Classical analog of quasilinear Landau-Zener tunneling

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(Received 19 September 2011; published 10 January 2012)

In this paper we develop an analytical framework to study the effect of nonlinearity on irreversible energy transfer in a system of two weakly coupled oscillators with time-dependent parameters, with special attention to an analogy between classical energy transfer and nonadiabatic quantum tunneling. For preciseness, we suppose that a linear oscillator with constant parameters is excited by an initial impulse but a coupled quasilinear oscillator with slowly varying parameters is initially at rest. It is shown that the equations of the slow passage through resonance in this system are identical to quasilinear equations of nonadiabatic Landau-Zener tunneling. Due to revealed equivalence, a recently found analogy between irreversible energy transfer in a classical linear system and conventional linear Landau-Zener tunneling can be extended to quasilinear systems. An explicit analytical solution of the quasilinear problem is found with the help of an iteration procedure, wherein the linear solution is chosen as an initial approximation. Correctness of the constructed approximations is confirmed by numerical simulations. The results presented in this paper, in addition to providing an analytical framework for understanding the transient dynamics of coupled oscillators, suggest an approximate procedure for solving the quasilinear Landau-Zener equations with arbitrary initial conditions over a finite time interval.

DOI: 10.1103/PhysRevE.85.016202 PACS number(s): 05.45.Xt, 02.30.Mv

I. INTRODUCTION

Targeted energy transfers (TET), where energy is directed from a source to a receiver in a one-way irreversible fashion, govern a broad range of physical phenomena, from multibody systems and waves in fluids and plasmas to photosynthesis and quantum computers. The theoretical basis of the TET analysis has been suggested in [1,2]; examples from diverse fields of applied mathematics, natural sciences, and engineering, and recent advances in the theory and applications of this phenomenon are discussed, e.g., in [3–11] and references therein.

Although a passage between two energy levels is an intrinsic feature of both quantum and classical transitions, a mathematical analogy between nonadiabatic classical and quantum energy transfer has been recently exposed only for linear system with time-dependent parameters [12–15]. As shown in [4–6], the equations for the slowly varying envelopes of near-resonance motion in a system of two weakly coupled linear oscillators with slowly varying parameters are asymptotically identical to the equations of the Landau-Zener tunneling problem [16,17], i.e., there exists a direct mathematical analogy between irreversible energy transfer in a linear oscillatory system and nonadiabatic quantum tunneling. The purpose of the current paper is to demonstrate similar asymptotic equivalence between the equations of the slow passage through resonance in a quasilinear oscillatory system and the equations of quasilinear Landau-Zener tunneling, thereby extending the previously found mathematical analogy to quasilinear systems. Note that a connection between classical and quantum energy transfer in nonlinear systems with constant parameters has been discussed earlier [18].

Equivalence of the mathematical descriptions implies that a classical system of weakly coupled oscillators may serve as a simple but adequate model of complicated physical processes. A comparison of corresponding equations shows that the contribution of time-dependent stiffness in irreversible energy transfer in a classical system is identical to the effect of external fields on quantum tunneling. Moreover, the mathematical equivalence, in principle, enables the substitution of mechanical modeling [4,5] for complicated and costly quantum experiments.

Nonadiabatic quantum transitions under slow driving have been investigated using various methods, e.g., in [19–23]. However, even in the linear case an exact solution to the Landau-Zener equation is too complicated for a straightforward analysis. It is not surprising, then, that since the appearance of the seminal Landau paper [16] the attention has been focused on quasi-stationary solutions at infinitely large times. In this paper we propose a rigorous asymptotic approach for studying transient processes and evaluating the effect of nonlinearity in both classical and quantum systems with arbitrary initial conditions on a finite time interval.

The paper is organized as follows. In Sec. II, a basic model of two weakly coupled oscillators with time-dependent frequency detuning is introduced. The earlier developed procedures [6] are used to derive the evolutionary equations describing the slowly-varying envelopes of near-resonance motion for both oscillators. It is shown that these equations are identical to the equations of nonlinear Landau-Zener tunneling. Since the analysis is focused on quasilinear systems, the notion of a quasilinear system is formalized. In Sec. III we suggest an iteration procedure, which provides an explicit approximate solution of the quasilinear Landau-Zener equations. The initial iteration in the form of the Fresnel integrals corresponds to the linear solution, while the successive iterations improve the accuracy of approximations and take into account the effect of nonlinearity. In particular, the approximate solution demonstrates a decrease of energy transfer with an increase of nonlinearity. In Sec. IV, the theoretical results are compared

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to the numerical solutions. A good agreement between the theoretical and numerical results for both fast and slow dynamics is observed.

II. MODEL AND MAIN EQUATIONS

The system considered consists of two oscillators connected with linear coupling. A linear oscillator with constant parameters is excited by an initial impulse, while a coupled nonlinear oscillator with time-dependent stiffness is initially at rest. We will demonstrate that the nonlinear oscillator acts as an energy sink and ensures a visible reduction of the amplitude of oscillations of the excited mass.

We denote by m_1 and m_2 the masses of the first and second oscillators, respectively; by c_1 and $C_2(t)$, the linear stiffness of the corresponding oscillator, by k_3, the coefficient of nonlinearity, by c_{12}, the stiffness of linear coupling, by P, the initial impulse acting on the nonlinear oscillator. The nonlinear equations of motion are given by

$$m_1\frac{d^2u_1}{dt^2}+c_1u_1+c_{12}(u_1-u_2)=0,$$
$$m_2\frac{d^2u_2}{dt^2}+C_2(t)u_2+k_3u_2^3+c_{12}(u_2-u_1)=0, \tag{2.1}$$

with initial conditions at $t=0$: $u_1=u_2=0$; $du_1/dt=P$, $du_2/dt=0$. The variables u_i $(i=1,2)$ refer to the absolute displacements of the oscillators; the time-variable stiffness $C_2(t)=c_2-(k_1-k_2t)$, $k_1>0$, $k_2>0$; nonlinearity $k_3>0$. It is assumed that $(c_1/m_1)^{1/2}=(c_2/m_2)^{1/2}=\omega$ but all other parameters are small compared to c_2, thus causing internal resonance in the system. It will be shown that the passage through resonance due to the change of the frequency will result in intense energy transfer from the excited to coupled oscillator.

If coupling $c_{12}\ll c_2$, then the equality $c_{12}/c_2=2\varepsilon\ll 1$ defines the small parameter of the system. Introducing the dimensionless parameters

$$c_{12}/c_r=2\varepsilon\lambda_r,\quad r=1,2;\quad k_1/c_2=2\varepsilon\sigma,$$
$$k_3/c_2=8\varepsilon\alpha,\quad k_2/c_2\omega=4\varepsilon^2\beta^2$$

and using the dimensionless time-scales $\tau_0=\omega t$, $\tau_1=\varepsilon\tau_0$, we rewrite the rescaled system as

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon\lambda_1(u_1-u_2)=0,$$
$$\frac{d^2u_2}{d\tau_0^2}+u_2+8\varepsilon\alpha u_2^3+2\varepsilon\lambda_2(u_2-u_1)-2\varepsilon\zeta(\tau_1)u_2=0, \tag{2.2}$$

with initial conditions $\tau_0=0$: $u_1=u_2=0$; $v_1=P/\omega=V_0$, $v_2=0$, $v_i=du_i/d\tau_0$. By definition, $\lambda_2\equiv1$ but for clarity both coupling parameters λ_1 and λ_2 are retained in Eqs. (2.2). The frequency modulation is denoted by $\zeta(\tau_1)=\sigma-2\beta^2\tau_1$. System (2.2) can be treated as resonant if the values of $\varepsilon\,|\zeta(\tau_1)|$ are small for small ε, that is $|\zeta(\tau_1)|$ is of $O(1)$ in the interval of consideration.

We now provide a rigorous definition of a quasilinear system. Consider a linear counterpart of system (2.2):

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon\lambda_1(u_1-u_2)=0,$$
$$\frac{d^2u_2}{d\tau_0^2}+u_2+2\varepsilon\lambda_2(u_2-u_1)-2\varepsilon\zeta(\tau1)u_2=0, \tag{2.3}$$

with the same initial conditions. It has been shown [12] that the phase portrait of the symmetric conservative system ($\zeta=0$, $\lambda_1=\lambda_2=\lambda$) is similar to the phase portrait of its linear counterpart if the parameter $k=3\alpha/4\lambda<0.5$. Hence, if this condition holds, the solution of the nonlinear system can be considered as close to that of a corresponding linear system with $k=0$. It is easy to prove that in the non-symmetric system with $\lambda_1\neq\lambda_2$ the above condition is replaced by the inequality $k=3\alpha/4\lambda_2<0.5$. Throughout this paper, we assume that the system is quasilinear in this sense.

As in the linear case [6], it is convenient to reduce Eqs. (2.2) to a single nonlinear integrodifferential equation. It follows from the first linear equation that

$$u_1=\omega_\varepsilon^{-1}V_0\sin\omega_\varepsilon\tau_0+2\varepsilon\omega_\varepsilon^{-1}\lambda_1I_\varepsilon(\tau_0),$$
$$I_\varepsilon(\tau_0)=\int_0^{\tau_0}u_2(s)\sin\omega_\varepsilon(\tau_0-s)ds, \tag{2.4}$$

where $\omega_\varepsilon=(1+2\varepsilon\lambda_1)^{1/2}=1+\varepsilon\lambda_1+\varepsilon^2\cdots$. Since the frequency detuning is of $O(\varepsilon)$, then, as shown in [13], $I_\varepsilon(\tau_0)$ is of $O(\varepsilon^{-1})$ in the time interval of $O(\varepsilon^{-1})$ and thus $\varepsilon I_\varepsilon(\tau_0)$ is of $O(1)$. This means that this term should be taken into account in the main approximation. Substituting Eqs. (2.4) into Eqs. (2.2) and taking into account all significant terms, we obtain the following system:

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon\lambda_1(u_1-u_2)=0,$$
$$\frac{d^2u_2}{d\tau_0^2}+(1+2\varepsilon\lambda_2)u_2+8\varepsilon\alpha u_2^3-2\varepsilon\zeta(\tau_1)u_2 \tag{2.5}$$
$$=2\varepsilon\omega_\varepsilon^{-1}\lambda_2V_0\sin\omega_\varepsilon\tau_0+4\varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\,I_\varepsilon(\tau_0).$$

The second equation in system (2.5) depends only on u_2 and can be solved separately. As in the linear case [4], the asymptotic analysis is performed with the help of the complexification-averaging technique [24]. The starting point of this method is the complex-valued change of variables

$$\psi=v_2+iu_2,\quad \psi^*=v_2-iu_2. \tag{2.6}$$

The substitution of Eqs. (2.6) into Eqs. (2.5) gives the equation for $\psi(\tau_0,\varepsilon)$:

$$\frac{d\psi}{d\tau_0}-i\psi-i\varepsilon[\lambda_2-\zeta(\tau_1)](\psi-\psi^*)+i\varepsilon\alpha(\psi-\psi^*)^3$$
$$=2\varepsilon\lambda_2V_0\omega_\varepsilon^{-1}\sin\omega_\varepsilon\tau_0-2i\varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\int_0^{\tau_0}[\psi(s,\varepsilon)$$
$$-\psi^*(s,\varepsilon)]\sin\omega_\varepsilon(\tau_0-s)ds,\psi(0)=0. \tag{2.7}$$

The solution of Eq. (2.7) is sought as

$$\psi(\tau_0,\varepsilon)=\varphi(\tau_0,\varepsilon)e^{i\omega_\varepsilon\tau_0}, \tag{2.8}$$

where the envelope $\varphi(\tau_0, \varepsilon)$ satisfies the equation

$$\frac{d\varphi}{d\tau_0} - i\varepsilon\{[\lambda_2 - \lambda_1 - \zeta(\tau_1)](\varphi - \varphi^* e^{-2i\omega_\varepsilon\tau_0}) - \lambda_2\varphi^* e^{-2i\omega_\varepsilon\tau_0} - \alpha\, e^{2i\omega_\varepsilon\tau_0}(\varphi - \varphi^* e^{-2i\omega_\varepsilon\tau_0})^3\}$$
$$= -i\varepsilon\lambda_2 V_0\omega_\varepsilon^{-1}(1 - e^{-2i\omega_\varepsilon\tau_0}) - \varepsilon^2\omega_\varepsilon^{-1}\lambda_1\lambda_2\int_0^{\tau_0}[\varphi(s.\varepsilon)(1 - e^{-2i\omega_\varepsilon(\tau_0 - s)}) + \varphi^*(s,\varepsilon)(1 - e^{2i\omega_\varepsilon(\tau_0 - s)})e^{-2i\omega_\varepsilon\tau_0}]ds. \quad (2.9)$$

Substituting into Eq. (2.9) the multiple-scale expansions

$$\varphi(\tau_0,\varepsilon) = \varphi_0(\tau_1) + \varepsilon\varphi_1(\tau_0,\tau_1) + \cdots; \quad \frac{d\varphi}{d\tau_0} = \frac{\partial\varphi}{\partial\tau_0} + \varepsilon\frac{\partial\varphi}{\partial\tau_1} + \cdots, \quad \tau_1 = \varepsilon\tau_0, \quad (2.10)$$

and then proceeding to the first-order approximation, we obtain the following equation:

$$\frac{\partial\varphi_0}{\partial\tau_1} + \frac{\partial\varphi_1}{\partial\tau_0} - i\Omega(\tau_1)(\varphi_0 - \varphi_0^* e^{-2i\omega_\varepsilon\tau_0}) + i\lambda_2\varphi_0^* e^{-2i\omega_\varepsilon\tau_0} + i\alpha e^{2i\omega_\varepsilon\tau_0}(\varphi_0 - \varphi_0^* e^{-2i\omega_\varepsilon\tau_0})^3$$
$$= -i\omega_0^{-1}\lambda_2 V_0(1 - e^{-2i\omega_\varepsilon\tau_0}) - \varepsilon\omega_0^{-1}\lambda_1\lambda_2\int_0^{\tau_0}[\varphi_0(\varepsilon s)(1 - e^{-2i\omega_\varepsilon(\tau_0 - s)}) + \varphi_0^*(\varepsilon s)(1 - e^{2i\omega_\varepsilon(\tau_0 - s)}e^{-2i\omega_\varepsilon\tau_0})]ds, \quad (2.11)$$

where $\Omega(\tau_1) = \rho + 2\beta^2\tau_1, \rho = \lambda_2 - \lambda_1 - \sigma, \omega_0 = \omega_\varepsilon|_{\varepsilon=0} = 1$. For brevity, we further omit the parameter $\omega_0 = 1$. In order to avoid the secular growth of φ_1 with respect to the fast time τ_0, i.e., avoid a response not uniformly valid with increasing time, we exclude non-oscillating terms from Eq. (2.11); as a result, we obtain the following integro-differential equation for the slowly varying envelope $\varphi_0(\tau_1)$:

$$\frac{d\varphi_0}{d\tau_1} - i\Omega(\tau_1)\varphi_0 - 3i\alpha\varphi_0|\varphi_0|^2$$
$$= -i\lambda_2 V_0 - \lambda_1\lambda_2\int_0^{\tau_1}\varphi_0(r)dr, \quad \varphi_0(0) = 0. \quad (2.12)$$

We recall that the multiple-scale expansion provides an accurate approximation with an error of $O(\varepsilon)$ if $\tau_1 = O(1)$ [25]. This condition agrees with the requirement $|\zeta(\tau_1)|\sim 1$.

In a similar way, we derive an asymptotic approximation for the response u_1. The complex-valued change of variables $y = v_1 + iu_1$, $y^* = v_1 - iu_1$ reduces the first equation in system (2.5) to an equation for the complex envelope y:

$$\frac{dy}{d\tau_0} - i\omega_\varepsilon y + i_\varepsilon\lambda_1 y^* = -i\varepsilon\lambda_1[\varphi e^{i\omega_\varepsilon\tau_0} - \varphi^* e^{-i\omega_\varepsilon\tau_0}],$$
$$y(0) = V_0, \quad (2.13)$$

where $y(\tau_0, \varepsilon) = \eta(\tau_0, \varepsilon)e^{i\omega_\varepsilon\tau_0}$, with $\eta(\tau_0, \varepsilon)$ satisfying the equation

$$\frac{d\eta}{d\tau_0} + i_\varepsilon\lambda_1\eta^* e^{-2i\omega_\varepsilon\tau_0} = -i\varepsilon\lambda_1[\varphi - \varphi_0^* e^{-2i\omega_\varepsilon\tau_0}],$$
$$\eta(0) = V_0. \quad (2.14)$$

An approximate solution of Eq. (2.14) is sought in the form of the multiple-scales expansion $\eta(\tau_0,\varepsilon) = \eta_0(\tau_1) + \varepsilon\eta_1(\tau_0,\tau_1) + \cdots$. After a series of transformations similar to Eqs. (2.10)–(2.12), the resulting equation for the slowly varying envelopes $\eta_0(\tau_1)$ becomes

$$\frac{d\eta_0}{d\tau_1} = -i\lambda_1\varphi_0(\tau_1), \quad \eta_0(\tau_1) = V_0 - i\lambda_1\int_0^{\tau_1}\varphi_0(r)dr. \quad (2.15)$$

Combining Eqs. (2.15) and (2.16), we obtain the system equivalent to the nonlinear Landau-Zener equations:

$$\frac{d\eta_0}{d\tau_1} = -i\lambda_1\varphi_0(\tau_1), \quad \eta_0(0) = V_0,$$
$$\frac{d\varphi_0}{d\tau_1} = i\Omega(\tau_1)\varphi_0 - i\lambda_2\eta_0 + 3i\alpha\varphi_0|\varphi_0|^2, \quad \varphi_0(0) = 0. \quad (2.16)$$

These equations are central to our investigation. Following the definition of quasilinearity, Eqs. (2.16) with $3\alpha < 2\lambda_2$ may be referred to as quasilinear Landau-Zener equations.

Once the solution $\varphi_0(\tau_1)$, $\eta_0(0)$ is found, the leading-order approximations $u_{r\ 0}$ and $v_{r\ 0}$ $(r = 1, 2)$ are calculated by Eqs. (2.6) and (2.8). As a result, we obtain

$$u_{10}(\tau_0, \tau_1) = |\eta_0(\tau_1)|\sin[\omega_\varepsilon\tau_0 + \delta(\tau_1)],$$
$$v_{10} = |\eta_0(\tau_1)|\cos[\omega_\varepsilon\tau_0 + \delta(\tau_1)], \quad (2.17)$$
$$\delta(\tau_1) = \arg[\eta_0(\tau_1)],$$

and

$$u_{20}(\tau_0, \tau_1) = |\varphi_0(\tau_1)|\sin[\omega_\varepsilon\tau_0 + \alpha(\tau_1)],$$
$$v_{20}(\tau_0,\tau_1) = |\varphi_0(\tau_1)|\cos[\omega_\varepsilon\tau_0 + \alpha(\tau_1)], \quad (2.18)$$
$$\alpha(\tau_1) = \arg\varphi_0(\tau_1).$$

Partial energy of the second oscillator is expressed as

$$e_{20}(\tau_1) = \tfrac{1}{2}\left(\langle u_{20}^2\rangle + \langle v_{20}^2\rangle\right) = \tfrac{1}{2}|\varphi_0(\tau_1)|^2, \quad (2.19)$$

where < > denotes the averaging over the "fast" period $T = 2\pi/\omega_\varepsilon$. Although a closed-form solution $\varphi_0(\tau_1)$ is unavailable, a straightforward analysis shows that in a sufficiently small initial time interval Eq. (2.12) can be approximated by the reduced equation

$$\frac{d\varphi_0}{d\tau_1} = -i\lambda_2 V_0, \quad \varphi_0(0) = 0,$$

with the solution

$$\varphi_0(\tau_1) = -i\lambda_2 V_0\tau_1, \quad e_{20}(\tau_1) = \tfrac{1}{2}(\lambda_2 V_0\tau_1)^2. \quad (2.20)$$

Partial energy of the excited oscillator is calculated as

$$e_{10}(\tau_1) = \left(\langle u_{10}^2\rangle + \langle v_{10}^2\rangle\right) = \tfrac{1}{2}\,|\eta_0(\tau_1)|^2. \qquad (2.21)$$

For small τ_1 we obtain

$$\eta_0(\tau_1) = V_0\left(1 - \tfrac{1}{2}\lambda_1\lambda_2\tau_1^2\right), \quad e_{10}(\tau_1) = \tfrac{1}{2}V_0^2\left(1 - \lambda_1\lambda_2\tau_1^2\right). \qquad (2.22)$$

It follows from Eqs. (2.20) and (2.22) that at the initial stage of motion the energy of the excited oscillator decreases while the energy of the coupled oscillator increases. Irreversibility of energy transfer is demonstrated in Sec. IV.

III. APPROXIMATE ANALYSIS OF ENERGY TRANSFER

The solution of the coupled problem can be significantly simplified if the integral terms in Eqs. (2.5) and (2.12) are negligible in the main approximation. As shown in [6], in the linear system this assumption corresponds to the weak coupling limit, for which $\lambda_1\lambda_2 \ll 2\beta^2$. Using this condition, we develop a relevant asymptotic procedure for a quasilinear system. It will be shown below that this formally derived inequality allows a simple physical interpretation: The transition time is less than the period of beating in the system with the zero detuning, or, in other words, this condition renders substantially nonadiabatic transition.

A. Symmetric system with weak coupling

In the symmetric system with $m_1 = m_2 = m$, $c_1 = c_2 = c$ weak coupling is defined by the equality $c_{12}/c = 2\varepsilon\lambda_0$; in addition, we introduce an additional parameter $\mu \ll 1$ and denote $\lambda_0 = \mu\lambda$, $\alpha = \mu\alpha_1$. If coupling is weak, then energy transfer may exist if the initial impact is large enough, that is $V_0 = \mu^{-1}V$ but $\lambda V = \lambda_0 V_0$. Under the given assumptions, Eq. (2.12) is rewritten as

$$\frac{d\varphi_0}{d\tau_1} - i\Omega(\tau_1)\varphi_0 - 3i\mu\alpha_1\varphi_0\,|\varphi_0|^2 = -i\lambda V - \mu\lambda^2\int_0^{\tau_1}\varphi_0(r)dr, \quad \varphi^0(0) = 0, \qquad (3.1)$$

where $\Omega(\tau_1) = -\sigma + 2\beta^2\tau_1$. Equation (3.1) is iterated in the standard way. Using the hypothesis of quasilinearity, we construct the initial iteration $\varphi_0^{(0)}(\tau_1)$ as a solution of the linear equation obtained from Eq. (3.1) at $\mu = 0$:

$$\frac{d\varphi_0^{(0)}}{d\tau_1} - i\Omega(\tau_1)\varphi_0^{(0)} = -i\lambda V, \quad \varphi_0^{(0)}(0) = 0, \qquad (3.2)$$

The initial iteration determines the shape of the solution, while the successive iterations improve the accuracy of approximations. We construct the first iteration $\varphi_0^{(0)}(\tau_1)$ as a solution of the truncated linearized equation, which approximately considers the effect of weak nonlinearity but ignores the integral term:

$$\frac{d\varphi_0^{(1)}}{d\tau_1} + i\Omega(\tau_1)\varphi_0^{(1)} - 3i\mu\alpha_1\left|\varphi_0^{(0)}(\tau_1)\right|^2\varphi_0^{(1)} = -i\lambda V, \quad \varphi_0^{(1)}(0) = 0. \qquad (3.3)$$

The improved iteration $\varphi_0^{(2)}(\tau_1)$ takes into account both nonlinearity and the integral term

$$\frac{d\varphi_0^{(2)}}{d\tau_1} + i\Omega(\tau_1)\varphi_0^{(2)} - 3i\mu\alpha_1\left|\varphi_0^{(0)}(\tau_1)\right|^2\varphi_0^{(2)} = -i\lambda V - \mu^2\lambda^2\int_0^{\tau_1}\varphi_0^{(0)}(r)dr, \quad \varphi_0^{(2)}(0) = 0. \qquad (3.4)$$

It is important to note that the equalities $\mu\lambda = \lambda_0$, $\lambda V = \lambda_0 V_0$ allow us to exclude the unknown parameter μ from calculations. Approximations $\eta_0^{(k)}(\tau_1)$ of the complex envelope $\eta_0(\tau_1)$ are then found from Eq. (2.15) in which φ_0 is replaced by $\varphi_0^{(k)}$ ($k = 0, 1, 2$).

Approximations (3.2)–(3.4) are valid for both classical and quantum problems. A deeper insight into the behavior of the classical system can be obtained if we relate Eqs. (3.2)–(3.4) to the dynamical equations of the oscillators. It is easy to deduce that, correct to first order, the initial iterations $\varphi_0^{(0)}$, $\eta_0^{(0)}$ describe the slow complex envelopes of the truncated linear system with the removed nonlinearity and integral term, namely,

$$\begin{aligned}&\frac{d^2u_1^{(0)}}{d\tau_0^2} + u_1^{(0)} + 2\varepsilon\lambda_0\left(u_1^{(0)} - u_2^{(0)}\right) = 0,\\ &\frac{d^2u_2^{(0)}}{d\tau_0^2} + u_2^{(0)} + 2\varepsilon\lambda_0 u_2^{(0)} - 2\varepsilon\zeta(\tau_1)u_2^{(0)} = 2\varepsilon\lambda_0 V_0\sin\tau_0,\end{aligned} \qquad (3.5)$$

The first iteration $\varphi_0^{(1)}$, $\eta_0^{(1)}$ describes the slow complex envelopes of the following truncated linearized system with the removed integral term but with an approximate nonlinear term:

$$\begin{aligned}&\frac{d^2u_1^{(1)}}{d\tau_0^2} + u_1^{(1)} + 2\varepsilon\lambda_0\left(u_1^{(1)} - u_2^{(1)}\right) = 0,\\ &\frac{d^2u_2^{(1)}}{d\tau_0^2} + u_2^{(1)} + 2\varepsilon\lambda_0 u_2^{(1)} - 2\varepsilon\zeta(\tau_1)u_2^{(1)} + 8\varepsilon\alpha\left|u_1^{(0)}(\tau_1)\right|2u_2^{(1)}\\ &= 2\varepsilon\lambda_0 V_0\sin\omega_\varepsilon\tau_0.\end{aligned} \qquad (3.6)$$

The improved iteration $\varphi_0{}^{(2)}$, $\eta_0{}^{(2)}$ depicts the slow complex envelopes of the following system:

$$\begin{aligned}&\frac{d^2u_1^{(2)}}{d\tau_0^2} + u_1^{(2)} + 2\varepsilon\lambda_0\left(u_1^{(2)} - u_2^{(2)}\right) = 0,\\ &\frac{d^2u_2^{(2)}}{d\tau_0^2} + u_2^{(2)} + 2\varepsilon\lambda_0 u_2^{(2)} - 2\varepsilon\zeta(\tau_1)u_2^{(2)} + 8\varepsilon a\left|u_1^{(0)}(\tau_1)\right|2u_2^{(2)}\\ &= 2\varepsilon\lambda_0 u_1^{(0)}.\end{aligned} \qquad (3.7)$$

System (3.7) takes into account both nonlinear and integral terms (in the main approximation).

Calculations of $\varphi_0^{(0)}(\tau_1)$. As shown in [6], the solution of Eq. (3.2) is given by

$$\begin{aligned}&\varphi_0^{(0)}(\tau_1) = -i\lambda_0 V_0\, e^{iB(\tau_1)}\int_0^{\tau_1} e^{-iB(s)}ds,\\ &B(s) = -\sigma s + (\beta s)^2,\end{aligned} \qquad (3.8)$$

where the equality $\lambda_0 V_0 = \lambda V$ is taken into account. Since $B(s) = (\beta s - \sigma/2\beta)^2 - \sigma^2/4\beta^2 = h^2(s) - \theta^2$, with $h(s) = \beta s - \sigma/2\beta$, $\theta = \sigma/2\beta$, then $e^{iB(\tau_1)} = e^{-i\theta^2 + i(\beta\tau_1 - \theta)^2}$ and, therefore,

$$\varphi_0^{(0)}(\tau_1) = -i\frac{\lambda_0 V_0}{\beta}\Phi_0(\tau_1)\, e^{i(\beta\tau_1-\theta)^2},$$
$$\Phi_0(\tau_1) = [C(\beta\tau_1 - \theta) + C(\theta)] - i[S(\beta\tau_1 - \theta) + S(\theta)], \tag{3.9}$$

where $C(x)$ and $S(x)$ are the cos- and sin-Fresnel integrals

$$C(x) = \int_0^x \cos t^2 dt, \quad S(x) = \int_0^x \sin t^2 dt.$$

It follows from Eqs. (3.8) and (3.9) that

(i) If $\beta\tau_1 \ll \theta$, that is, $\tau_1 \ll \sigma/2\beta^2$, then $B(\tau_1) \approx 0$ and

$$\left|\varphi_0^{(0)}(\tau_1)\right| \approx \lambda_0 V_0 \tau_1. \tag{3.10}$$

(ii) If $\beta\tau_1 \gg \theta$, that is, $\tau_1 \gg \sigma/2\beta^2$, then the following asymptotic representations hold [26]:

$$C(\beta\tau_1 - \theta) \approx \left(\sqrt{\frac{\pi}{2}} + \frac{\sin(\beta\tau_1)^2}{\beta\tau_1}\right) + O\left(\frac{1}{(\beta\tau_1)^2}\right),$$
$$S(\beta\tau_1 - \theta) \approx \frac{1}{2}\left(\sqrt{\frac{\pi}{2}}\frac{\cos(\beta\tau_1)^2}{\beta\tau_1}\right) + O\left(\frac{1}{(\beta\tau_1)^2}\right). \tag{3.11}$$

Therefore,

$$\left|\varphi_0^{(0)}(\tau_1)\right| \to |\bar{\Phi}_0| = \frac{\sqrt{\pi}\lambda_0 V_0}{2\beta} \quad \text{as} \quad \tau_1 \to \infty. \tag{3.12}$$

Although an analysis as $\tau_1 \to \infty$ is formal, expression (3.12) indicates the main feature of energy transfer: A transition of the coupled oscillator from the initial rest state to quasi-stationary oscillations. The energy of quasistationary oscillations is defined as $\bar{e}_{20} = \frac{1}{2}|\bar{\Phi}_0|^2$; the residual energy of the first oscillator $\bar{e}_{10}$ is calculated using the solution (2.15).

We now highlight a correlation between the parameters λ and β. First, we consider the symmetric linear system (2.3) with the zero detuning ($\zeta = 0$) and symmetric coupling $\lambda_1 = \lambda_2 = \lambda_0$. It is well known that in the system of two weakly coupled harmonic oscillators any finite amount of energy injected in one of the oscillators while the other is initially at rest oscillates between two oscillators; the period of beating (with respect to the slow time τ_1) is $T_b = \pi/\lambda_0$. Secondly, we analyze expressions (3.9). It can be easily shown that in the system with $\sigma = 0$ the transition time T_1 defined by the first maximum of $|\varphi_0^{(0)}(\tau_1)|$ lies in the interval $\pi/2 < (\beta T_1)^2 < \pi$. Therefore, the condition $\lambda_1\lambda_2 \ll 2\beta^2$ can be rewritten as $T_1^2 \ll 2T_b^2/\pi < T_b^2$. In other words, the transition time is supposed to be far less than the period of beating in the relevant system of harmonic oscillators, and the system under consideration demonstrates fast energy transfer. Additional weak nonlinearity and weak asymmetry does not spoil this conclusion.

Calculations of $\varphi_0^{(1)}(\tau_1)$. The iteration $\varphi_0^{(1)}(\tau_1)$ is given by

$$\frac{d\varphi_0^{(1)}}{d\tau_1} - if(\tau_1)\varphi_0^{(1)} = -i\lambda_0 V_0, \quad \varphi_0^{(1)}(0) = 0, \tag{3.13}$$

where $f(\tau_1) = -\sigma + 2\beta^2\tau_1 + 3\alpha|\varphi_0^{(0)}(\tau_1)|^2 = -\sigma + 2\beta^2\tau_1 + 6\alpha e_{20}(\tau_1)$, with $e_{20}(\tau_1) = |\varphi_0^{(0)}(\tau_1)|^2/2$. The solution of Eq. (3.13) is obviously similar to Eq. (3.8), that is,

$$\varphi_0^{(1)}(\tau_1) = -i\lambda_0 V_0 \Phi_1(\tau_1) e^{iF(\tau_1)}, \quad \left|\varphi_0^{(1)}(\tau_1)\right| = \lambda V|\Phi_1(\tau_1)|, \tag{3.14}$$

with

$$\Phi_1(\tau_1) = \int_0^{\tau_1} e^{-F(s)} ds, F(s) = \int_0^s f(t)dt$$
$$= -\sigma s + (\beta s)^2 + 6\alpha\int_0^s e_{20}(t)dt.$$

It is easy to prove that $|\varphi_0^{(1)}(\tau_1)| \approx \lambda_0 V_0 \tau_1$ if $0 \leqslant \tau_1 \ll \sigma/2\beta^2$. Although a closed form of $\Phi_1(\tau_1)$ is unavailable, the stationary phase method [27] can be used to obtain the Fresnel-type approximation for large τ_1. First, using the definitions of the functions F and f, we find the stationary phases ϕ from the equation

$$F'(\phi) = -\sigma + 2\beta^2\phi + 6\alpha e_{20}(\phi) = 0. \tag{3.15}$$

Assuming that Eq. (3.15) has a single solution $\phi = \phi_1$ and then expanding $F(s)$ in the Taylor series with two first nonzero terms, we obtain

$$F(s) \approx F(\phi_1) + \kappa^2(s - \phi_1)^2,$$
$$\kappa^2 = \frac{1}{2}F''(\phi_1) = \beta^2 + 3\alpha\left[\left.\frac{de_{20}}{ds}\right|_{s=\phi_1}\right]. \tag{3.16}$$

The substitution of Eq. (3.16) into Eqs. (3.13) and (3.14) gives the following Fresnel-type approximation:

$$\Phi_1(\tau_1) = \int_0^{\tau_1} e^{-iF(s)} ds \approx e^{-iF(\phi_1)}\int_0^{\tau_1} e^{-i[\kappa(s-\phi_1)]^2} ds$$
$$= \frac{1}{\kappa}\int_{-k\phi_1}^{k(\tau_1-\phi_1)} e^{-ih^2} dh = \frac{1}{\kappa}(\{C[\kappa(\tau_1 - \phi_1)] + C(\kappa\phi_1)\}$$
$$- i\{S[\kappa(\tau_1 - \phi_1)] + S(\kappa\phi_1)\}). \tag{3.17}$$

Using the same arguments as above, we obtain from Eqs. (3.14) and (3.17) an asymptotic limit akin to Eq. (3.11):

$$\left|\varphi_0^{(1)}(\tau_1)\right| \to |\bar{\Phi}_1| = \frac{\sqrt{\pi}\lambda_0 V_0}{2\kappa} \quad \text{as} \quad \tau_1 \to \infty. \tag{3.18}$$

It is important to note that Eq. (3.15) does not allow an explicit solution. However, if we assume that the approximation $e_{20}(\tau_1) \approx \tilde{e}_{20}(\tau_1) = \frac{1}{2}(\lambda_0 V_0 \tau_1)^2$ holds for all $\tau_1 \leqslant \phi_1$ and then substitute $\tilde{e}_{20}$ for e_{20} in Eqs. (3.15) and (3.16), we derive the following quadratic equation for the approximate stationary phase $\tilde{\phi}_1$:

$$3\alpha(\lambda_0 V_0 \tilde{\phi}_1)^2 + 2\beta^2\tilde{\phi}_1 - \sigma = 0,$$
$$\tilde{\phi}_1 = \frac{\beta^2}{a} + \sqrt{\frac{\beta^4}{a^2} + \frac{\sigma}{a}}, \quad a = 3\alpha(\lambda_0 V_0)^2, \tag{3.19}$$

and, therefore,

$$\kappa^2 = \beta^2 + 3\alpha(\lambda_0 V_0)^2\tilde{\phi}_1. \tag{3.20}$$

It follows from Eqs. (3.19) and (3.20) that the coefficient $1/\kappa$ decreases as $\alpha^{-1/2}$ and, therefore, the resulting energy of the trap also decreases with an increase of nonlinearity. This

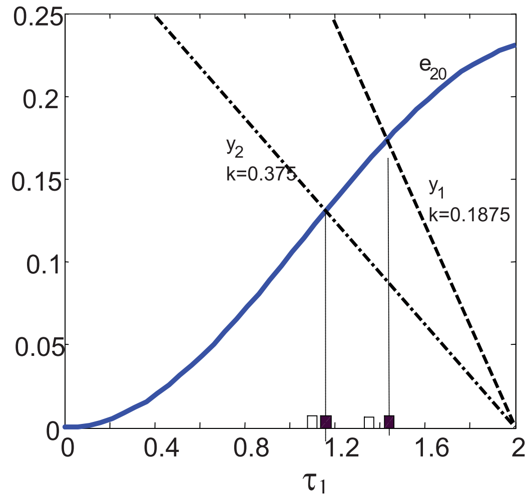

FIG. 1. (Color online) Exact (black squares) and approximate (open squares) stationary phases.

conclusion is in good agreement with the results of numerical simulation (Sec. IV). A similar effect for quantum systems (in terms of tunneling probability at $\tau_1 \to \infty$) was demonstrated earlier [18] for another type of the Landau-Zener equations.

B. Asymmetric system

A similar iterative scheme can be developed for an asymmetric system with $m_1 \neq m_2$, $c_1 \neq c_2$. Let $m_2 = \mu m_1$, where $\mu \ll 1$ is a known parameter. Since $c_2/m_2 = c_1/m_1 = \omega^2$, then $c_1 = m_1 c_2/m_2 = c_2/\mu$ and $c_{12}/c_1 = 2\varepsilon\lambda_1 = 2\varepsilon\mu\lambda_2$, or $\lambda_1 = \mu\lambda_2$. In addition, it is supposed that $\alpha = \mu\alpha_1$, $\lambda_2 = \lambda \sim O(1)$, $\lambda_1 = \mu\lambda$. Under these assumptions, the integro-differential equation (2.12) coincides with Eq. (3.1). This implies that in the asymmetric system the initial iteration $\varphi_0^{(0)}(\tau_1)$ is also determined by the linear equation (3.2), in which $\Omega(\tau_1) = \lambda - \sigma + 2\beta^2\tau_1$; the subsequent iterations $\varphi_0^{(1)}(\tau_1)$ and $\varphi_0^{(2)}(\tau_1)$ are calculated by Eqs. (3.3) and (3.4), respectively.

IV. NUMERICAL RESULTS

We discuss the results of numerical simulations for system (2.2) with coefficients of nonlinearity from the admissible interval $k \in (0, 0.5)$. We use the following parameters of the numerical simulations:

$$\varepsilon = 0.05, \quad \lambda_0 = 1, \quad V_0 = 1, \quad \sigma = 2.25, \quad 2\beta^2/\sigma = 1,$$
$$(a): \alpha = 0.25 \quad \text{or} \quad (b): \alpha = 0.5. \tag{4.1}$$

A simple calculation proves that $\lambda_0^2 < 2\beta^2$, $k = 3\alpha/4\lambda = 0.1875$ in case (a) and $k = 0.375$ in case (b). Therefore, the hypotheses of weak coupling and weak nonlinearity hold.

Formula (3.19) gives the approximate stationary phases $\tilde{\phi}_1 = 1.39$ for $k = 0.1875$ and $\tilde{\phi}_1 = 1.18$ for $k = 0.375$. In the numerical simulation the stationary phase is determined as a point of intersection of the plot $e_{20}(\tau_1)$ with the straight line $y = (\sigma - 2\beta^2\tau_1/6\alpha)$, where $e_{20}(\tau_1) = \frac{1}{2}|\varphi_0^{(0)}(\tau_1)|^2$ is calculated by Eq. (3.2). The points of intersections of the curve $e_{20}(\tau_1)$ with the lines y_1 and y_2 corresponding to $k = 0.1875$ and $k = 0.375$, respectively, lie on the initial rising branch of the curve e_{20} (Fig. 1). This implies the uniqueness of the intersection for each set of the parameters. As seen in Fig. 1, the difference between the exact and approximate phases Φ_1 and $\tilde{\phi}_1$ does not exceed 5%, and, therefore, the approximate solution (3.19) provides a fairly good approximation of the stationary phase. In accordance with the theoretical prediction, the value of the stationary phase diminishes if nonlinearity α increases.

The numerical solutions of the original nonlinear system (2.2) are shown in Fig. 2. Almost irreversible transfer from the excited oscillator to the sink is observed for both $k = 0.1875$ [Fig. 2(a)] and $k = 0.375$ [Fig. 2(b)] but a distinct decrease of the amplitude of the excited oscillator for $k = 0.1875$ is obvious [Fig. 2(a)].

Figures 3 and 4 allows us to compare the solutions u_1 and u_2 of the original system (2.2), the approximate solutions of the truncated systems (3.6), and the solutions of linear system (2.3) for the parameters $k = 0.1875$ and $k = 0.375$.

As seen in Figs. 3 and 4, an increase of nonlinearity makes a difference between the nonlinear and linear dynamics more

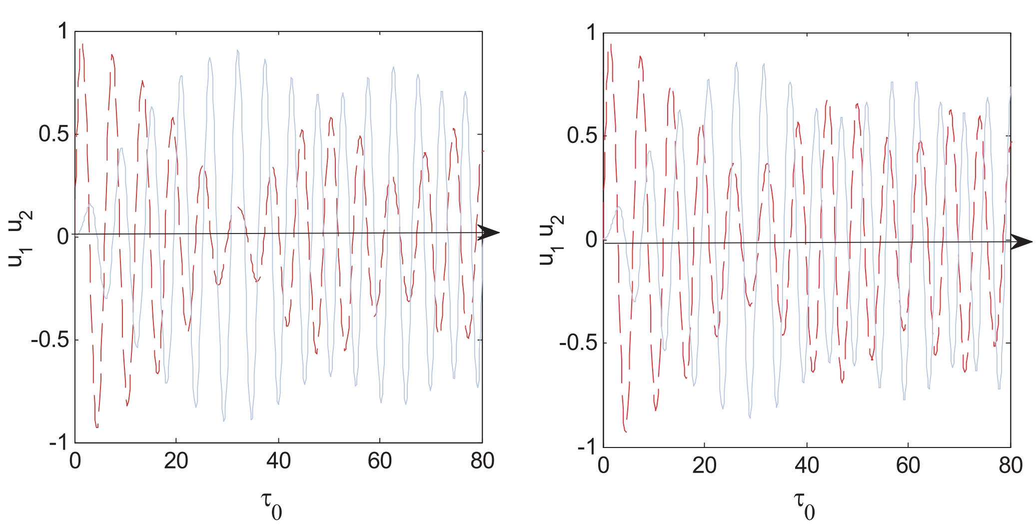

FIG. 2. (Color online) Numerical solutions u_1 [red (dark gray) dashed lines] and u_2 [solid (light gray) lines] of system (2.2): (a) $k = 0.1875$, (b) $k = 0.375$.

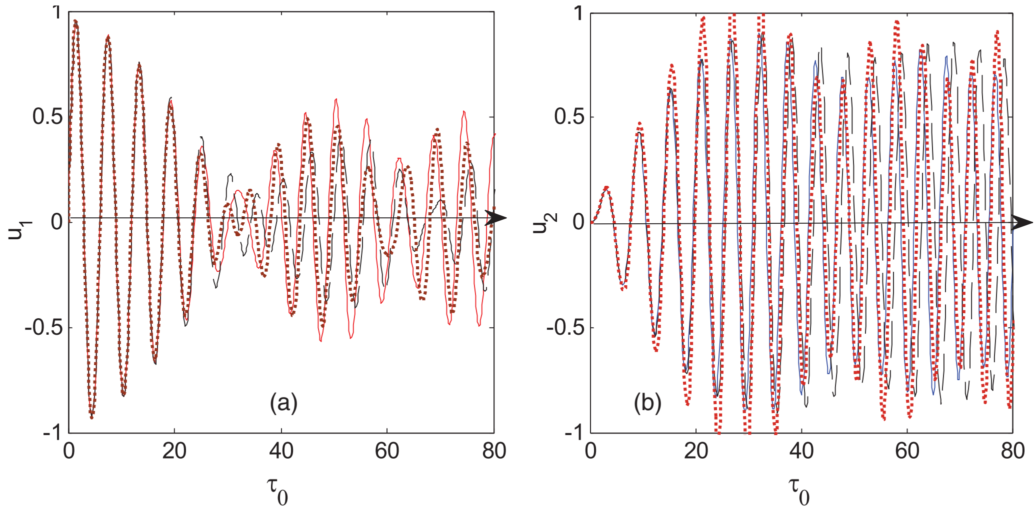

FIG. 3. (Color online) Numerical solutions u_1 (a), and u_2 (b) for $k = 0.1875$: Solid lines: Solutions of the full nonlinear system (2.2); dashed lines: Solutions of the full linear system (2.3); dotted lines: Solutions of the truncated system (3.7).

pronounced but the approximate solution of the truncated system remains in good agreement with the precise solution. Moreover, the linear oscillator [Figs. 3(a) and 4(a)] is more sensitive to the influence of nonlinearity than the nonlinear trap [Figs. 3(b) and 4(b)].

Figure 5 depicts the energy of the oscillator (1) and the trap (2) in the systems with $k = 0.1875$ and $k = 0.375$. Energy of the nonlinear system is calculated by the asymptotically precise formulas $e_1 = \frac{1}{2}|\eta_0|^2$ and $e_2 = \frac{1}{2}|\varphi_0|^2$ with η_0 and φ_0 satisfying Landau-Zener equations (2.16) rescaled to the fast time scale τ_0.

A reduction of energy transfer with an increase of nonlinearity is obvious.

V. CONCLUSIONS

In this paper we have studied energy transfer in a system of two weakly coupled oscillators in which the first oscillator with constant parameters is excited by an initial impulse whereas the coupled quasilinear oscillator with a time-dependent frequency is initially at rest but then acts as an energy trap. It has been shown that the equations for the slowly varying envelopes of near-resonance motion in this system are identical to the equations of nonlinear Landau-Zener tunneling. This asymptotic equivalence allows a unified approach to the study of physically different processes such as energy transfer in a classical oscillatory system with variable parameters and nonadiabatic quantum Landau-Zener tunneling.

Since the asymptotic analysis is restricted to the study of a quasilinear system, the notion of quasilinearity and the formal limitations ensuring the desired quasilinear dynamics have been introduced. An approximate solution of the quasilinear problem has been constructed using the iteration procedure, in which the linear solution is chosen as an initial approximation. Although not quantitatively exact, an explicit approximate solution gives an understanding of the influence of nonlinearity

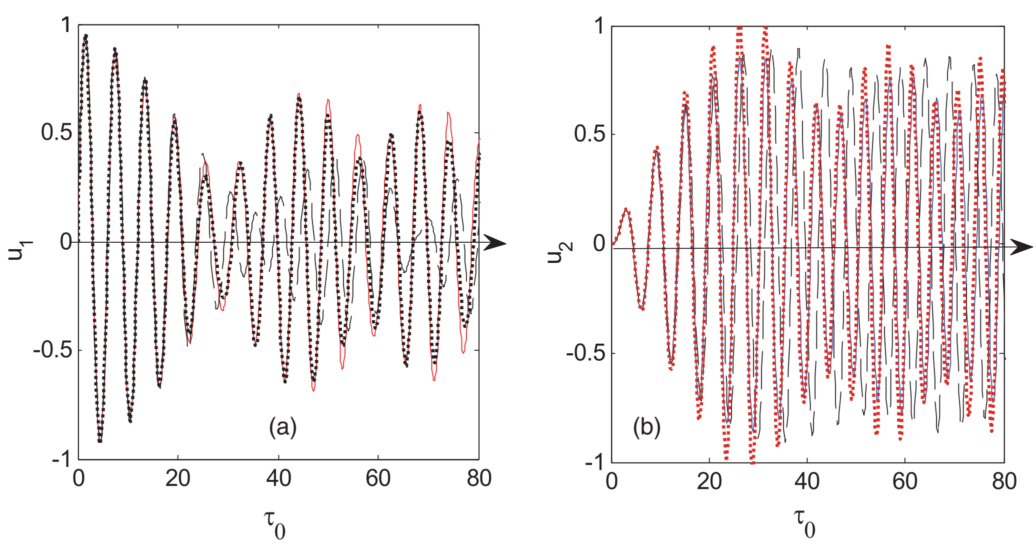

FIG. 4. (Color online) Numerical solutions u_1 (a), and u_2 (b) for $k = 0.375$: Solid lines: Solutions of the full nonlinear system (2.2); dashed lines: Solutions of the full linear system (2.3); dotted lines: Solutions of the truncated system (3.7).

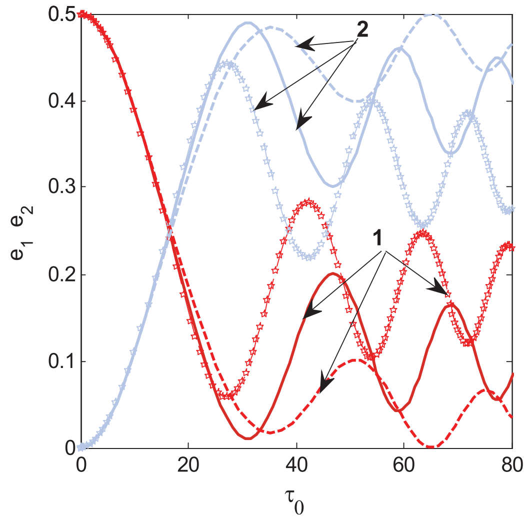

FIG. 5. (Color online) Energy of the excited oscillator [group 1 of red (dark gray) lines 1] and the trap [group 2 of light blue (light gray) lines] for the linear system (dashed lines); the nonlinear systems with $k = 0.1875$ (solid lines) and $k = 0.375$ (starred lines).

on energy transfer. In particular, it has been shown that an increase of nonlinearity diminishes the intensity of energy transfer.

In view of a profound mathematical analogy between energy transfer in a classical oscillatory system with variable parameters and nonadiabatic quantum Landau-Zener tunneling, the results of this paper, in addition to providing an analytical framework for understanding the transient dynamics of coupled oscillators, suggest an adequate approximate procedure for solving the nonlinear Landau-Zener problem with arbitrary initial conditions over a finite time-interval. Furthermore, this analogy paves the way for a simple mechanical simulation of complicated quantum effects.

The results concerning energy transfer in strongly nonlinear system is beyond the scope of the current paper. They will appear in forthcoming authors' papers.

ACKNOWLEDGMENTS

This work is partially supported by the Russian Foundation for Basic Research through RFBR Grant No. 10-01-00698.

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