (Color online) The geometry used in evaluating the 2D Poisson contour integral in the boundary element method: (a) for a straight contour and (b) for an angular edge.

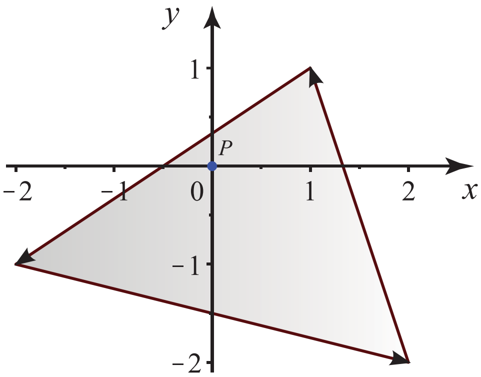

FIG. 3. (Color online) The 2D triangular region for the evaluation of the hypersingular integrals in the 3D Poisson problem with a singularity located at (0,0) is shown. The 2D integrals over the triangle calculated using the expressions given by Refs. [18,19] are compared with our 1D line integral along the edges of the triangle in Table I.

where ΔS is an element containing the singularity and

$$G_{ij} = \left[\frac{(3 - 3ik\varrho - k^2\varrho^2)\varrho_i\varrho_j}{\varrho^5} - \frac{\delta_{ij}(1 - ik\varrho)}{\varrho^3}\right]e^{ik\varrho}. \tag{55}$$

The singular integrals in $\int G_{ij}\,dS$ are

$$K_1 = \int_{\Delta S}\left(\frac{3\varrho_i^2}{\varrho^5} - \frac{1}{\varrho^3}\right)dS, \quad K_2 = \int_{\Delta S}\frac{\varrho_i\varrho_j}{\varrho^5}\,dS, \quad (i \neq j), \tag{56}$$

where we have expanded the exponential $\exp(ik\varrho) \simeq (1 + ik\varrho)$ for small ϱ to isolate the singular terms. If we take the region of integration to be a circle around the singularity, we find $K_1 = \pi/R$ is the simplest ISI with $d = 3, D = 2$, and $K_2 = 0$ as is evident from Eq. (A13). In the full calculation, we have to take the integral within and outside the circular region separately; we then note that the integral over the exterior of the circle is a regular integral and can be computed directly. These identifications of the finite parts should substantially simplify the computational modeling of electromagnetic scattering.

B. 2D scattering

In 2D scattering, the Green's function is given by [28]

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik\varrho}}{\sqrt{\varrho}}, \tag{57}$$

where $\varrho = |\mathbf{r} - \mathbf{r}'|$. Similarly, the components of $\nabla\nabla G$ are

$$\begin{aligned}\partial_i\partial_j G &= \left[\left(\tfrac{5}{4}\,\varrho^{-\frac{9}{2}} - 2ik\,\varrho^{-\frac{7}{2}} - k^2\,\varrho^{-\frac{5}{2}}\right)\varrho_i\varrho_j \right.\\ &\quad \left. + \left(-\tfrac{1}{2}\,\varrho^{-\frac{5}{2}} + ik\,\varrho^{-\frac{3}{2}}\right)\delta_{ij}\right]e^{ik\varrho}\\ &= \tfrac{5}{4}\,\varrho_i\varrho_j\,\varrho^{-\frac{9}{2}} - \tfrac{1}{2}\,\delta_{ij}\varrho^{-\frac{5}{2}}\\ &\quad + ik\left(-\tfrac{3}{4}\,\varrho_i\varrho_j\,\varrho^{-\frac{7}{2}} + \tfrac{1}{2}\,\delta_{ij}\varrho^{-\frac{3}{2}}\right) + O\left(\varrho^{-\frac{1}{2}}\right).\end{aligned} \tag{58}$$

It can be easily checked that the singular terms in $\int G_{ij}\,dS$ also sum to ISIs. For example, the leading terms of $\int G_{ij}\,dS$ given by

$$\begin{aligned}K_3 &= \int_{\varrho<R}\left(\frac{1}{\varrho^{5/2}} - \frac{5}{2}\frac{\varrho_i^2}{\varrho^{9/2}}\right)dS,\\ K_4 &= \int_{\varrho<R}\left(\frac{1}{\varrho^{3/2}} - \frac{3}{2}\frac{\varrho_i^2}{\varrho^{7/2}}\right)dS,\end{aligned} \tag{59}$$

where $K_3 = \pi R^{-1/2}$ is the simplest ISI with $d = 5/2, D = 2$, and $K_4 = \pi R^{1/2}$ is the simplest ISI with $d = 3/2, D = 2$.

VI. CONCLUDING REMARKS

We have used dimensional continuation in the evaluation of integrals of the well-known Green's functions and their derivatives. We have identified the rules for obtaining consistent results through the use of such methods for the hypersingular integrals occurring in the BEM, potential theory, electromagnetic scattering, and fracture analysis (see Appendix B). We have provided a systematic approach to the identification of the singularities in typical integrals and shown how to isolate them using the dimensional continuation method. We have identified new representations for the Dirac δ function in D dimensions that are not stated in the standard literature. These results are then used in the calculation of examples of such integrals occurring in physical applications. A summary of the integrable singular integrals is given in tabular form in Appendix C. The theorem presented in Sec. III shows how the potential problems in three dimensions, which are reduced to 2D boundary integrals by Green's theorem with hypersingularities, can be further reduced to 1D finite integrals. This provides a concrete example of the strength of our approach through the further reduction in dimensionality afforded the application of the theorem.

It is hoped that the present approach will provide an effective, practical method of evaluating the so-called hypersingular integrals in computational science and engineering applications. Our tabulated ISIs will lead to an automated computation of the physical quantities of interest without having to recalculate finite parts of integrals for each specific occurrence.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under grant number ECS-0725427, and by the Defense Advanced Research Projects Agency (DARPA), administered by AFRL/WPAFB under grant number FA8650-10-1-7046.

APPENDIX A: CLASSIFICATION OF SINGULAR INTEGRALS

For convenience, we use the substitutions

$$\begin{aligned}\rho^2 &= r^2 + \epsilon^2, \quad \mathbf{r} = (x_1,x_2,\ldots,x_D),\\ \delta &= D - d, \quad A_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}.\end{aligned} \tag{A1}$$

We will always use d as the order of the singularity of the integrand, i.e., the power of r in the denominator of the integrand, and D as the dimension of the multidimensional integration. It will be shown below that whether an integral is singular, and if so the type of the infinite part, is determined by $\delta = D - d$. Here A_D is the surface area of the D-dimensional unit hypersphere.

1. The basic integral $I_0(R;d,\delta) = \int_{|\mathbf{r}|<R} \frac{1}{|\mathbf{r}|^d}\, d^D\mathbf{r}$

Doing the "angular" integrations in D dimensions, we note that $d^D\mathbf{r} = A_D\, r^{D-1} dr$. We change $|\mathbf{r}|$ in the denominator to $\rho = \sqrt{r^2+\epsilon^2}$ to write

$$I_0(R;d,\delta) \Rightarrow A_D \int_0^R \rho^{-d}\, r^{D-1}\, dr.$$

The integral then becomes

$$\frac{I_0(R;d,\delta)}{A_D} = \int_0^\infty \rho^{-d}\, r^{D-1}\, dr - \int_R^\infty \rho^{-d}\, r^{D-1}\, dr. \tag{A2}$$

The first integral can be expressed in terms of gamma functions,

$$\int_0^\infty \frac{r^{D-1}}{\rho^d}\, dr = \epsilon^\delta \left[\frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)}\right], \tag{A3}$$

and the second integral can be expressed as a hypergeometric function,

$$\int_R^\infty \frac{r^{D-1}}{\rho^d}\, dr = -\left(\frac{R^\delta}{\delta}\right) {}_2F_1\left(\frac{d}{2}, -\frac{\delta}{2}; 1-\frac{\delta}{2}; -\frac{\epsilon^2}{R^2}\right), \tag{A4}$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(\xi)_n = \xi(\xi+1)(\xi+2)\cdots(\xi+n-1)$, $(\xi)_0 = 1$. From the series expansion of the hypergeometric function, to leading order in ϵ, we have

$$\int_R^\infty \frac{r^{D-1}}{\rho^d} dr = -\frac{R^\delta}{\delta}[1 + O(\epsilon^2)]. \tag{A5}$$

Therefore,

$$\frac{I_0(R;d,\delta)}{A_D} = \epsilon^\delta \frac{\Gamma(-\delta/2)\,\Gamma(D/2)}{2\,\Gamma(d/2)} + \frac{R^\delta}{\delta}. \tag{A6}$$

2. Integrals of the type $I_1(R;d,\delta,n) = \int_{|\mathbf{r}|<R} \frac{x^n}{|\mathbf{r}|^{d+n}}\, d^D\mathbf{r}$

We suppose that the x above coincides with one of the x_i in Eq. (A1). In this case, we make the usual substitution into hyperspherical coordinates:

$$\begin{aligned} x_1 &= r\cos\theta_1, \quad 0 \leqslant \theta_1 \leqslant \pi,\\ x_2 &= r\sin\theta_1\,\cos\theta_2, \quad 0 \leqslant \theta_2 \leqslant 2\pi,\\ &\vdots \quad \vdots\\ x_D &= r\sin\theta_1\,\sin\theta_2\,\cdots\,\sin\theta_{D-1}\\ d^D\mathbf{r} &= r^{D-1}\sin^{D-2}\theta_1\cdots dr\, d\theta_1\cdots d\theta_{D-1}, \end{aligned} \tag{A7}$$

and shift the denominator from $|\mathbf{r}|$ to $\rho = \sqrt{r^2+\epsilon^2}$ (with $x \equiv x_1$ without loss of generality, and $\theta_1 = \theta$) to write

$$\begin{aligned} I_1(R;d,\delta,n) &\Rightarrow A_{D-1}\int_0^\pi \int_0^R \frac{(r^n\cos^n\theta)}{\rho^{d+n}}\, r^{D-1}\sin^{D-2}\theta\, dr\, d\theta\\ &= A_{D-1}\int_0^R \rho^{-(d+n)}\, r^{[(D+n)-1]}\, dr \int_0^\pi \cos^n\theta\,\sin^{D-2}\theta\, d\theta\\ &= I_0(R;d+n,\delta)\cdot\frac{A_{D-1}}{A_D}\int_0^\pi \cos^n\theta\,\sin^{D-2}\theta\, d\theta. \end{aligned} \tag{A8}$$

The angular integrals have been suppressed into A_{D-1}, A_D, which are the surface areas of the unit hypersphere in $D-1$ and D dimensions. The last integral in Eq. (A8) is a beta function, and we have

$$\frac{A_{D-1}}{A_D}\int_0^\pi \cos^n\theta\sin^{D-2}\theta\, d\theta = \begin{cases} 0, & \text{for } n \text{ odd},\\ \frac{\Gamma(\frac{D}{2})\,\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{D+n}{2})}, & \text{for } n \text{ even}. \end{cases} \tag{A9}$$

When n is even, the gamma functions can be simplified further to obtain

$$I_1(R;d,\delta,n) = \frac{(n-1)(n-3)\cdots 1}{(D+n-2)(D+n-4)\cdots D}\, I_0(R;d+n,\delta). \tag{A10}$$

Since the type of infinity just depends on δ, $I_1(R;d,\delta,n)$ has the same singular behavior as $I_0(R;d,\delta)$. For example, the most commonly occurring nonzero case in typical applications is when $n=2$, for which we obtain

$$I_1(R;d,\delta,2) = \int_{|\mathbf{r}|<R} \frac{x^2}{|\mathbf{r}|^{d+2}}\, d^D\mathbf{r} = \frac{1}{D}\, I_0(R;d+2,\delta). \tag{A11}$$

A. Integrals of the type $I_2(R;d,\delta,m,n) = \int_{|\mathbf{r}|<R} \frac{x_1^m x_2^n}{|\mathbf{r}|^{d+m+n}}\, d^D\mathbf{r}$

For such integrals we again make the substitutions into hyperspherical coordinates

$$x_1 = r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2, \quad d^D\mathbf{r} = A_{D-2}r^{D-1}\sin^{D-2}\theta_1\,\sin^{D-3}\theta_2\, dr\, d\theta_1\, d\theta_2, \tag{A12}$$

and as usual change $|\mathbf{r}|$ in the denominator to $\rho = \sqrt{r^2+\epsilon^2}$ to obtain

$$\begin{aligned} I_2(R;d,\delta,m,n) &\Rightarrow A_{D-2}\int_0^\pi\int_0^\pi\int_0^R \frac{r^{m+n}\cos^m\theta_1\sin^n\theta_1\cos^n\theta_2}{\rho^{d+m+n}} r^{D-1}\sin^{D-2}\theta_1\sin^{D-3}\theta_2\,dr\,d\theta_1\,d\theta_2 \\ &= I_0(R;d+m+n,\delta)\frac{A_{D-2}}{A_D}\int_0^\pi \cos^m\theta_1\sin^{D+n-2}\theta_1\,d\theta_1\int_0^\pi\cos^n\theta_2\sin^{D-3}\theta_2\,d\theta_2 \\ &= \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\pi\,\Gamma\left(\frac{D+m+n}{2}\right)} I_0(R;d+m+n,\delta), \end{aligned} \tag{A13}$$

when both m and n are even, and $I_2 = 0$ otherwise. Thus I_2 also has the same singular behavior as $I_0(R;d,\delta)$.

4. Integrals of the type $I_k(R;d,\delta,\{n_i\}_{i=1}^k) = \int_{|\mathbf{r}|<R}\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{|\mathbf{r}|^{d+N}}\,d^D\mathbf{r};\ N=\sum_{i=1}^k n_i$

Using the same approach as above, we can obtain a general formula

$$I_k\left(R;d,\delta,\{n_i\}_{i=1}^k\right) = \int_{|\mathbf{r}|<R}\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{|\mathbf{r}|^{d+N}}\,d^D\mathbf{r} = \frac{\prod_{i=1}^k(n_i-1)!!}{(D+N-2)(D+N-4)\cdots D} I_0(R;d+N,\delta), \tag{A14}$$

when all n_i are even, and $I_k = 0$ otherwise. Also, $(n_i-1)!! = (n_i-1)(n_i-3)\cdots 1$, and $(-1)!! = 1$.

5. Integral of the derivative of Green's function

Theorem. If the Green's function $G(\mathbf{r})$ can be expressed as the following series expansion:

$$G(\mathbf{r}) = \sum_M\sum_{\{n_i\}} a_{M,\{n_i\}}\frac{r_1^{n_1}r_2^{n_2}\cdots r_D^{n_D}}{r^M}, \tag{A15}$$

then we have the following equality:

$$\int_{r\leqslant R}\frac{\partial G(\mathbf{r})}{\partial r_i}\,d^D\mathbf{r} = \frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R} r_i\,G(\mathbf{r})\,d^D\mathbf{r}, \tag{A16}$$

which is always a finite number. Here M is any real number and the n_i are non-negative integers.

We can prove this result for each of the terms in G. Let

$$g = \frac{r_1^{n_1}r_2^{n_2}\cdots r_D^{n_D}}{r^M}, \tag{A17}$$

and without loss of generality we take its derivative respect to r_1 to obtain

$$\partial_1 g = \left(\frac{n_1}{r_1} - \frac{Mr_1}{r^2}\right) g. \tag{A18}$$

Its integral is

$$\int_{r\leqslant R}\partial_1 g\,d^D\mathbf{r} = \int_{r\leqslant R}\frac{n_1}{r_1}g\,d^D\mathbf{r} - \int_{r\leqslant R}\frac{Mr_1}{r^2}g\,d^D\mathbf{r}. \tag{A19}$$

If n_1 is even or any of the other n_i is odd, the multidimensional integral vanishes because of symmetry, and the theorem holds trivially. Otherwise, notice that the two terms on the right-hand side of Eq. (A19) are, respectively, given by

$$g_1 = \int_{r\leqslant R}\frac{n_1}{r_1}g\,d^D\mathbf{r} = n_1\int_{r\leqslant R}\frac{r_1^{n_1-1}r_2^{n_2}\cdots r_D^{n_D}}{r^M}\,d^D\mathbf{r} = n_1\,I_D(R;d,\delta,\{n_1-1,n_2,\ldots,n_D\}), \tag{A20}$$

and

$$g_2 = -\int_{r\leqslant R}\frac{Mr_1}{r^2}g\,d^D\mathbf{r} = -M\int_{r\leqslant R}\frac{r_1^{n_1+1}r_2^{n_2}\cdots r_D^{n_D}}{r^{M+2}}\,d^D\mathbf{r} = -M\,I_D(R;d,\delta,\{n_1+1,n_2,\ldots,n_D\}), \tag{A21}$$

where $d = M-N+1$, $N=\sum_{i=1}^D n_i$, and $\delta = D-d = D-M+N-1$.

Recalling Eq. (23), we note that it applies to g_1 and g_2 with $k = D$, now that we have n_1 odd and all other n_i even. We then obtain

$$g_1 + g_2 = \frac{n_1!!\ \prod_{i=2}^{D}(n_i-1)!!}{D(D+2)\cdots(D+N-1)} A_D R^{D-M+N-1}, \tag{A22}$$

which holds for all δ and is always a finite number.

From the right-hand side of Eq. (A16) we have

$$\int_{r\leqslant R} r_1\, g\, d^D\mathbf{r} = \int_{r\leqslant R} \frac{r_1^{n_1+1} r_2^{n_2}\cdots r_D^{n_D}}{r^M} d^D\mathbf{r} = I_D(R;d,\delta,\{n_1+1,n_2\ \ldots,n_D\}) = \frac{n_1!!\ \prod_{i=2}^{D}(n_i-1)!!}{D(D+2)\cdots(D+N-1)} I_0(R;M,\delta), \tag{A23}$$

where $d = M - N - 1$ and $\delta = D - M + N + 1$. The derivative with respect to R applies only to I_0, so that

$$\frac{\partial}{\partial R} I_0(R;M,D-M+N+1) = A_D R^{D-M+N}, \tag{A24}$$

which is a finite number, independent of whether I_0 is a singular integral or not. Thus we have

$$\int_{r\leqslant R} \frac{\partial g(\mathbf{r})}{\partial r_1} d^D\mathbf{r} = \frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R} r_1\, g(\mathbf{r})\, d^D\mathbf{r}. \tag{A25}$$

This equation holds if we take linear combination of different g, and we can also change r_1 to any r_i. Therefore, in general we have

$$\int_{r\leqslant R} \frac{\partial G(\mathbf{r})}{\partial r_i} d^D\mathbf{r} = \frac{1}{R}\frac{\partial}{\partial R}\int_{r\leqslant R} r_i\, G(\mathbf{r})\, d^D\mathbf{r}. \tag{A26}$$

APPENDIX B: FRACTURE ANALYSIS

It is important to show the generality of our method. The theory of crack energetics again illustrates the issue of resolving hypersingular integrals using dimensional continuation. For the sake of completeness we briefly describe the relation appearing in fracture analysis. The relation between surface displacements $u_i(P)$ and tractions $\tau_i(P)$ for a smooth crack is given by the integral equation [8,31,32]

$$u_j(P) = 2\int_{\partial C} [U_{ij}(P,Q)\tau_i(Q) - T_{ij}(P,Q)u_i(Q)]\, ds_Q, \tag{B1}$$

where ∂C is the crack surface. A sum over repeated indices is implied. The displacement $U_{ij}(P,Q)$ and traction $T_{ij}(P,Q)$ at the observation point P due to source point Q are given by Kelvin's solution,

$$U_{ij} = \frac{1}{16\pi\, r\,(1-\nu)\,G}[(3-4\nu)\delta_{ij} + \partial_i r\,\partial_j r] \tag{B2}$$

and

$$T_{ij} = -\frac{1}{8\pi\, r^2\,(1-\nu)}\left\{[(1-2\nu)\,\delta_{ij} + 3\,\partial_i r\,\partial_j r]\frac{\partial r}{\partial n} + (1-2\nu)(n_j\,\partial_i r - n_i\,\partial_j r)\right\}, \tag{B3}$$

where $r = |\mathbf{r}_P - \mathbf{r}_Q|$, ν is Poisson's ratio, and G is the shear modulus. With the normal force $\mathbf{N} = N_i\,\mathbf{e}_i$, the traction τ is given by

$$\tau_i(P) = G\left[(\partial_j u_i + \partial_i u_j)N_j + \frac{2\nu}{1-2\nu} N_i\,\partial_k u_k\right]. \tag{B4}$$

The derivative of u_i can be obtained from Eq. (B1) to be substituted here, and we have

$$\begin{aligned}\tau_i(P) = {}& 2GN_j\int_{\partial C}\{[\,\partial_j U_{mi}(P,Q) + \partial_i U_{mj}(P,Q)]\,\tau_m(Q) - [\partial_j T_{mi}(P,Q) + \partial_i T_{mj}(P,Q)]\,u_m(Q)\,\}\,ds_Q\\ &+ \frac{4\nu}{1-2\nu} GN_i\int_{\partial C}[\tau_m(Q)\,\partial_k U_{mk}(P,Q) + u_m(Q)\,\partial_k T_{mk}(P,Q)]\,ds_Q.\end{aligned} \tag{B5}$$

We assume the boundary condition that the traction $\tau_m(Q) = 0$ on the crack, so the above integral is simplified to

$$0 = -2GN_j\int_{\partial C}[\partial_j T_{mi}(P,Q) + \partial_i T_{mj}(P,Q)]\,u_m(Q)\,ds_Q - \frac{4\nu}{1-2\nu} GN_i\int_{\partial C} u_m(Q)\,\partial_k T_{mk}(P,Q)\,ds_Q, \tag{B6}$$

and $\partial_k T_{ij}$ is given by

$$\partial_k T_{ij}(P,Q) = \frac{1}{8\pi\,(1-\nu)\,r^3}\left\{3(\delta_{jk}\,\partial_i r + \delta_{ik}\,\partial_j r - 5\,\partial_i r\,\partial_j r\,\partial_k r)\frac{\partial r}{\partial n} + 3n_k\,\partial_i r\,\partial_j r + (1-2\nu)\left[\delta_{ij}n_k - \delta_{jk}n_i + \delta_{ik}n_j + 3\left(n_i\,\partial_j r\,\partial_k r - n_j\,\partial_i r\,\partial_k r - \delta_{ij}\,\partial_k r\,\frac{\partial r}{\partial n}\right)\right]\right\}. \quad \text{(B7)}$$

We will show that the first integral of Eq. (B6) is a singular integral and can be resolved by the ISI method. The same technique can be applied for the second integral. Invoking the finite element method, we assume the crack surface is flat over a small element ΔS, and choose the local coordinate system so that the normal direction of ΔS is $\mathbf{e}_3$. Here ΔS contains the singular point, so that P and Q are points in ΔS. On this element we have $\mathbf{n} = \mathbf{e}_3$ and the normal force $\mathbf{N} = N_3\mathbf{e}_3$. Hence the first integral in Eq. (B6) becomes

$$-2GN_3\int_{\Delta S}[\partial_3 T_{mi}(P,Q) + \partial_i T_{m3}(P,Q)]\,u_m(Q)\,ds_Q, \quad \text{(B8)}$$

and $\partial_k T_{ij}$ becomes

$$\partial_k T_{ij}(P,Q) = \left[\frac{1}{8\pi\,(1-\nu)\,r^3}\right]\{3(\delta_{jk}\,\partial_i r + \delta_{ik}\,\partial_j r - 5\,\partial_i r\,\partial_j r\,\partial_k r)\,\partial_3 r + 3\delta_{3k}\,\partial_i r\,\partial_j r + (1-2\nu)[\delta_{ij}\delta_{3k} - \delta_{jk}\delta_{3i} + \delta_{ik}\delta_{3j} + 3(\delta_{3i}\,\partial_j r\,\partial_k r - \delta_{3j}\,\partial_i r\,\partial_k r - \delta_{ij}\,\partial_k r\,\partial_3 r)]\}. \quad \text{(B9)}$$

We further assume that $u_m(Q)$ is a constant u_m over the small element ΔS, and consider the integral in Eq. (B6) to be over a small circle centered at P. We then have the singular integral

$$J_{im} = 8\pi\,(1-\nu)\int_{r<R}[\partial_3 T_{mi}(P,Q) + \partial_i T_{m3}(P,Q)]\,ds_Q = \delta_{im}\int_{r<R}\left[\frac{(3+12\delta_{3m})(\partial_3 r)^2+3(\partial_m r)^2-30(\partial_3 r)^2(\partial_m r)^2}{r^3} + (1-2\nu)\,\frac{2-3(\partial_3 r)^2-3(\partial_m r)^2}{r^3}\right]d^2\mathbf{r}, \quad \text{(B10)}$$

with no sum over m. Notice that we take the integral to be on the xy plane, and the z direction is actually the direction along which we shift the origin. We therefore write $\partial_3 r = z/r = \epsilon/r$. Since J_{im} is zero for $i \neq m$, we are left with

$$J_{mm} = \int_{r<R}\left[\frac{3r_m^2}{r^5} + \frac{3\epsilon^2}{r^5} - \frac{30\epsilon^2 r_m^2}{r^7} + (1-2\nu)\left(\frac{2}{r^3} - \frac{3r_m^2}{r^5} - \frac{3\epsilon^2}{r^5}\right)\right]d^2\mathbf{r}, \quad \text{for } m \neq 3 \quad \text{(B11)}$$

and

$$J_{33} = \int_{r<R}\left[\frac{18\epsilon^2}{r^5} - \frac{30\epsilon^4}{r^7} + (1-2\nu)\left(\frac{2}{r^3} - \frac{6\epsilon^2}{r^5}\right)\right]d^2\mathbf{r}. \quad \text{(B12)}$$

When $m \neq 3$, J_{mm} is a linear combination of the integrals

$$J_1 = \int_{r<R}\left(\frac{1}{r^3} - \frac{3r_m^2}{r^5}\right)dS, \quad J_2 = \int_{r<R}\left(\frac{1}{r^3} - \frac{3\epsilon^2}{r^5}\right)dS, \quad J_3 = \int_{r<R}\left(\frac{1}{r^5} - \frac{5r_m^2}{r^7}\right)dS. \quad \text{(B13)}$$

Here, J_1, J_2 and J_3 are all ISIs with no singularities. In the above, $J_1 = \pi R^{-1}$ is the simplest ISI with $d = 3, D = 2$, $J_2 = -2\pi R^{-1}$ is an ϵ^2-ISI with $d = 3, D = 2$, and $J_3 = \pi R^{-3}$ is the simplest ISI with $d = 5, D = 2$. In fact, we have

$$J_{mm} = -J_1 + J_2 + 6\epsilon^2 J_3 + (1-2\nu)(J_1 + J_2) = 2(\nu - 2)\pi R^{-1}. \quad \text{(B14)}$$

Returning to J_{33} we see that it is a linear combination of J_2 and J_4 given by

$$J_4 = \int_{r<R}\left[\frac{1}{r^5} - \frac{5}{3}\frac{\epsilon^2}{r^5}\right]dS. \quad \text{(B15)}$$

Here $J_4 = -(2\pi R^{-3}/3)$ is an ϵ^2-ISI with $d = 5, D = 2$. So we have

$$J_{33} = 18\epsilon^2 J_4 + 2(1-2\nu)J_2 = -4(1-2\nu)\pi R^{-1}. \quad \text{(B16)}$$

In all the above integrals the finite parts are explicitly determined by the ISI method. We thus see again that dimensional continuation provides a unified approach to all hypersingular integrals, making it easy to isolate the singularities, which actually cancel, leaving a well-defined finite part.

APPENDIX C: TABLE OF INTEGRABLE SINGULAR INTEGRALS

1. A table of the simplest ISI: $\int_{r<R}(\frac{1}{r^d} - \frac{x^2 d}{r^{d+2}})d^D\mathbf{r} = \frac{A_D}{D}\, R^{-(d-D)}$

TABLE II. A table of integrable singular integrals for spatial dimension $D = \{1,\ldots,5\}$, with singular denominators r^{-d}, with $d = \{1,\ldots,5,d\}$.

Integral	D	d	Value
$\int_{r<R}\left(\frac{1}{r} - \frac{x^2}{r^3}\right)dx$	1	1	2
$\int_{r<R}\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right)dx$	1	2	$2R^{-1}$
$\int_{r<R}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)dx$	1	3	$2R^{-2}$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)dx$	1	d	$2R^{-(d-1)}$
$\int_{r<R}\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right)dS$	2	2	π
$\int_{r<R}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)dS$	2	3	πR^{-1}
$\int_{r<R}\left(\frac{1}{r^4} - \frac{4x^2}{r^6}\right)dS$	2	4	πR^{-2}
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)dS$	2	d	$\pi R^{-(d-2)}$
$\int_{r<R}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)dV$	3	3	$4\pi/3$
$\int_{r<R}\left(\frac{1}{r^4} - \frac{4x^2}{r^6}\right)dV$	3	4	$4\pi R^{-1}/3$
$\int_{r<R}\left(\frac{1}{r^5} - \frac{5x^2}{r^7}\right)dV$	3	5	$4\pi R^{-2}/3$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)dV$	3	d	$4\pi R^{-(d-3)}/3$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)d^4\mathbf{r}$	4	d	$\pi^2 R^{-(d-4)}/2$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)d^5\mathbf{r}$	5	d	$8\pi^2 R^{-(d-5)}/15$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d\,x^2}{r^{d+2}}\right)d^D\mathbf{r}$	D	d	$A_D R^{-(d-D)}/D$

2. A table of the integrals: ϵ^2-ISI $\int_{r<R}(\frac{1}{r^d} - \frac{d}{d-D}\,\frac{\epsilon^2}{r^{d+2}})d^D\mathbf{r} = -\frac{A_D}{d-D}\, R^{-(d-D)}$

TABLE III. A table of ϵ^2-integrable singular integrals for spatial dimension $D = \{1,\ldots,5\}$, with singular denominators r^{-d}, with $d = \{1,\ldots,5,d\}$.

Integral	D	d	Value
$\int_{r<R}\left(\frac{1}{r^2} - 2\,\frac{\epsilon^2}{r^4}\right)dx$	1	2	$-2R^{-1}$
$\int_{r<R}\left(\frac{1}{r^3} - \frac{3}{2}\,\frac{\epsilon^2}{r^5}\right)dx$	1	3	$-2R^{-2}/2$
$\int_{r<R}\left(\frac{1}{r^4} - \frac{4}{3}\,\frac{\epsilon^2}{r^6}\right)dx$	1	4	$-2R^{-3}/3$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-1}\,\frac{\epsilon^2}{r^{d+2}}\right)dx$	1	d	$-2R^{-(d-1)}/(d-1)$
$\int_{r<R}\left(\frac{1}{r^3} - 3\,\frac{\epsilon^2}{r^5}\right)dS$	2	3	$-2\pi R^{-1}$
$\int_{r<R}\left(\frac{1}{r^4} - 2\,\frac{\epsilon^2}{r^6}\right)dS$	2	4	$-\pi R^{-2}$
$\int_{r<R}\left(\frac{1}{r^5} - \frac{5}{3}\,\frac{\epsilon^2}{r^7}\right)dS$	2	5	$-2\pi R^{-3}/3$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-2}\,\frac{\epsilon^2}{r^{d+2}}\right)dS$	2	d	$-2\pi R^{-(d-2)}/(d-2)$
$\int_{r<R}\left(\frac{1}{r^4} - 4\,\frac{\epsilon^2}{r^6}\right)dV$	3	4	$-4\pi R^{-1}$
$\int_{r<R}\left(\frac{1}{r^5} - \frac{5}{2}\,\frac{\epsilon^2}{r^7}\right)dV$	3	5	$-2\pi R^{-2}$
$\int_{r<R}\left(\frac{1}{r^6} - 2\,\frac{\epsilon^2}{r^8}\right)dV$	3	6	$-4\pi R^{-3}/3$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-3}\,\frac{\epsilon^2}{r^{d+2}}\right)dV$	3	d	$-4\pi R^{-(d-3)}/(d-3)$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-4}\,\frac{\epsilon^2}{r^{d+2}}\right)d^4\mathbf{r}$	4	d	$-2\pi^2 R^{-(d-4)}/(d-4)$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-5}\,\frac{\epsilon^2}{r^{d+2}}\right)d^5\mathbf{r}$	5	d	$-8\pi^2 R^{-(d-5)}/3(d-5)$
$\int_{r<R}\left(\frac{1}{r^d} - \frac{d}{d-D}\,\frac{\epsilon^2}{r^{d+2}}\right)d^D\mathbf{r}$	D	d	$-A_D R^{-(d-D)}/(d-D)$

3. The Dirac δ function in ISI: $\delta^{(D)}(\mathbf{r}) = \frac{D}{A_D}\lim_{\rho\to r^+}(\frac{1}{\rho^D} - \frac{r^2}{\rho^{D+2}})$, $(\rho^2 = r^2 + \epsilon^2)$

TABLE IV. A table of Dirac δ functions in spatial dimension $D = \{1,\ldots,4\}$ arising in integrable singular integrals.

Dimension	δ function
1	$\delta^{(1)}(\mathbf{r}) = \frac{1}{2}\lim_{\rho\to r^+}\left(\frac{1}{\rho} - \frac{r^2}{\rho^3}\right)$
2	$\delta^{(2)}(\mathbf{r}) = \frac{1}{\pi}\lim_{\rho\to r^+}\left(\frac{1}{\rho^2} - \frac{r^2}{\rho^4}\right)$
3	$\delta^{(3)}(\mathbf{r}) = \frac{3}{4\pi}\lim_{\rho\to r^+}\left(\frac{1}{\rho^3} - \frac{r^2}{\rho^5}\right)$
4	$\delta^{(4)}(\mathbf{r}) = \frac{2}{\pi^2}\lim_{\rho\to r^+}\left(\frac{1}{\rho^4} - \frac{r^2}{\rho^6}\right)$

[1] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[2] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vols. I and II (Academic Press, New York, 1964); M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1968); H. Bremermann, *Distributions, Complex Variables, and Fourier Transforms* (Addison-Wesley, Reading, MA, 1965).

[3] N. N. Bogoliubov and D. V. Shirkov, in *Introduction to the Theory of Quantized Fields* (Wiley-Interscience, New York, 1959), p. 177.
[4] G. 'tHooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).
[5] P. A. Martin and F. J. Rizzo, Proc. R. Soc. London A **421**, 341 (1989).
[6] W. L. Wendland and E. P. Stephan, Arch. Ration. Mech. Anal. **112**, 363 (1990).
[7] Q. Huang and T. A. Cruse, Int. J. Numer. Meth. Eng. **36**, 2643 (1993).
[8] L. J. Gray, L. F. Martha, and A. R. Ingraffea, Int. J. Numer. Meth. Eng. **29**, 1135 (1990).
[9] L. J. Gray, J. M. Glaeser, and T. Kaplan, SIAM J. Sci. Comput. **25**, 1534 (2004).
[10] L. A. de Lacerda and L. C. Wrobel, Int. J. Numer. Meth. Eng. **52**, 1337 (2001).
[11] A. Frangi and M. Guiggiani, Int. J. Numer. Meth. Eng. **46**, 1845 (1999).
[12] S. Mukherjee, Eng. Anal. Bound. Elem. **24**, 767 (2000).
[13] A. M. Korsunsky, Proc. R. Soc. London A **458**, 2721 (2002).
[14] Y.-S. Chan, A. Fannjiang, G. Paulino, and B.-F. Feng, Dynam. Cont. Discrete Impulsive Syst. A **14**, 264 (2007).
[15] J. J. Granados and R. Gallego, Eng. Anal. Bound. Elem. **25**, 165 (2001).
[16] P. Kim and U. J. Choi, Int. J. Numer. Meth. Eng. **56**, 469 (2003).
[17] L. R. Ram-Mohan, in *Finite Element and Boundary Element Applications in Quantum Mechanics*, Chap. 16–18 (Oxford University Press, Oxford, 2002), p. 408.
[18] S. Nintcheu Fata, Int. J. Numer. Meth. Eng. **78**, 32 (2009).
[19] S. Nintcheu Fata and L. J. Gray, J. Comput. Appl. Math. **231**, 561 (2009).
[20] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1975).
[21] M. S. Tong and W. C. Chew, Microw. Opt. Tech. Lett. **49**, 1383 (2007).
[22] The nomenclature, integrable singular integral, is in analogy with integrable end-point singularities. For example, in $\int_0^R \ln(x)\,dx$ the argument has a singularity at $x = 0$ but is integrable.
[23] As mentioned earlier, only in quantum field theory do we encounter a need for renormalization of physical parameters such as mass, charge, or coupling constants entering the calculations. This is not the issue in engineering applications and in potential theory.
[24] We do not want to use $I_0(R; d + N, \delta)$ because this is an integral in $D + N$ dimension.
[25] The results here may be compared with those given in Ref. [2], Vol. I, p. 29.
[26] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. (McGraw-Hill, New York, 1976).
[27] A. Carini and A. Salvadori, Comput. Mech. **28**, 177 (2002).
[28] P. M. Morse and H. Feshbach, in *Methods of Theoretical Physics*, Parts I and II (McGraw-Hill, 1953), pp. 1770, 1374.
[29] J. Jin, in *The Finite Element Method in Electromagnetics*, 2nd ed. (Wiley, New York, 2002), pp. 443–446.
[30] J. L. Volakis, A. Chatterjee, and L. C. Kempel, in *Finite Element Method for Electromagnetics* (IEEE Press, New York, 1998), pp. 10–11.
[31] F. J. Rizzo, Q. Appl. Math. **25**, 83 (1967).
[32] T. L. Anderson, in *Fracture Mechanics Fundamentals and Applications*, 2nd ed. (CRC Press, New York, 1995), p. 570.