Langevin equations for competitive growth models

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Langevin equations for several competitive growth models in one dimension are derived. For models with crossover from random deposition (RD) to some correlated deposition (CD) dynamics, with small probability p of CD, the surface tension ν and the nonlinear coefficient λ of the associated equations have linear dependence on p due solely to this random choice. However, they also depend on the regularized step functions present in the analytical representations of the CD, whose expansion coefficients scale with p according to the divergence of local height differences when $p \to 0$. The superposition of those scaling factors gives $\nu \sim p^2$ for random deposition with surface relaxation (RDSR) as the CD, and $\nu \sim p$, $\lambda \sim p^{3/2}$ for ballistic deposition (BD) as the CD, in agreement with simulation and other scaling approaches. For bidisperse ballistic deposition (BBD), the same scaling of RD-BD model is found. The Langevin equation for the model with competing RDSR and BD, with probability p for the latter, is also constructed. It shows linear p dependence of λ, while the quadratic dependence observed in previous simulations is explained by an additional crossover before the asymptotic regime. The results highlight the relevance of scaling of the coefficients of step function expansions in systems with steep surfaces, which is responsible for noninteger exponents in some p-dependent stochastic equations, and the importance of the physical correspondence of aggregation rules and equation coefficients.

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I. INTRODUCTION

The technological applications of thin films and multilayers motivated intense theoretical study of growth models in the last decades [1–3]. Many processes show evidence of a competition between different aggregation dynamics. For instance, this occurs in deposition of diamond-like carbon by plasma, where growth is mainly due to aggregation of slow radicals but ion bombardment is essential to create sp^3 bonds [4]. When physicochemical conditions are continuously changed, such as in cyclical electrodeposition or dissolution of metals, competing dynamics are also present [5]. Consequently, many competitive growth models were already proposed, with microscopic aggregation rules representing the atomistic dynamics. They are usually defined on lattices, such as those with aggregation of different species of particles [6–8] and those mixing different aggregation rules for the same species [9–19]. They usually show crossover effects from one dynamics at small times t and short length scales L to another dynamics at long t or large L, and in special cases anomalous roughening is present [20,21].

A widely studied group of models is that showing crossover from random (uncorrelated) deposition (RD) to some correlated deposition (CD) process, hereafter called RD-CD models. For RD-CD in general, aggregation of each incident particle follows the rules of the CD process with probability p and those of RD with probability $1 - p$ (other models might also show the same crossover with a parameter that is not a probability [18]). Another group of relevant models show crossover from Edwards-Wilkinson (EW) [22] to Kardar-Parisi-Zhang (KPZ) [23] scaling. A representative model in this latter group is the competition between ballistic deposition (BD) [24] and random deposition with surface relaxation (RDSR, or Family model) [25], respectively with probabilities p and $1 - p$. In all cases, the crossover appears for small p.

For several reasons, the association of those models with stochastic growth equations of the Langevin type is a problem of central interest. First, it facilitates finding asymptotic properties which are frequently unclear in numerical works on lattice models [26–28]. Second, renormalization study of the growth equation may relate unexpected numerical results to crossover or instability effects [29]. Finally, improvement in atomistic modeling of thin film growth may be achieved from the advance on the stochastic equation analysis.

Langevin equations for some of those competitive growth models were derived in Refs. [15,17,30] and scaling features of KPZ and EW equations were discussed by several authors [10,18,19]. For RD-RDSR and RDSR-BD models, Muraca *et al.* [15] suggested quadratic dependence of equation coefficients on the probability p in the crossover regime, which was in good agreement with available numerical data. They argued that the time for the less probable process to occur scales as $1/p$ and that the corresponding increase in local height is also proportional to p [Eqs. (1) and (2) of Ref. [15]]. The results for RD-RDSR were confirmed by scaling arguments in Refs. [17,18]. However, the claim on the universal quadratic form of vanishing coefficients [15] is ruled out by the study of a restricted solid-on-solid (RSOS) [31] model with deposition and erosion, which shows linear p dependence of the nonlinear term in the KPZ equation [30], and by the RD-BD model [17,18], which shows $p^{3/2}$ scaling of that term.

In this work, Langevin-type equations associated with various competitive lattice models showing RD-CD and EW-KPZ crossovers are derived through a standard van Kampen expansion of the master equation, followed by a proper choice of the jump moments. From this approach, the form of the equation coefficients is remarkably different from the one proposed by Muraca *et al.* [15]: in all cases, the random choice of the asymptotically dominant process gives a linear dependence on p for the coefficients that vanish

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as $p \to 0$, instead of the quadratic dependence. However, for RD-CD models, average local slopes diverge as $p \to 0$; thus the optimal regularization of step functions (present in the transition rates of all discrete models) have lowest order coefficients that scale as p or $p^{1/2}$, depending on the aggregation mechanism of the CD. The combination of those scaling relations give equation coefficients of EW and KPZ equations vanishing as p, p^2, or $p^{3/2}$. In all cases, they agree with simulations and other scaling approaches [17,18]. For the RDSR-BD model, we show that a nonasymptotic regime with quadratic scaling is present, which is associated with the dominant effect of subsequent BD events, while the asymptotic linear relation is predicted for p much smaller than that of previous simulations [14].

The rest of this work is organized as follows. In Sec. II, three lattice models with the RD-CD crossover are defined and the approach to predict amplitudes of roughness scaling is reviewed. In Sec. III, details of the method to derive Langevin equations are presented and the equation for the RD-RDSR model is obtained. In Sec. IV, the equations for the RD-BD model and for the bidisperse ballistic deposition (BBD) are constructed. In Sec. V, the equation for the RDSR-BD model is presented and the crossover from quadratic to linear scaling is discussed. In Sec. VI we summarize our results and present our conclusions.

II. LATTICE MODELS WITH COMPETITION OF CORRELATED AND UNCORRELATED DEPOSITION

In these models, growth begins with a flat d-dimensional substrate with cubic symmetry and L adsorption sites (or columns) in each direction, with a total of L^d sites. Cubic particles of lateral size $a_\parallel$ (parallel to the substrate plane) and vertical size $a_\perp$ (parallel to the average growth direction) are sequentially released at randomly chosen columns above the deposit and fall vertically toward the substrate. The time interval for deposition of one layer of atoms [L^d atoms] is τ. Each incident particle may irreversibly stick at the top of the column of incidence, with probability $1 - p$ (RD), or move and stick following some aggregation rule that takes into account the neighboring column heights (mimicking physical processes such as diffusion, desorption, or bond formation), with probability p (CD).

The competitive models where RDSR and BD are the correlated components were introduced by Albano and coworkers [10,11]. The rules of RDSR [25] are illustrated in Fig. 1(a): The particle sticks at the top of the column of incidence if no neighboring column has a smaller height; otherwise it sticks at the top of the column with the smallest height among the neighbors (if two or more neighbors have the same height, one of them is randomly chosen). In BD, which is illustrated in Fig. 1(b), the incident particle aggregates at the first site where it finds a nearest neighbor occupied site (lateral or below it) [24], which generates a porous deposit.

BBD was introduced in Ref. [32] and is itself defined as a competitive model. Taking the $d = 1$ case for illustration, particles of two different sizes are incident toward the surface: single-site particles (lateral size $a_\parallel$, vertical size $a_\perp$) with probability $1 - F$ and double-site particles (dimers with lateral size $2a_\parallel$, vertical size $a_\perp$) with probability F. Any

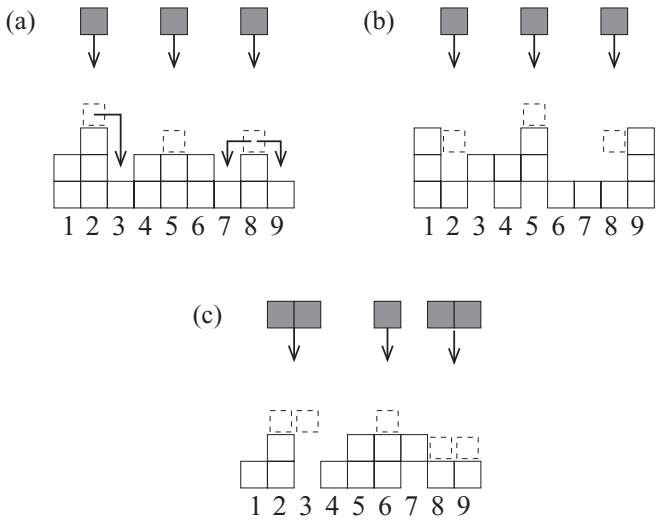

FIG. 1. Microscopic rules of the models (a) RDSR, (b) BD, and (c) BBD. The incident particles are shaded squares, the deposited particles are empty squares with solid boundaries, and the squares with dashed boundaries indicate the points of first contact with the surface. In RDSR, the incident particle may move from the point of first contact, as illustrated by the arrows. In the other models, aggregation occurs at that point.

incident particle permanently sticks at the first position where it encounters a previously deposited particle below it. The aggregation of a dimer leads to pore formation when the neighboring columns have different heights, similarly to the lateral aggregation of BD. The rules of BBD are illustrated in Fig. 1(c).

In these models, the surface configuration is the array of discrete height variables $\mathbf{H} = \{h_i\}$, where $i = 1,2,3,\ldots,L^d$. In all deposits (compact or porous), the height variable is the one of the highest particle at that substrate site; thus $\mathbf{H}$ always refers to the outer surface of the deposit.

The global roughness of the surface is defined as the rms fluctuation of the height variables, whose average is $\overline{h}$:

$$W(L,t) \equiv \left\langle \frac{1}{L^d} \sum_k (h_k - \overline{h})^2 \right\rangle^{1/2}. \quad (1)$$

Here the overbars indicate spatial averages over the height variables, the sum is taken over all the L^d substrate sites, and the angular brackets indicate configurational averages. At short times, RD dominates; thus the roughness increases as

$$W_{RD} \approx a_\perp (t/\tau)^{1/2}. \quad (2)$$

After a crossover time t_c, the CD determines the universality class of the process. The roughness follows Family-Vicsek (FV) scaling [33] as

$$W(L,t) \approx A L^\alpha f\left(\frac{t}{t_\times}\right), \quad (3)$$

where α is the roughness exponent, f is a scaling function such that $f \sim 1$ in the regime of roughness saturation ($t \to \infty$), and $t_\times$ is the characteristic time of crossover to saturation, which scales as

$$t_\times \approx B L^z, \quad (4)$$

where z is the dynamic exponent. For $t \ll t_\times$ (but $t \gg t_c$), the roughness scales as

$$W \approx Ct^\beta, \tag{5}$$

where $\beta = \alpha/z$ is the growth exponent. In this growth regime, $f(x) \sim x^\beta$ in Eq. (3).

The exponents α, β, and z depend on the basic symmetries of the CD, but the amplitudes A, B, and C are model dependent. For small p, they scale as

$$\begin{aligned} A &\sim p^{-\delta}, \\ B &\sim p^{-y}, \\ C &\sim p^{-\gamma}, \end{aligned} \tag{6}$$

where the convention of crossover exponents (δ, y, γ) of Albano and co-workers [10] was used. FV scaling implies

$$y\beta - \delta + \gamma = 0. \tag{7}$$

The scaling approaches of Refs. [18,19] explain the values of the exponents obtained in simulations of lattice models and show that only one exponent [usually y or δ; Eq. (6)] is sufficient to characterize the crossover.

For $p \ll 1$, most deposited atoms attach to the top of the randomly chosen site (RD). Thus, from Eq. (2), the height difference of neighboring sites is of order $\Delta h_{RD} \sim a_\perp(\Delta t/\tau)^{1/2}$ after a time interval Δt. On the other hand, the average time for a correlated event (probability p) to occur at a given column is $\tau_c \sim \tau/p$.

In the case of ballistic-like models (e.g., BD or BBD), the frequent lateral aggregation [e.g., at columns 2 and 8 in Fig. 1(b)] immediately creates correlations between the neighboring columns. Thus, the time of crossover from random to correlated growth is

$$t_c \sim \tau_c \sim p^{-1}\tau. \tag{8}$$

This applies to other BD-like models, as discussed, e.g., in Ref. [34].

However, in the case of solid-on-solid (SOS) models, such as RDSR, the column height changes by a single lattice unit at each time step. A single CD event does not cancel the random height fluctuation Δh of neighboring columns; instead, it only reduces that fluctuation by one lattice unit [e.g., the particle aggregating at column 3 in Fig. 1(a)]. The height fluctuation produced by RD will be suppressed only when the number of correlated events N_c is of order $\Delta h_{RD}/a_\perp$. At the crossover time t_c, this number is $N_c = t_c/\tau_c \sim t_c p/\tau$; thus

$$t_c \sim p^{-2}\tau. \tag{9}$$

In both cases, all time scales of the purely correlated system ($p = 1$) are also changed by the scaling factor t_c/τ, such as the saturation time $t_\times$ [Eq. (4)]. Thus we have $y = 1$ for ballistic-like models and $y = 2$ for SOS models with single particle deposition attempts [Eqs. (6)]. This result does not depend on the universality class of the CD neither on the substrate dimension.

The average height difference between neighboring columns saturates at

$$\Delta h \sim a_\perp(t_c/\tau)^{1/2} \sim a_\perp p^{-y/2} \tag{10}$$

for all models (ballistic-like and SOS). This is the scaling factor for global height fluctuations [Eqs. (3) and (6)]; thus

$$\delta = y/2. \tag{11}$$

Combined with Eq. (7), it shows that a single exponent (y) fully characterizes the crossover.

III. THE RD-RDSR MODEL

The equation associated with the RD-RDSR model is constructed through a van Kampen expansion of the master equation [35], as discussed in Refs. [28,36,37].

The transition rate $W(\mathbf{H}';\mathbf{H})$ from the height configuration $\mathbf{H} = \{h_i\}$ to the configuration $\mathbf{H}' = \{h_i'\}$ is

$$W(\mathbf{H}';\mathbf{H}) = \frac{1}{\tau}\sum_k w_k \delta(h_k', h_k + a_\perp)\prod_{j\neq k}\delta(h_j', h_j), \tag{12}$$

where the δ-function product represents the condition that $\mathbf{H}$ and $\mathbf{H}'$ differ by the deposition of one only particle, and w_k is the rate at which the process $h_k \to h_k + a_\perp$ occurs.

Let $K_i^{(1)}$ and $K_{ij}^{(2)}$ be respectively the first and second jump moments of W, computed through [35,38]:

$$K_i^{(1)}(\mathbf{H}) = \sum_{\mathbf{H}'}(h_i' - h_i)W(\mathbf{H}';\mathbf{H}) \tag{13}$$

and

$$K_{ij}^{(2)}(\mathbf{H}) = \sum_{\mathbf{H}'}(h_i' - h_i)(h_j' - h_j)W(\mathbf{H}';\mathbf{H}). \tag{14}$$

According to a theorem of Kurtz [39,40], later revisited by Fox and Keiser in the context of a macrovariable description for noisy trajectories [41] (see also Ref. [42]), we expect that

$$\frac{\partial h_i}{\partial t} = K_i^{(1)}(\mathbf{H}) + \eta_i(t) \tag{15}$$

gives the macroscopic description of $\mathbf{H}$ in the hydrodynamic limit. If all the conditions imposed in Ref. [35] and in Refs. [39–41] are met, the fluctuations in Eq. (15) must obey

$$\langle \eta_i(t)\rangle = 0 \tag{16}$$

and

$$\langle \eta_i(t)\eta_j(t')\rangle = a_\perp K_i^{(1)}\delta_{ij}\delta(t'-t), \tag{17}$$

where we used the identity

$$K_{ij}^{(2)} = a_\perp K_i^{(1)}\delta_{ij} \tag{18}$$

between the first and second jump moments in Eq. (17). In fact, all higher order jump moments are proportional to $K_i^{(1)}$, as can be seen from direct calculation.

For the RD-RDSR model (and related competitive models) in $d = 1$, the first jump moment [Eq. (13)] can be cast to the form

$$K_i^{(1)} = p\frac{a_\perp}{\tau}\left(\omega_i^{(0)} + \omega_{i+1}^{(1)} + \omega_{i-1}^{(2)}\right) + (1-p)\frac{a_\perp}{\tau}, \tag{19}$$

where each $\omega_j^{(k)}$ gives the conditions for a particle incident at column j to move and stick to one of its neighbors or to stick at the incidence column. Those conditions depend on the local height configuration.

The functions $\omega_i^{(k)}$ are called *aggregation rules*. For the RDSR model, $\omega_i^{(0)}$ represents the conditions for the particle incident at site i to stick at i; $\omega_i^{(1)}$, the conditions for the particle to relax to its left site $(i-1)$; and $\omega_i^{(2)}$, the conditions for the particle to relax to its right site $(i+1)$. The aggregation rules can be written in terms of discrete step functions as [15]

$$\begin{aligned}\omega_i^{(0)} &= \theta_i^{i+1}\theta_i^{i-1},\\ \omega_i^{(1)} &= \tfrac{1}{2}\left(1+\theta_i^{i+1}\right)\left(1-\theta_i^{i-1}\right),\\ \omega_i^{(2)} &= \tfrac{1}{2}\left(1+\theta_i^{i-1}\right)\left(1-\theta_i^{i+1}\right),\end{aligned} \tag{20}$$

where $\theta_k^j = \Theta(h_j - h_k)$, and $\Theta(x)$ is the unit step function, defined at our convenience to be $\Theta(x) = 1$ for $x \geqslant 0$ and $\Theta(x) = 0$ for $x < 0$.

In order to pass from the discrete model to its continuum limit, we assume there exists a continuous function $\Psi(x,t)$ that interpolates all points $h_i(t)$ of the substrate, while $a_\parallel$ is kept small but finite. This is possible if we can write

$$h_{i\pm n} - h_i = \sum_{k=1}^{\infty}\left(\frac{\partial^k \Psi}{\partial x^k}\right)\frac{(\pm a_\parallel n)^k}{k!}, \tag{21}$$

for some $\Psi(x,t)$.

We assume further there also exists an analytical representation of the step function Θ (see for example Refs. [27,43]), and we define $\Delta_k^j \equiv h_j - h_k$, so that the function $\Theta(\Delta_k^j)$ can be expanded in a power series of the height differences as

$$\Theta\left(\Delta_k^j\right) = 1 + A_1\Delta_k^j + A_2\left(\Delta_k^j\right)^2 + \cdots, \tag{22}$$

where the expansion coefficients have to be chosen according to the rules of the lattice model to be represented [27–29,44].

Equation (19) is inserted in Eq. (15), and step functions and height differences are expanded according to Eqs. (22) and (21). Retaining terms up to the leading order in $a_\parallel$ and $a_\perp$, and in the limit of small p, we obtain the EW equation [22]

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \eta(x,t), \tag{23}$$

where

$$\nu = \frac{2a_\perp a_\parallel^2}{\tau}A_1 p \tag{24}$$

and

$$F = \frac{a_\perp}{\tau}. \tag{25}$$

These coefficients differ from those of Ref. [15], which gave $\nu = (2a_\parallel^2/\tau)A_1 p^2$ and $F = (a_\perp/\tau)[(1-p)^2 + p^2]$. That work proposes that the height at a given column increases by a factor proportional to p $(1-p)$ after a time interval τ/p $[\tau/(1-p)]$ characteristic of the RDSR (RD) process. This gives the quadratic dependence of ν on p. However, this hypothesis also leads to a flux F depending on p, which is not true. Instead, the model is SOS and no deposition attempt is rejected; thus the flux is independent of p, as given in Eq. (25): One layer of atoms of height $a_\perp$ is deposited during time τ.

On the other hand, numerical work [10] and scaling arguments [17,18] give $\nu \sim p^2$ for small p, which apparently disagrees with Eq. (24). As will be explained below, an additional p factor is hidden in the coefficient A_1 of Eq. (22), which is known to be model dependent.

Since the step function is limited to values 0 and 1, the sum of terms in the right-hand side of Eq. (22) is expected to be of order 1 or smaller. The step at the origin indicates that the first-order term $A_1\Delta_k^j$ is finite and nonzero. Indeed, when A_j is computed through some continuous representation of the step function, as in Ref. [15,45], the first-order term is of order 1 for any model.

For pure correlated models, such as RDSR, height differences Δ_k^j are of order 1; thus we expect A_1 is of that order. On the other hand, in RD-RDSR with small p, typical neighboring height differences Δ_k^j diverge as p^{-1} [Eq. (10) with $y = 2$]. Thus A_1 must vanish as

$$A_1 \sim p \tag{26}$$

for a correct regularization of the step function for small p.

Substituting this result in Eq. (24), we obtain $\nu \sim p^2$. Thus, a first factor p in the surface tension coefficient ν comes from the random choice of correlated events, while a second factor p comes from the reduced smoothing effect of each correlated event in surface steps of depth p^{-1}, as discussed in Sec. II. This interpretation also differs from that of Muraca *et al.* [15], that relates the complete p^2 factor to the random choice of RDSR.

Analogous arguments apply to the step function expansion of RD-CD models in general. From Eq. (10), they give

$$A_1 \sim p^\delta. \tag{27}$$

Since the second-order term of the expansion in Eq. (22) must also be of order one, we expect

$$A_2 \sim p^{2\delta}. \tag{28}$$

Eventually the second-order term in Eq. (22) is zero or converges to zero as $p \to 0$, which would imply that A_2 is zero or has a higher power in p.

Another important question is a possible crossover effect on the noise term of the stochastic equations. However, it can be shown that the noise amplitude is independent of the parameter p. At the end of Sec. IV, a detailed discussion is presented for all models with the RD-CD crossover.

IV. THE RD-BD AND THE BBD MODEL

For the one-dimensional RD-BD model, the first jump moment is

$$\begin{aligned}K_i^{(1)} = \frac{p}{\tau}\left[\omega_i^{(3)}(h_{i-1}-h_i) + \omega_i^{(4)}(h_{i+1}-h_i) + \omega_i^{(5)}a_\perp\right]\\ + (1-p)\frac{a_\perp}{\tau},\end{aligned} \tag{29}$$

with the aggregation rules given by

$$\begin{aligned}\omega_i^{(3)} &= \theta_i^{i-1}\theta_{i+1}^{i-1} - \tfrac{1}{2}\left(1-\theta_{i-1}^i\right)\delta_{i+1}^{i-1},\\ \omega_i^{(4)} &= \theta_i^{i+1}\theta_{i-1}^{i+1} - \tfrac{1}{2}\left(1-\theta_{i+1}^i\right)\delta_{i-1}^{i+1},\\ \omega_i^{(5)} &= \theta_{i-1}^i\theta_{i+1}^i,\end{aligned} \tag{30}$$

where $\delta^i_j = \theta^i_j + \theta^j_i - 1$ is the Kronecker delta function. Note that $\omega^{(3)}_i + \omega^{(4)}_i + \omega^{(5)}_i \neq 1$ when $h_i = h_{i-1}$ or $h_i = h_{i+1}$. However, the corresponding jump moment expression [Eq. (29)] is correct, since $\omega^{(3)}_i$ and $\omega^{(4)}_i$ are multiplied by $h_{i-1} - h_i$ and $h_{i+1} - h_i$, respectively. Also note that $\omega^{(3)}_i$ and $\omega^{(4)}_i$ account for lateral aggregation, while $\omega^{(5)}_i$ refers to aggregation at a local maximum (see Fig. 1).

The same steps of the previous model are then followed: Eq. (29) is inserted into Eq. (15), with step functions and height differences expanded according to Eqs. (22) and (21). Retaining terms up to the leading order in $a_\parallel$ and $a_\perp$, and in the small p limit, we obtain the KPZ equation

$$\frac{\partial h}{\partial t} = F + \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t), \tag{31}$$

with

$$\nu = \frac{a_\parallel^2}{\tau} p, \tag{32}$$

$$\lambda = \frac{10 a_\parallel^2}{\tau} A_1 p, \tag{33}$$

and

$$F = \frac{a_\perp}{\tau}. \tag{34}$$

A naive inspection of Eqs. (32) and (33) suggests $\nu \sim p$ and $\lambda \sim p$. However, the scaling of A_1 and A_2 in Eqs. (27) and (28), with $y = 1$ for ballistic-like models, gives

$$\nu \sim p, \quad \lambda \sim p^{3/2}, \tag{35}$$

which agrees with simulation [11] and scaling arguments [17,18].

Thus, this model also shows that the scaling of the equation coefficients on p depends not only on the random choice of BD but also on the scaling of height differences. Moreover, our analysis shows that equation coefficients which are noninteger powers of p, such as $p^{3/2}$, can be predicted by construction of growth equations from the microscopic rules, with the noninteger exponent related to the step function regularization.

The coefficient of the surface tension term [Eq. (32)] is always positive. Indeed, the lateral aggregation rules of BD [$\omega^{(3)}_i$ and $\omega^{(4)}_i$ in Eq. (30)] reduce local height differences, as illustrated by deposition at columns 2 and 8 in Fig. 1(b), which is the role of surface tension. This balances the negative contribution to the surface tension from aggregation at a local maximum, illustrated by deposition at column 5 in Fig. 1(b).

This result differs from what is obtained with the aggregation rules of Ref. [15]: $\nu = -(a_\parallel^2 a_\perp/\tau)A_1 p$, which is negative for (expected) positive A_1. With the aggregation rules presented in that work, lateral aggregation does not contribute to surface tension; thus aggregation at a local maximum renders ν negative, as can be inferred from the $a_\perp$ factor appearing in that formula for ν (in Ref. [15], it follows only from the expression of $\omega^{(5)}_i$).

The preceding discussion shows that it is essential to check the consistency of the equation coefficients and the geometry of the model when strong approximations (such as regularization of step functions) are involved. When the surface is rough, pure lateral aggregation at edges and steps tends to bring the surface to a plain surface state, through both nonconservative (taming of a height difference larger than $a_\perp$) and conservative mechanisms (taming of a step of height $a_\perp$). Indeed, we chose aggregation rules for BD such that the pure lateral aggregation gives a positive contribution to the Laplacian term, as it is expected. That was only possible by allowing $\omega^{(3)}_i + \omega^{(4)}_i + \omega^{(5)}_i \neq 1$ in Eq. (30) and avoiding products between terms such as $1 - \theta^i_j$, which will be subsequently regularized.

Now we consider the BBD model. The first jump moment is

$$\begin{aligned} K^{(1)}_i = \frac{p}{\tau}\big[&(\omega^{(6)}_i + \omega^{(9)}_{i-1})(a_\perp + h_{i-1} - h_i) \\ &+ (\omega^{(7)}_i + \omega^{(8)}_{i+1})(a_\perp + h_{i+1} - h_i) + a_\perp\omega^{(6)}_{i+1} \\ &+ a_\perp\omega^{(7)}_{i-1} + a_\perp\omega^{(9)}_i + a_\perp\omega^{(8)}_i\big] + (1-p)\frac{a_\perp}{\tau} \end{aligned} \tag{36}$$

with

$$\begin{aligned} \omega^{(6)}_i &= (1/2)\theta^{i-1}_i, \\ \omega^{(7)}_i &= (1/2)\theta^{i+1}_i, \\ \omega^{(8)}_i &= (1/2)\left(1 - \theta^{i-1}_i\right), \\ \omega^{(9)}_i &= (1/2)\left(1 - \theta^{i+1}_i\right), \end{aligned} \tag{37}$$

where the factors $1/2$ correspond to the equal probability for the two possible orientations of the dimers on the incidence site. The KPZ equation [Eq. (31)] is also obtained from this rules. In the small p limit, it has coefficients

$$\nu = \frac{a_\parallel^2}{\tau} p, \tag{38}$$

$$\lambda = 4\frac{a_\parallel^2}{\tau} A_1 p, \tag{39}$$

and

$$F = \frac{a_\perp}{\tau}(1 + p). \tag{40}$$

The p dependence of A_1 for ballistic-like models [Eq. (27) with $\delta = 1/2$] again gives $\nu \sim p$ and $\lambda \sim p^{3/2}$, in agreement with Ref. [34], which combined scaling properties of the KPZ equation in one dimension and numerical results.

The relation between the first and second jump moments [Eq. (18)] shows that this scaling picture can also provide the noise term of the growth equation. We follow Ref. [1] and rewrite Eq. (17) as $\langle \eta_i(t)\eta_j(t')\rangle = D\delta_{ij}\delta(t' - t)$ to have $D = a_\perp K^{(1)}_i$, where D is the amplitude of the noise correlations. For each model, we expand the step functions and height differences [Eqs. (22) and (21)] and retain terms up to the leading order in $a_\perp$ and $a_\parallel$. This gives $D \sim a_\perp F$ for all models, as can be found from inspection of Eqs. (24) and (25), (32) to (34), (38) to (40), and (44) to (46), since all time and space derivatives of h are finite in the limiting process. That means $D = a_\perp^2/\tau$ plus terms of order $a_\perp a_\parallel^2$ or $a_\perp^2 a_\parallel^2$ in all RD-CD models. For BBD, $D = a_\perp^2(1 + p)/\tau + O(a_\perp a_\parallel^2)$, due to the particular choice of the time unit for that model.

These results show that, in the small p limit, there is no effect of this parameter on the noise amplitude. Consequently, all the crossover effects depend on the coefficients ν and

λ (in contrast to what is observed in other growth models [46]).

V. THE RDSR-BD MODEL

This model was introduced in Ref. [9] and involves the competition of BD (KPZ class), with probability p, and RDSR (EW class), with probability $1-p$. In Ref. [14], scaling properties were studied numerically, with the coefficient of the nonlinear term scaling as $\lambda \sim p^2$ for $0.2 \leqslant p \leqslant 0.5$. That quadratic dependence was proposed analytically by Muraca *et al.* [15].

The first jump moment in this case is

$$K_i^{(1)} = p\mathcal{K}_i^{BD} + (1-p)\mathcal{K}_i^{RDSR}, \tag{41}$$

where $\mathcal{K}_i^{RDSR}$, $\mathcal{K}_i^{BD}$ are the first jump moments

$$\mathcal{K}_i^{RDSR} = \frac{a_\perp}{\tau}\left(\omega_i^{(0)} + \omega_{i+1}^{(1)} + \omega_{i-1}^{(2)}\right) \tag{42}$$

and

$$\mathcal{K}_i^{BD} = \frac{1}{\tau}\left[\omega_i^{(3)}(h_{i-1}-h_i) + \omega_i^{(4)}(h_{i+1}-h_i) + \omega_i^{(5)}a_\perp\right], \tag{43}$$

where $\omega_i^{(k)}$, $k = 0,\ldots,5$, are the aggregation rules given in Eqs. (20) and (30). Following the same approach of the other models, we obtain the KPZ equation with coefficients

$$\nu = \frac{a_\parallel^2}{\tau}p + \frac{2a_\parallel^2 a_\perp}{\tau}A_1\left(1-\frac{3}{2}p\right), \tag{44}$$

$$\lambda = \frac{2a_\parallel^2}{\tau}A_1\left(5 - a_\perp A_1\right)p, \tag{45}$$

and

$$F = \frac{a_\perp}{\tau}. \tag{46}$$

Both RDSR and BD have correlated kinetics which lead to finite average values of local slopes, even in the steady states. Consequently, the leading coefficients of the step functions (A_1, A_2) do not vanish in the limit $p \to 0$, in contrast with the models with crossover from RD. Thus, Eq. (45) gives $\lambda \sim p$ as $p \to 0$, while the other coefficients remain nonzero.

This result disagrees with the quadratic dependence observed in simulations of Ref. [14] and suggested in analytical work of Ref. [15]. In order to understand this discrepancy, the conditions where the BD component generates nonlinearity in the RDSR-BD model have to be analyzed.

In Fig. 1(b), deposition at columns 2 and 8 shows the condition in which lateral aggregation (characteristic of BD) leads to excess velocity: deposition occurs at a column i which has at least one neighbor with height larger by $2a_\perp$ or more. This leads to formation of a hole in column i. However, for small p, pure RDSR dominates. We simulated the one-dimensional RDRS model in lattice sizes $L = 256$ and $L = 512$ and found that the number of columns where excess velocity is possible is $P \approx 0.044$. This fraction is small because RDSR produces a very smooth surface, with a very small number of high steps. On the other hand, in pure BD, our simulations show that this probability is near $1/2$. Thus, in the competitive model with small p, the fraction of columns which have lateral growth (i.e., nonlinear growth) is approximately Pp.

On the other hand, the BD model itself creates conditions for two neighboring sites to have height difference $2a_\perp$ or more: If a column i has one larger neighbor j ($h_i < h_j$), a BD event at j followed by another BD event at i leads to lateral aggregation with formation of a hole. For instance, this would correspond to the deposition in column $j = 5$, shown in Fig. 1(b), followed by deposition in column $i = 4$ (not shown). In the pure RDSR surface, the fraction of columns with at least one higher neighbor is $Q \approx 0.44$, also obtained from simulation. Thus, the probability of this column having nonlinear growth due to those subsequent BD events is approximately Qp^2.

From the point of view of Eq. (45), pure RDSR corresponds to $A_1 \sim P$ in the regularization of step functions, while a BD event corresponds to $A_1 \sim Qp$.

For small enough p, we certainly have $Pp > Qp^2$; thus the crossover EW-KPZ is dominated by BD events taking place on a nearly pure RDSR surface. This occurs for $p < P/Q \approx 0.10$. In this regime, the linear dependence of λ on p [Eq. (45)] is expected. However, simulation results of Ref. [14] are for $p \geqslant 0.2$. In the lower limit $p = 0.2$, we have $Pp \approx 0.0088$ and $Qp^2 \approx 0.0176$. This means that Qp^2 is twice as large as Pp, and their difference is enhanced for larger p. Consequently, the simulated range of p favors nonlinearities arising from two subsequent BD events at neighboring columns, which explains the observed quadratic dependence of the coefficient λ on p.

The arguments of Ref. [15] for the $\lambda \sim p^2$ behavior were also based on the association of the p^2 factor to the random choice of BD. However, it is also a double counting of the factor p, which is reasonable for the simulated range of p but fails for very small p.

The linear scaling of the coefficient λ on p was also found in the RSOS model of deposition and erosion of Ref. [30], both in simulations and in the derivation of the associated KPZ equation. The constraint on the neighboring height difference of the RSOS model leads to rejection of deposition and erosion attempts, which is the mechanism to generate nonlinear growth. That rejection occurs with a probability much larger than the probability P for the RDSR; thus the linear dependence on p was easily observed in simulations [30].

VI. CONCLUSION

Langevin equations associated with various competitive lattice models were derived. The approach is based on a van Kampen expansion of the master equation, but the correct assessment of how characteristic times and lengths scale with the competing parameter plays a central role if we are to find the true dependence of the equation coefficients in the crossover regimes. Moreover, it is essential to choose representations of aggregation rules (using, e.g., step and delta functions) that lead to physically reasonable equation coefficients, as RDSR and BD models illustrate.

We considered a series of models with crossover from random deposition to correlated growth (RD-CD), with probability p for the latter, and a model with EW-KPZ crossover,

with probability p for the KPZ component. All coefficients that vanish as $p \to 0$ show a linear p dependence arising from the random choice of aggregation rules. However, in the RD-CD case, neighboring height differences diverge in that limit, which leads to the p scaling of the parameters of the optimal regularization of step functions. Thus, the coefficients depending on those parameters show scaling as p, p^2, and $p^{3/2}$, in all cases in agreement with simulation results and other scaling approaches. For the model with EW-KPZ crossover, the quadratic dependence of the nonlinear term coefficient, observed in simulations, is explained as a crossover behavior due to particular model features, while linear p dependence is expected for very small p. Although the scaling properties derived here are similar to previous works on those models [15,17,18], the interpretation is very different and the applicability of the method is broader, for instance being extendable to higher dimensions.

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