Power-law behavior in a cascade process with stopping events: A solvable model

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The present paper proposes a stochastic model to be solved analytically, and a power-law-like distribution is derived. This model is formulated based on a cascade fracture with the additional effect that each fragment at each stage of a cascade ceases fracture with a certain probability. When the probability is constant, the exponent of the power-law cumulative distribution lies between −1 and 0, depending not only on the probability but the distribution of fracture points. Whereas, when the probability depends on the size of a fragment, the exponent is less than −1, irrespective of the distribution of fracture points. The applicability of our model is also discussed.

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I. INTRODUCTION

A power-law distribution is found ubiquitously, ranging from physical systems, such as critical phenomena in phase transitions, aggregation, fracture, and earthquakes, to economic systems, such as distributions of income and sales, and price fluctuations [1]. A lot of physical approaches, such as fractal growth [2], self-organized criticality [3], and the "rich get richer" mechanism [4], have derived power-law distributions in various systems, but our understanding of power laws is still insufficient in view of their diversity. In the present paper, we propose a simple stochastic model which is analytically solvable and produces a power-law distribution. The model incorporates two elements: a multiplicative stochastic cascade process and random stopping.

Fracture is one of the most typical phenomena deeply related to a power law. A number of experiments have confirmed that fragment size distributions mainly follow power laws [5–7]. Theoretically, various models have attempted to derive power-law distributions [8–11], but there seems to be no decisive model which simply and analytically explains a power-law distribution of fragment sizes without using specific breaking mechanisms. Fracture phenomena have offered us many interesting properties and behaviors, not only a power-law distribution of fragment sizes. For example, the shattering transition occurs when fragmentation of smaller fragments becomes increasingly fast and a finite fraction of mass falls into a dust phase of zero-size fragments [12]; also, the damage-fragmentation transition observed in collisional fragmentation yields scaling relations similar in percolation theory [13].

Before we introduce our model, we briefly refer to a simple multiplicative stochastic process, using a model of cascade fracture [14]. In this model, one rod of length L breaks into two fragments at a randomly chosen point, and each of the two fragments again breaks into two subfragments, and so on [see Fig. 1(a)]. The length of one of the fragments after the nth stage of fracture is expressed as $\xi_1\xi_2\cdots\xi_n L$, where $\xi_1,\xi_2,\ldots,\xi_n$ are random numbers between 0 and 1. This process is referred to as "multiplicative," because the length of a fragment is given by multiplying the previous length by ξ_i. The fragment size distribution in this case is not a power-law distribution but a lognormal one, which is proved by the central limit theorem for $\log\xi_i$.

II. MODEL AND ANALYSIS

Our model proposed in this paper also starts with one rod of length L, and fragments repeatedly break into two subfragments. A fracture point is given by a random number $\xi\in(0,1)$ drawn from a probability density function $g(\xi)$. The difference from the above simple multiplicative model is that each fragment ceases fracture with probability ρ, which we call the "stopping probability" [see Fig. 1(b)]. Whether each fragment stops fracture or not is determined independently; once a fragment ceases fracture, it never restarts fracture any more, and we call such a fragment "inactive." Previously, a similar stochastic model of cascade fracture adopting a random-stopping event has been proposed in Ref. [15], but their analyses treat only a simple situation where $g(\xi)\equiv 1$ in our notation. In this paper, we solve the model for a general probability density $g(\xi)$. Furthermore, we consider the two cases: (i) ρ is constant, and (ii) ρ depends on the length of a fragment.

Case I (*ρ is constant*). We focus on the cumulative number $N_L(x)$ of fragments, which represents the expected number of the inactive fragments larger than x. $N_L(x)$ can be computed as follows. With probability ρ, the initial rod ceases fracture; one fragment (i.e., the initial rod itself) is larger than x. With probability $1-\rho$, by contrast, the initial rod breaks into two fragments at a random point given by ξ; each of the two fragments can experience further fracture, and these subfracture processes are both similar to the whole process. Hence, the expected number of the fragments larger than x in this case is given by $N_{\xi L}(x)+N_{(1-\xi)L}(x)$. By taking into account that the fracture point ξ is drawn from a distribution g, $N_L(x)$ satisfies the following equation:

$$N_L(x)=\rho\times 1+(1-\rho)\int_0^1\{N_{\xi L}(x)+N_{(1-\xi)L}(x)\}g(\xi)d\xi. \tag{1}$$

If we rescale our length scale by a factor $\alpha(>0)$ and observe fracture processes, the length of the initial rod is αL in the new scale, and the cumulative number $N_L(x)$ turns to $N_{\alpha L}(\alpha x)$. Hence, a scaling relation $N_{\alpha L}(\alpha x)=N_L(x)$ or $N_{\alpha L}(x)=N_L(x/\alpha)$ is satisfied. Using this relation to convert all subscripts in Eq. (1) into L, and introducing

$$\tilde{N}_L(x)=N_L(x)+\frac{\rho}{1-2\rho} \tag{2}$$

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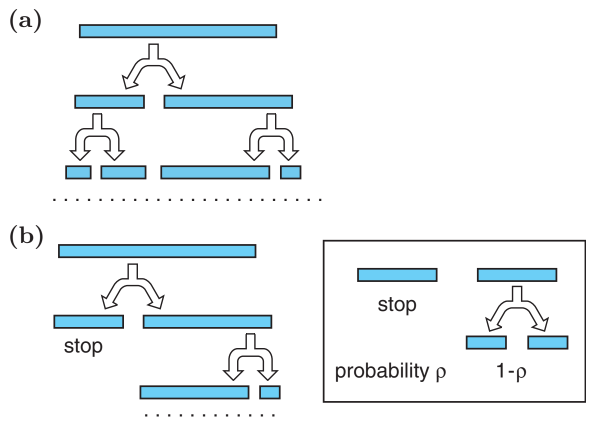

FIG. 1. (Color online) (a) The simple model of cascade fracture, where the fragments at each stage break into two subfragments. The resulting fragment size distribution is a lognormal one. (b) Our proposed model, which is different from (a) in that each fragment ceases fracture with probability ρ.

in order to eliminate the inhomogeneous term "ρ" at the right-hand side of Eq. (1), we obtain a homogeneous equation of $\tilde{N}_L$:

$$\tilde{N}_L(x) = (1-\rho)\int_0^1 \left\{\tilde{N}_L\left(\frac{x}{\xi}\right) + \tilde{N}_L\left(\frac{x}{1-\xi}\right)\right\} g(\xi)d\xi. \tag{3}$$

We assume a power-law form $\tilde{N}_L(x) = Cx^{-\beta}$, where C and $\beta(>0)$ are both independent of x. Then, we have

$$(1-\rho)\int_0^1 \{\xi^\beta + (1-\xi)^\beta\} g(\xi)d\xi = 1. \tag{4}$$

The exponent β is determined by this equation; hence β generally depends on both ρ and g. The coefficient C is determined from Eq. (2) by considering $N_L(L) = \rho$ with $\tilde{N}_L(L) = CL^{-\beta}$ as

$$C = \frac{2\rho(1-\rho)}{1-2\rho}L^\beta.$$

Eventually, the complete solution is

$$N_L(x) = \frac{2\rho(1-\rho)}{1-2\rho}L^\beta x^{-\beta} - \frac{\rho}{1-2\rho}$$
$$= \frac{\rho}{1-2\rho}\left\{2(1-\rho)\left(\frac{x}{L}\right)^{-\beta} - 1\right\},$$

coupled with Eq. (4) for determination of β. Note that $\tilde{N}_L(x)$ is an exact power law of x, but N_L is not exactly because of the presence of the second term "-1." Nonetheless, $N_L(x)$ can be approximated by a power law if the second term is negligible, i.e., $2(1-\rho) \gg (x/L)^\beta$, or if x is sufficiently smaller than L and ρ is also small.

In above calculation, we postulate the power-law form $\tilde{N}_L(x) = Cx^{-\beta}$, but the validity of this hypothesis is not obvious at first sight. First of all, the uniqueness of solutions of an integral equation (3) is not quite evident, and other types of solutions may exist. We prove in the Appendix that Eq. (3) can admit only power-law solutions.

The solution β of Eq. (4) cannot be expressed explicitly for a general probability density g. Now, we provide three examples of calculations of β. The simplest instance is $g(\xi) = \delta(\xi - 1/2)$, where δ is the Dirac δ function. In other words, the fracture points are always at the middle of the fragments. Equation (4) in this case is reduced to

$$2(1-\rho)\left(\frac{1}{2}\right)^\beta = 1,$$

and the solution is $\beta = 1 + \log(1-\rho)/\log 2$.

In the second example, the fracture point is distributed uniformly over each fragment, i.e., $g(\xi) = 1$ for all $\xi \in (0,1)$. Equation (4) becomes

$$(1-\rho)\int_0^1 \{\xi^\beta + (1-\xi)^\beta\}d\xi = 2(1-\rho)\frac{1}{1+\beta} = 1,$$

and the solution is $\beta = 1 - 2\rho$, which reproduces the previous result in Ref. [15].

The third example is $g(\xi) = 6\xi(1-\xi)$. (The coefficient "6" comes from the normalization $\int_0^1 g(\xi)d\xi = 1$.) Also in this case, we can calculate β explicitly as

$$\beta = \frac{\sqrt{49-48\rho} - 5}{2},$$

which is a completely untrivial result.

In the above three examples, two limiting values $\beta \nearrow 1$ as $\rho \searrow 0$ and $\beta \searrow 0$ as $\rho \nearrow 1/2$ can be easily obtained in common. Since β is a decreasing function with respect to ρ in these examples, we conclude that the reasonable ranges are $0 < \beta < 1$ and $0 < \rho < 1/2$. Moreover, the same constraints for β and ρ can be derived for a general $g(\xi)$. Indeed, the relations $\lim_{\rho\searrow 0}\beta = 1$ and $\lim_{\rho\nearrow 1/2}\beta = 0$ and monotonity of β with respect to ρ hold in a general g.

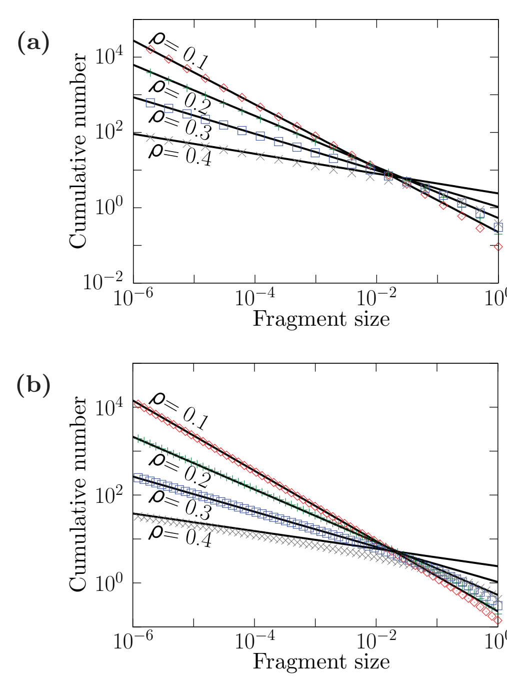

FIG. 2. (Color online) Numerical results of cumulative number $N_L(x)$ for $L = 1$ and $\rho = 0.1, 0.2, 0.3$, and 0.4, generated by counting only the inactive fragments, and averaging 1000 samples each. Each straight line indicates the corresponding $\tilde{N}_L$ that follows an exact power law. The fracture points are (a) at the middle of the fragments $g(\xi) = \delta(\xi - 1/2)$, and (b) distributed uniformly $g(\xi) = 1$.

Here, we show the results of a numerical check of N_L in Fig. 2. The parameters are $L = 1$, and $\rho = 0.1,0.2,0.3,0.4$. The probability density g for the fracture points are $g(\xi) = \delta(\xi - 1/2)$ in (a) and $g(\xi) = 1$ in (b). Each sample of the calculations was performed until all the active fragments became smaller than 10^{-6}, and we counted only the inactive fragments, and each data set in the figure is the average of 1000 samples. An exact power law $\tilde{N}_L$ is also shown with a solid line. Power laws fail in larger fragment sizes, as mentioned above.

Case II (*ρ depends on a fragment size*). It is noted that the above model provides only $\beta < 1$. However, in real experiments of fracture, some results correspond to $\beta > 1$ [5,16]. Here we propose a modification of the above model in order to realize $\beta > 1$ by treating the stopping probability as a function of a fragment size. In particular, we give here the stopping probability of a fragment of size ℓ as

$$\rho(\ell) = \begin{cases} \left(\frac{\lambda}{\ell}\right)^{\gamma}, & \ell \geqslant \lambda \\ 1, & \ell \leqslant \lambda, \end{cases} \tag{5}$$

where λ is a characteristic length and $\gamma > 0$ is a constant. The stopping probability (5) represents an effect that smaller fragments have more difficulty experiencing further fracture. Obviously, a fragment becomes inactive whenever its size becomes smaller than λ; hence the parameter λ is the lower bound of the fragment sizes. We employ the assumption $\lambda \ll L$ in the following analysis.

As in Case I, the cumulative number $N_{L,\lambda}(x)$, including two parameters L and λ this time, plays an important role. In the same way as Eq. (1), $N_{L,\lambda}$ satisfies the following equation:

$$\begin{aligned} N_{L,\lambda}(x) &= \{1 - \rho(L)\} \int_0^1 \{N_{\xi L,\lambda}(x) + N_{(1-\xi)L,\lambda}(x)\} g(\xi) d\xi + \rho(L) \\ &= \left\{1 - \left(\frac{\lambda}{L}\right)^{\gamma}\right\} \int_0^1 \{N_{\xi L,\lambda}(x) \\ &\quad + N_{(1-\xi)L,\lambda}(x)\} g(\xi) d\xi + \left(\frac{\lambda}{L}\right)^{\gamma} \\ &\simeq \int_0^1 \{N_{\xi L,\lambda}(x) + N_{(1-\xi)L,\lambda}(x)\} g(\xi) d\xi, \end{aligned} \tag{6}$$

where we used the approximation $\lambda/L \simeq 0$. (the symbol "$\simeq$" is used only in this sense.)

A scaling relation $N_{\alpha L,\alpha\lambda}(\alpha x) = N_{L,\lambda}(x)$ is again obtained. Moreover, by the definition of the cumulative number,

$$N_{L,\lambda}(x) = \int_x^L \rho(\ell)\nu(\ell) d\ell = \int_x^L \left(\frac{\lambda}{\ell}\right)^{\gamma} \nu(\ell) d\ell \underset{\sim}{\propto} \lambda^{\gamma},$$

where $\nu(\ell)d\ell$ is the average number of active fragments whose lengths are within $[\ell, \ell + d\ell)$. Thus, another scaling relation $N_{L,\alpha\lambda}(x) \simeq \alpha^{\gamma} N_{L,\lambda}(x)$ is derived for $x \gg \lambda$ and $\alpha > 0$. We guess a power-law form $N_{L,\lambda}(x) = Cx^{-\beta}$, and substitute into Eq. (6) together with the above two scaling relations, which yields

$$\int_0^1 \{\xi^{\beta-\gamma} + (1-\xi)^{\beta-\gamma}\} g(\xi) d\xi = 1.$$

Since the normalization $\int_0^1 g(\xi) d\xi = 1$ holds, the solution is $\beta = 1 + \gamma$. $\beta > 1$ is attained because $\gamma > 0$. A remarkable point is that the exponent $\beta = 1 + \gamma$ is universal over any probability density g governing the fracture points. (Compare this with the case of a constant stopping probability, where β depends on g.)

The coefficient C is $\lambda^{\gamma} L$, derived from the consistency of two expressions $N_{L,\lambda}(L) = CL^{-\beta} = CL^{-(1+\gamma)}$ and $N_{L,\lambda}(L) = \rho(L) = (\lambda/L)^{\gamma}$. Finally, the complete solution is expressed as

$$N_{L,\lambda}(x) = \lambda^{\gamma} L x^{-(1+\gamma)} = \left(\frac{\lambda}{L}\right)^{\gamma} \left(\frac{x}{L}\right)^{-(1+\gamma)}. \tag{7}$$

The calculation is based on $x \gg \lambda$; consequently, this solution breaks down if $x \lesssim \lambda$.

Numerical confirmation is shown in Fig. 3, where we set $L = 1$ and $\lambda = 10^{-6}$. The numerically obtained cumulative numbers clearly lie on the power-law solutions (solid lines) over a wide range of larger fragment sizes. Also, the data points deviate from the power laws in a fragment size close to or less than λ, as expected theoretically.

III. DISCUSSION

One can straightforwardly extend the model so that each fragment breaks into n subfragments at a single fracture, where n can be either a fixed or random number. A fragment size

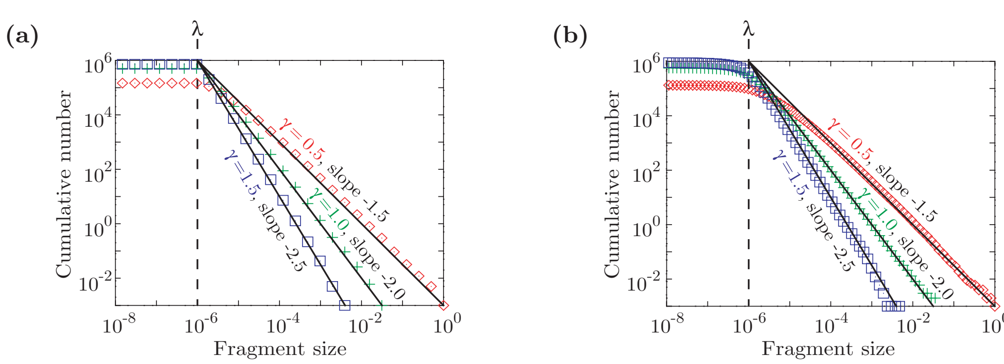

FIG. 3. (Color online) Numerical results of N_L for $L = 1$. The parameter values of the stopping probability are $\lambda = 10^{-6}$ and $\gamma = 0.5,1$, and 1.5. Each data set was generated by averaging 1000 samples. Solid lines indicate the corresponding solutions (7). The probability densities for the fracture points are, respectively, $g(\xi) = \delta(\xi - 1/2)$ in (a), and $g(\xi) = 1$ in (b).

distribution in this case is again like a power law; the exponent β is less than 1 under a constant stopping probability, and $\beta = 1 + \gamma (>1)$ under the stopping probability as in Eq. (5). A special case like the Sierpinski fractal is found in Refs. [17,18] without pointing out the sensitivity of β against g.

Now we discuss the relevance and applicability to other systems. As described in Sec. I, the natural distribution related to a multiplicative process is a lognormal distribution. Thus, some additional "tricks" are needed to generate a power law from a multiplicative process, such as a reset event [19], additive noise [20], and a boundary constraint [21]. The trick for a power law in our models is random stopping. Moreover, our model is related to a stochastic branching process [22]. In fact, in the case where the stopping probability is constant ρ, the "genealogical tree" of the fragments is simply a kind of the Galton-Watson branching process, in the sense that each fragment at each stage of a cascade has either two "children" of subfragments with probability ρ or no children with $1 - \rho$. Such a stochastic process has been investigated exhaustively, but we stress that a stochastic branching process alone is not associated with a power-law distribution.

For clarity, we formulate and analyze the model in terms of the fracture of a rod. However, we require neither the properties of materials nor specific breaking mechanisms; hence, the model is not limited to the fracture of materials in a narrow sense. For instance, at the simplest level, a power-law distribution of income, often referred to as the Pareto distribution [23], can be thought as the consequence of hierarchical partitioning of profit or wealth, and a power-law distribution of file sizes [24] is caused by the partitioning of bulk data.

It has been pointed out that a lognormal distribution can be confused easily with a power-law distribution [25]. Our result gives one theoretical basis for their connection; the difference is whether the stopping probability exists or not. Some experiments also have shown that the two types of distributions can possess a common origin. In fact, a fragment size distribution qualitatively changes according to impact energy [26] (or falling height [27]): it exhibits a lognormal distribution under lower energy, and a power-law distribution under higher energy. We can roughly explain the experiments as follows. Let us consider that the stopping probability is given by Eq. (5), where the cumulative number follows a power law as in Eq. (7). In the low-impact-energy limit, a fracture process corresponds to a cascade limited to the first several stages, where most fragments are far larger than λ, so $\rho(\ell) \simeq 0$ holds. This is almost equivalent to the simple multiplicative process free from stopping events [Fig. 1(a)]; therefore, the fragment size distribution in this limit becomes rather like a lognormal distribution.

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APPENDIX: THE VALIDITY OF A POWER-LAW SOLUTION OF EQ. (3)

We prove here that Eq. (3) has only a power-law solution $\tilde{N}_L(x) = Cx^{-\beta}$. First, Eq. (3) changes to

$$(1-\rho)\int_0^1 \left\{ \frac{\tilde{N}_L\left(\frac{x}{\xi}\right)}{\tilde{N}_L(x)} + \frac{\tilde{N}_L\left(\frac{x}{1-\xi}\right)}{\tilde{N}_L(x)} \right\} g(\xi)d\xi = 1$$

by dividing both sides by $\tilde{N}_L(x)$. The left-hand side is a function of L and x, whereas the right-hand side is a constant. Hence, $\tilde{N}_L(x/\xi)/\tilde{N}_L(x)$ depends only on ξ in reality, and we set $\Phi(\xi) := \tilde{N}_L(x)/\tilde{N}_L(x/\xi)$. Equivalently, we can write $\tilde{N}_L(xy) = \tilde{N}_L(x)\Phi(y)$, and also $\tilde{N}_L(xy) = \tilde{N}_L(y)\Phi(x)$ by interchanging x and y. Then, we have $\tilde{N}_L(x)\Phi(y) = \tilde{N}_L(y)\Phi(x)$ for any x and y, which concludes $\Phi(x) = c\tilde{N}_L(x)$ for some constant c. (To be precise, c is a function of L.) Finally, a closed relation

$$\tilde{N}_L(xy) = c\tilde{N}_L(x)\tilde{N}_L(y)$$

is obtained. By differentiating with respect to y and then putting $y = 1$, we have a differential equation

$$x\tilde{N}_L'(x) = c\tilde{N}_L'(1)\tilde{N}_L(x),$$

whose solution is a power law $\tilde{N}_L(x) = Cx^{-\beta}$, where C and $\beta := -c\tilde{N}_L'(1)$ are constants. As in the main part of the paper, the coefficient C is determined by a boundary condition, and the exponent β by Eq. (4). In conclusion, the integral equation (3) with the boundary condition $\tilde{N}_L(L) = \rho$ has the unique power-law solution shown here.

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