Time needed to board an airplane: A power law and the structure behind it

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A simple model for the boarding of an airplane is studied. Passengers have reserved seats but enter the airplane in arbitrary order. Queues are formed along the aisle, as some passengers have to wait to reach the seats for which they have reservation. We label a passenger by the number of his or her reserved seat. In most cases the boarding process is much slower than for the optimal situation, where passenger and seat orders are identical. We study this dynamical system by calculating the average boarding time when all permutations of N passengers are given equal weight. To first order, the boarding time for a given permutation (ordering) of the passengers is given by the number s of sequences of monotonically increasing values in the permutation. We show that the distribution of s is symmetric on $[1,N]$, which leads to an average boarding time $(N+1)/2$. We have found an exact expression for s and have shown that the full distribution of s approaches a normal distribution as N increases. However, there are significant corrections to the first-order results, due to certain correlations between passenger ordering and the substrate (seat ordering). This occurs for some cases in which the sequence of the seats is partially mirrored in the passenger ordering. These cases with correlations have a boarding time that is lower than predicted by the first-order results. The large number of cases with reduced boarding times have been classified. We also give some indicative results on the geometry of the correlations, with sorting into geometry groups. With increasing N, both the number of correlation types and the number of cases belonging to each type increase rapidly. Using enumeration we find that as a result of these correlations the average boarding time behaves like N^{α}, with $\alpha \simeq 0.69$, as compared with $\alpha = 1.0$ for the first-order approximation.

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I. INTRODUCTION

Many systems display large variability: in geometrical structure, in response to a constant driving, in temporal behavior. In several cases, the probability distribution functions of relevant quantities have unusual, highly asymmetric forms. Complexity is a broad and imprecise term often used for these systems.

Theoretical models often contain a large number of particles that move according to local or global rules (interactions). A classical example is diffusion-limited aggregation, a pattern-growth model that mimics several experimental situations realistically [1,2]. In many cases analytical results are not available, and only numerical simulations can give some information on how relevant quantities depend on system size.

Furthermore, if the aim is to develop approaches that are general, and not only useful for specific systems, one needs general ways to characterize the particle configurations. These configurations may constitute initial conditions or arise during the dynamics. In most cases the particles are indistinguishable. An important one-dimensional variant is the asymmetric exclusion process [3,4]. We consider here, however, *distinguishable* particles in one dimension. This has several consequences: The number of initial configurations of N particles is $N!$. Even though this is a large number, the configurations may be organized systematically and in an identical way for several models. The substrate on which the process operates can in some cases be described similarly, as for the present model. This allows one to study *complexity as a matching problem.*

This article is organized as follows. In Sec. II we describe the passenger boarding problem, discuss briefly previous work, and define the model to be studied. The main quantity of interest to us, the average boarding time, is discussed in Sec. III. Calculations underlying a first-order result for the average boarding time are given in Sec. IV. Second-order corrections arise due to correlations between passenger ordering in the queue and the row ordering, as discussed in Sec. V. Some concluding remarks are given in Sec. VI.

II. THE MODEL SYSTEM

We consider the time needed to board an airplane. The passengers have reserved seats, but enter the airplane in arbitrary order. As a result, many passengers must wait in the aisle while their fellow travelers load their carry-on luggage. Several approaches have been suggested to shorten the total boarding time. These include observations of boarding designs that are being used, estimation of parameters, simulations, and mathematical modeling at various levels of sophistication and for specified airplane layouts [5–16].

Here we consider a simplified situation, with only one seat in each row. In this model there will therefore be no bottlenecks in the boarding process due to already seated passengers that must get up to allow a passenger in an inside seat. Moreover, we assume that the time needed to advance along the aisle is negligible compared to the time needed to place carry-on luggage and get seated. In each time step all passengers that have reached their reserved seats get seated. The seats are numbered sequentially starting with 1 near the entrance door, and we label the passengers with the number of the passenger's

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reserved seat. We assume that a passenger standing in the aisle queue takes up the same space as one seat.

The number of time steps needed for boarding N passengers, T, depends sensitively on the sequence in which the passengers are lined up. T takes all integer values from 1 to N. The value $T = 1$ corresponds to having the passengers lined up as $1,2,3,\ldots,N$, with passenger N in front, so that there is a perfect match between the queue and the row structure. $T = N$ corresponds to the inverse ordering.

In Fig. 1 some boarding configurations for $N = 5$ are shown. Figures 1(a) and 1(e) illustrate the configurations with minimum and maximum boarding times, respectively. An intermediate case is shown in Fig. 1(b). The ordering 12 435, with 5 in front, consists of two sequences with monotonically increasing values. Each sequence gets seated in a single time step, so in this case $T = 2$. There is a large number of intermediate cases, in which some but not all passengers are seated simultaneously.

Our model is simple, with a (uniform) time delay for a passenger to take a seat and $N!$ queue orderings as the only ingredients. In attempts to get closer to the real situation, previous studies have incorporated effects like assigning to each passenger one, two, or three pieces of luggage and assuming increasing loading times as bins fill up [5]; minimizing passenger interferences using groups [6,8,16]; allowing a passenger in the queue to move one row forward per time step if there is free space [7]; assigning to each seat an energy that characterizes how desirable it is to passengers (no seat reservation) [10]; defining a parameter for the aisle length occupied by a passenger [15]. As a consequence of this variety, it is difficult to compare results from previous models, an observation made earlier in Ref. [9].

III. THE AVERAGE BOARDING TIME

In contrast to most previous studies, we are not primarily interested in finding optimal boarding designs. Instead, we determine $\langle T(N)\rangle$, the *average* number of time steps needed to seat N passengers at N reserved seats, when all permutations of passengers are equally likely. We use $\langle T(N)\rangle$ as a simple characterization of the boarding model described in Sec. II above. Specifically, we are interested in the functional dependence of this average time upon N.

The calculation of the average boarding time involves two stages: To first order, T for a given configuration (permutation) is equal to the number s of sequences of increasing values along the direction of motion (which is to the right in Fig. 1). For the configuration in Fig. 1(b), for example, $T = s = 2$. The average boarding time can then be obtained directly from the probability distribution function for s. We show in Sec. IV below that this probability distribution function is symmetrical on $[1,N]$. As a result, the average boarding time $\langle T(N)\rangle$ is

$$\langle T(N)\rangle = \frac{N+1}{2} \propto N. \qquad (1)$$

However, due to certain structures in the queues, the boarding time may be lower than the number s of increasing sequences. An example is the configuration in Fig. 1(c), for which $T = s = 3$, but the actual boarding time $t = 2$. The reason is that the passenger labeled 1 is seated together with the sequence 35 in front. As a result, the two sequences 2 and 14 are reduced to one (24). One notes that passengers 1 and 3 have the same relative order and distance (in the queue) as their reserved seats have (on the "substrate"). We reserve T for the boarding time of one particular permutation in terms of the initial number of sequences of increasing label values. For any case, $T = s$. The actual boarding time is t. For any permutation, $t \leqslant T$.

These *correlations*, that is, correlations between queue and substrate, lead to reduced boarding times for some of the permutations (queue configurations). The symmetrical probability distribution function for boarding times is thus altered into an asymmetrical one, and the average boarding time is no longer given by the simple formula in Eq. (1). A more systematic treatment of correlations is given in Sec. V.

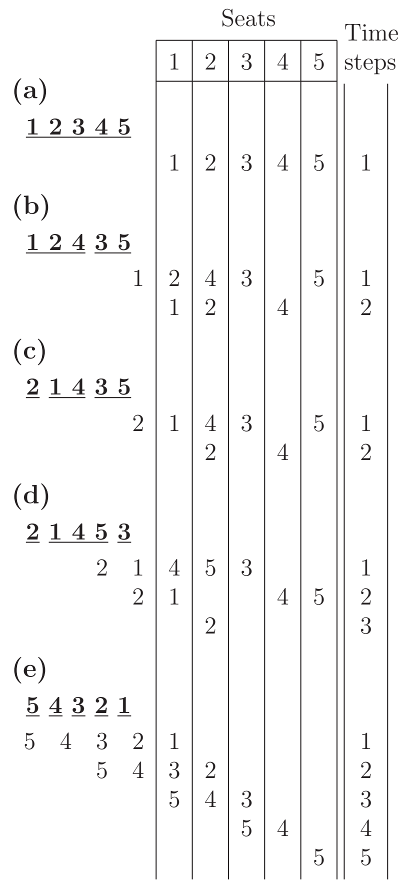

FIG. 1. Five configurations (queues) for boarding, with $N = 5$ passengers. The direction of motion is to the right. Each line represents one time step, during which one or more passengers are being seated, leaving others to wait. Seated passengers are removed from the queue, rendering it possible for other passengers to reach they reserved positions (seats) in the succeeding time step. Each passenger is identified (labeled) by the number of the reserved seat. On the left hand, each configuration is shown before entering the aisle, with sequences of increasing values (along the direction of motion) underlined (see text). The time to complete boarding ranges from $T = 1$ in (a) to $T = 5$ in (e), with some intermediate cases shown in (b), (c), and (d).

TABLE I. Values of $a(N,s)$ for $N=1$ (top line) to $N=6$ (bottom line), with s increasing from 1 to N in each line.

N					1					
↓				1		1				
			1		4		1			
		1		11		11		1		
	1		26		66		26		1	
1		57		302		302		57		1
					$s\rightarrow$					

IV. FIRST ORDER: SEQUENCES

As discussed above, the number of time steps needed for boarding to be completed is at most equal to the number s of increasing sequences in the permutation of $1,2,\ldots,N$, corresponding to the passenger lineup before entering the cabin. In Sec. IV A we establish a recursion relation and from it the symmetry property that leads to Eq. (1). Furthermore, we determine the distribution of s values for a given N. This is an interesting problem by itself and will yield an upper bound for t and its average. In Sec. IV B we show that this distribution approaches the normal distribution when N increases, and in Sec. IV C we give an explicit expression for the distribution.

A. Probability distribution properties

By $a(s,N)$ we denote the number of permutations of N integers containing s sequences of increasing numerical values. When all $N!$ permutations of the integers are assumed to be equally probable, the probability $p(s,N)$ to find a given value of s equals

$$p(s,N)=\frac{a(s,N)}{N!}. \tag{2}$$

We noted in Sec. II that there is merely one permutation with $s=1$, and one with $s=N$. Thus,

$$a(1,N)=a(N,N)=1. \tag{3}$$

By straight enumeration it is easy to determine $a(s,N)$ for small values of N (see Table I). A striking symmetry, $a(s,N)=a(N+1-s,N)$, is apparent for each of these values of N.

We may establish a recursion relation for $a(s,N)$ in the following way. Consider the effect of adding the integer 1 to all permutations of $2,3,\ldots,N$. The emerging permutations of N integers with s increasing sequences result from $(N-1)$-permutations with s or $s-1$ such sequences. When 1 is inserted to the *left* of a sequence, it merges with that sequence resulting in no increase in the number of sequences. When 1 is inserted elsewhere, the number of sequences in the $(N-1)$-permutation is increased by unity. Thus, we have

$$a(s,N)=s\,a(s,N-1)+(N+1-s)\,a(s-1,N-1). \tag{4}$$

Using induction, we prove from this recursion relation the symmetry apparent in Table I,

$$a(s,N)=a(N+1-s,N). \tag{5}$$

From (4) we have

$$\begin{aligned}a(N+1-s,N)&=(N+1-s)\,a(N+1-s,N-1)\\&\quad+s\,a(N-s,N-1).\end{aligned} \tag{6}$$

We assume that the symmetry (5) holds for $N-1$ so that $a(s,N-1)=a(N-s,N-1)$. Using this on the right-hand side of (6) gives the result

$$\begin{aligned}a(N+1-s,N)&=(N+1-s)\,a(s-1,N-1)\\&\quad+s\,a(s,N-1).\end{aligned} \tag{7}$$

Comparison with (4) yields $a(s,N)=a(N+1-s,N)$, the symmetry holds for N. Since the symmetry clearly is valid for some N, $N=3$, for example, it must hold for all values of N.

The symmetry property gives at once the average

$$\langle s\rangle_N=(N+1)/2. \tag{8}$$

Since under the first-order perspective T for any case (permutation) is equal to s, Eq. (1) follows.

B. Limit distribution

We determine now further properties of the symmetric distribution of s, beyond the average value (8), by calculating the second and fourth moments. The variance to be determined we denote by

$$\Delta=\langle(s-\langle s\rangle)^2\rangle=\langle s^2\rangle-\langle s\rangle^2\,. \tag{9}$$

When needed, a subscript will denote the value of N (as in Eqs. (11) and (12) below).

The recursion (4) implies the following relation between the probabilities (2):

$$\begin{aligned}&N[p(s,N)-p(s-1,N-1)]\\&\quad=sp(s,N-1)-(s-1)p(s-1,N-1).\end{aligned} \tag{10}$$

Multiplication by $(s-1)^2$ and summation over all values of s yields

$$N[\langle s^2\rangle_N-2\langle s\rangle_N+1-\langle s^2\rangle_{N-1}]=-2\langle s^2\rangle_{N-1}+\langle s\rangle_{N-1}. \tag{11}$$

Introducing the variance and the average values (8), we end up with

$$N\,\Delta_N-(N-2)\,\Delta_{N-1}=N/4. \tag{12}$$

For $N=2$ we have the variance $\Delta_2=1/4$. With this initial value Eq. (12) determines the variances for all N to be

$$\Delta_N=(N+1)/12. \tag{13}$$

In a similar way one may determine the fourth-order moment,

$$F_N=\langle(s-\langle s\rangle)^4\rangle_N. \tag{14}$$

Multiplying the probability relation (10) by s^4 and summing over all values of s we obtain after some manipulations the following one-step recursion relation:

$$NF_N-(N-4)F_{N-1}=(6N^2-N)/48. \tag{15}$$

The solution of (15) is

$$\begin{aligned}\langle(s-\langle s\rangle)^4\rangle_N=F_N&=\frac{(N+1)^2}{48}-\frac{N+1}{120}\\&=3\Delta_N^2\left(1-\frac{2}{5(N+1)}\right).\end{aligned} \tag{16}$$

The special value for $N = 4$, $F_4 = 23/48$, can be checked by direct calculation. Since (16) satisfies (15), it must be valid for all N.

Note that for large N we have

$$\lim_{N\to\infty} \frac{\langle (s - \langle s\rangle)^4\rangle_N}{\Delta_N^2} = 3, \tag{17}$$

the same value as for any Gaussian distribution. This indicates strongly that for large N the symmetric probability distribution of s values is well approximated by the Gaussian

$$P(s,N) = (2\pi\,\Delta_N)^{-1/2}\, e^{-(s-\langle s\rangle_N)^2/2\Delta_N^2}. \tag{18}$$

One may support this conclusion by comparing the maximum values of $p(s,N)$ and $P(s,N)$.

C. Analytic expression for the distribution

We now determine the probabilities $p(s,N)$. To that end define a generating function

$$g_N(u) = \sum_s a(s,N)\, u^s. \tag{19}$$

By multiplying the recursion equation (4) by u^s and summing over all s, we obtain

$$g_N(u) = \left[Nu + (u - u^2)\frac{d}{du}\right]\, g_{N-1}(u). \tag{20}$$

Starting with $g_1(u) = u$, we find $g_2 = u + u^2$, $g_3 = u + 4u^2 + u^3$, etc., where the values of $a(s,N)$ are seen as coefficients. By introducing

$$h_N(u) = (1-u)^{-N-1}\, g_N(u), \tag{21}$$

(20) simplifies to

$$h_N(u) = \left(u\frac{d}{du}\right)^N\, h_{N-1}(u), \tag{22}$$

with solution

$$h_N(u) = \left(u\frac{d}{du}\right)^N \frac{u}{(1-u)^2}, \tag{23}$$

since $h_1 = (1-u)^{-2} g_1(u) = u(1-u)^{-2}$.

Using the power series $u(1-u)^{-2} = \sum_{n=1}^{\infty} n u^n$ and (21), we find

$$g_N(u) = (1-u)^{N+1} \sum_{n=1}^{\infty} n^N\, u^n. \tag{24}$$

Inserting the binomial expansion,

$$(1-u)^{N+1} = \sum_{k=0}^{N+1} \binom{N+1}{k} (-u)^k, \tag{25}$$

into (24) and introducing $s = n + k$, we have

$$g_N(u) = \sum_s \sum_{k=0}^{s-1} (-1)^k \binom{N+1}{k} (s-k)^N u^s. \tag{26}$$

Thus, the coefficient of u^s is

$$a(s,N) = \sum_{k=0}^{s-1} (-1)^k \binom{N+1}{k} (s-k)^N. \tag{27}$$

This provides an explicit analytic expression for the probabilities $p(s,N) = a(s,N)/N!$. The values in Table I can be checked against this formula.

V. SECOND ORDER: CORRELATIONS

Above we obtained for the average boarding time the expression (1), $\langle T(N)\rangle = (N+1)/2$. That result is an upper bound, and in this section we show that for many permutations of the incoming passengers, the actual boarding time t is considerably lower than the first-order result T. The basic reason for the increased efficiency is that when a queue is waiting in the aisle, some members behind the leading sequence may be able to take their assigned seats at the same time as the members of the leading sequence are seated. In some cases this leads to a merger of two sequences behind the leading one and reduced boarding time. We refer to a permutation for which this happens as a *reduction case*. For N values with reduction cases, the average boarding time $\langle t(N)\rangle$ is consequently lower than the first-order result $\langle T(N)\rangle$.

We start by discussing in Sec. V A the simple case $N = 4$. The remaining part of this section is devoted to enumeration of all reduction cases for a given N. The enumerations are exact for small N, and we present also a few approximate enumerations for larger N. In Sec. V D we summarize the numerical results by estimating the N dependence of the average boarding time. The data is consistent with a power law,

$$\langle t(N)\rangle \propto N^\alpha, \tag{28}$$

with an exponent α considerably smaller than the first-order result $\alpha = 1$.

A. The case $N = 4$

Consider the incoming passengers having the order 2143, moving toward the right, with $T = 3$. When passenger 3 has reached seat number 3, passenger 1 will be standing at seat 1. Hence, passengers 3 and 1 are able to be seated during the first time step. The remaining two passengers form the increasing sequence 24, and consequently both may be seated during the second time step. The boarding time is therefore $t = 2$ in this case, reduced by 1 compared with the first-order result.

For $N = 4$ this is, in fact, the only case with $t < T$. When averaging over all 4! permutations, there will be one less case with boarding time 3 and one additional case with boarding time 2, compared with the first-order results. Thus, we obtain

$$\langle t(4)\rangle = \langle T(4)\rangle - \frac{1}{4!}\,3 + \frac{1}{4!}\,2 = \frac{59}{24}, \tag{29}$$

less than $\langle T(4)\rangle = 5/2$.

B. Classes of reduction cases

We refer to a merger of two sequences as *a single reduction*. For the reduction case discussed in Sec. V A there is one single reduction. A single reduction reduces the boarding time for the permutation by an amount 1. In general, a reduction case may have $r = 1,2,\ldots$ single reductions. The value of s, the initial number of sequences in the passenger queue, varies significantly from one reduction case to another.

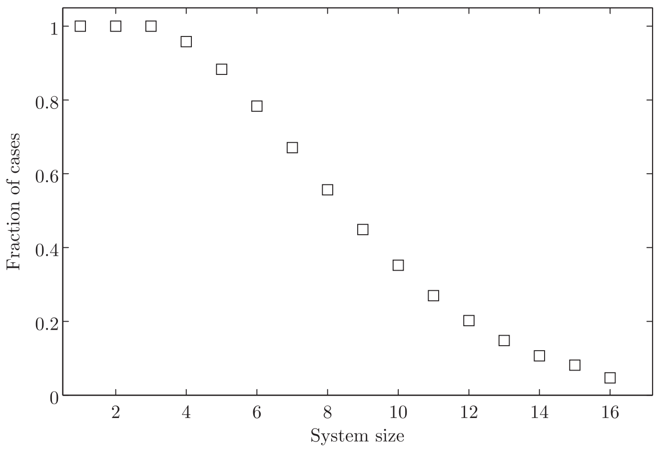

FIG. 2. The fraction of permutations (cases) that do not contain any reductions, as a function of system size N.

As N increases, the number of reduction cases increases rapidly. Figure 2 shows that the fraction of cases without reductions falls off as N increases. As an example of how the number of reduction cases $n(r,s)$ varies, detailed results for $N = 8$ are given in Table II.

Table II shows that there are no reduction cases with 1, 2, and N sequences. Moreover, reductions cannot produce a boarding time $t = 1$. Consequently, cases with r reductions can only occur when $s \geqslant r + 2$. These results are valid for all N. The reduction cases are concentrated at intermediate s values. The fraction of cases that are reduction cases increases monotonically with s up to $s = 7$.

By enumeration the distribution of boarding times $\nu(t)$ was obtained directly. It may also be calculated from the distribution $n(r,s)$, the number of cases with r reductions, and s increasing sequences. Since $t = s - r$, the number of cases $\nu(t)$ with boarding time t equals

$$\nu(t) = a(t) - \sum_{r\geqslant 1} n(r,t) + \sum_{r\geqslant 1} n(r,t + r). \tag{30}$$

The second term subtracts from the number $a(t)$ of cases with t sequences those that contain reductions.

TABLE II. Exact enumeration for $N = 8$. Here $a(s,8)$ is the number of passenger permutations with a given number s of increasing sequences. The last three columns record the number of cases with 1, 2, and 3 reductions. The bottom line lists the total number $B(r,N)$ of cases with r reductions for $N = 8$ passengers.

s	$a(s,8)$	$n(1,s)$	$n(2,s)$	$n(3,s)$
1	1	0	0	0
2	247	0	0	0
3	4293	303	0	0
4	15 619	4981	58	0
5	15 619	8203	985	1
6	4293	2337	803	17
7	247	125	57	1
8	1	0	0	0
		15 949	1903	19

TABLE III. Distribution of boarding times for $N = 8$. The second line is the distribution of the number of sequences in the passenger permutations (first column in Table II), and the third line gives the distribution $\nu(t)$ when reductions are taken into account. The second order is exact.

Boarding time	1	2	3	4	5	6	7	8
First order	1	247	4293	15 619	15 619	4293	247	1
Second order	1	609	9973	19 587	8824	1261	64	1

Table III gives for $N = 8$ the number of cases with a given boarding time to first and second order. To first order the distribution is symmetric, as explained in Sec. IV. With reductions present, the boarding time distribution is unsymmetrical, tilted toward shorter times. The average boarding time can be determined from this distribution, or calculated from the values of $B(r,N)$ (last line in Table II), the total number of cases with r reductions,

$$\langle t(N)\rangle = \frac{N + 1}{2} - \frac{1}{N!}\sum_{r=1}^{N} r B(r,N), \tag{31}$$

which should be compared to Eq. (29). In this case we obtain $\langle t(8)\rangle = 13\,469/3360 = 4.0086$, considerably less than the first-order result $\langle T(8)\rangle = 4.5$.

In the same way we have performed an exact determination of the reduction cases for values of N up to 14.

C. Geometry of reductions

So far we have only been concerned with the *number* of reduction cases as a function of N. However, the reduction cases can be sorted into geometrical categories, which is briefly illustrated here. In Table IV we list for $N = 5$ all permutations with reduction and, in addition, a few interesting cases for $N = 6$. As indicated in Table V, both the number of cases for each reduction type and the number of types increase rapidly with N.

Above we have sorted cases with reductions, as obtained from enumeration, into different types. We now start from the opposite side by building up systematically all permutations that contain a given reduction type. This involves restrictions on where some passenger labels may be placed, while others can be varied freely.

Consider the simplest type (1,2), which consists of correlations 1_3, 2_4, etc. Note that 1_2 does not give a reduction since in this case passengers 1 and 2 will not be seated during the same time step. For correlations $k\_(k + 2)$ there are restrictions on the label values behind and between, that is, on l and m in $lkm(k + 2)$. A sequence must end just behind the correlation; thus, we must have $l > k$. Moreover, a sequence must also end at the intermediate site, which requires $m > k + 2$. Finally, these two sequences must merge when the correlated structure leaves the queue, which requires $m > l$.

Determination of the number of permutations that give rise to a reduction of type (1,2) consists of two steps. The first step is to determine the number of $l\ k\ m\ (k + 2)$ blocks that fulfill the requirements above for each N value. We refer to these structures of fixed label values as 4-blocks. For $N = 4$, only

TABLE IV. All cases with reductions for $N = 5$ passengers, and some cases for $N = 6$, are listed. The permutations are listed in the first column, the number of sequences s in the second, and the boarding time t in the third. The fourth column shows the disconnected passenger groups that are seated simultaneously, thereby generating a reduction. An underlined blank denotes a label value (passenger) not being seated during the same time step. A decomposition of this group into correlations is given in the fifth column. The sixth column gives a classification in the following format: (number of blanks, difference between the numbers in the fifth column). Here c stands for a chain of correlations and g for reductions generated or modified after a previous time step.

Permutation	s	t	Group	Correlation	Type
1 3 2 5 4	3	2	2_4	2_4	(1,2)
2 1 3 5 4	3	2	1_,_4	1_,_4	(2,3)
2 1 4 3 5	3	2	1_35	1_3 + 1_,_5	(1,2 + 2,4)
2 1 5 3 4	3	2	1_34	1_3 + 1_,_4	(1,2 + 2,3)
2 1 5 4 3	4	3	1_4	1_4	(1,3)
2 3 1 5 4	3	2	1_4	1_4	(1,3)
2 4 1 5 3	3	2	1_3	1_3	(1,2)
3 1 2 5 4	3	2	12_4	12_4	(1,2/3)
3 1 5 4 2	4	3	1_4	1_4	(1,3)
3 2 1 5 4	4	3	1_4	1_4	(1,3)
3 2 5 4 1	4	3	2_4	2_4	(1,2)
4 1 5 3 2	4	3	1_3	1_3	(1,2)
4 2 1 5 3	4	3	1_3	1_3	(1,2)
5 2 1 4 3	4	3	1_3	1_3	(1,2)
2 1 4 3 6 5	4	2	1_3_5	1_3_5	(1,2c1,2)
2 1 5 3 4 6	3	2	1_346	1_3 + 1,_4 + 1,_,_6	(1,2 + 2,3 + 3,5)
3 1 2 6 4 5	3	2	12_45	12_4 + 12_,_5	(1,2/3 + 2,3/4)
3 1 6 2 5 4	4	3	1_5	g1_5	(g1,4)
3 2 6 1 5 4	4	3	2_5	g2_5	(g1,3)

the reduction _1_3 is possible, and only for one choice of label values on the remaining two sites: 2 1 4 3. This is the only case with reduction for $N = 4$ (see also Sec. V A). All four label values are here used inside the 4-block. For $N = 5$, the 4-block 2 1 4 3 is again possible under _1_3. Additional 4-blocks under _1_3 and one further 4-block under _2_4 are shown in Table VI. This table also show all 4-blocks for $N = 6$.

There is a simple pattern in the generation of 4-blocks from $(N-1)$ to N in Table VI. All 4-blocks for $(N-1)$ are also valid for N. Furthermore, there is one additional 4-block for N for each class that was present for $(N-1)$. A class is here defined as the set of 4-blocks with the same label value at the hindmost site (the site to the left, as shown). An example is 2 1 4 3 and 2 1 5 3 for $N = 5$, followed by 2 1 4 3, 2 1 5 3, and 2 1 6 3 for $N = 6$. There is also one 4-block in a new class for N, like 5 1 6 3 for $N = 6$. Finally, one new correlation of type _k_$(k+2)$ is possible when N is increased by 1, like _3_5 from $N = 5$ to $N = 6$.

An expression for the total number of 4-blocks can be found based on the pattern just described. For one correlation, the number of 4-blocks is always given by a triangle number, with each class as one line in the triangle representation. Consider as an example $N = 6$ and _1_3, where one has 3 (from 2-class) plus 2 (from 4-class) plus 1 (from 5-class). The base of the triangle number is $(N-3)$ for the correlation _1_3, $(N-4)$ for _2_4, ..., 1 for the last correlation that is possible. Thus,

TABLE V. Reduction types, with number of cases, for $N = 4$, $N = 5$, and $N = 6$. These types correspond to the last column in Table IV. The reduction types are listed under: Simple, which may be subdivided into uncompressed (same distance in queue and on substrate) and compressed (lower distance in initial queue than on substrate), where only two passengers leave the queue simultaneously; Interlaced, where two or more types overlap; Extended, which to some degree are reductions symmetrical to the interlaced ones, but where further decomposition is not possible; and Others, which consists of further, more complex reduction types. Note that in this table, the components of the interlaced cases are also counted under the main types. An example is the (1,2 + 2,3) case for $N = 5$, which is counted as one (1,2) case and one (2,3) case under Simple, uncompressed.

Reduction type	$N = 4$	$N = 5$	$N = 6$
Simple, uncompressed			
(1,2)	1	8	58
(2,3)		2	22
(3,4)			3
Simple, compressed			
(1,3)		4	42
(2,4)		1	16
(3,5)			3
(1,4)			14
(2,5)			8
Extended			
(1,2/3)		1	8
(1,2/4)			4
(2,3/4)			2
(1,3/4)			4
(2,4/5)			1
(1,2/3/4)			1
Others			
(1,2c1,2)			1
(g1,2)			1
(g1,3)			2
(g1,4)			1
(2,3g1,3)			1
Interlaced			
(1,2 + 2,3)		1	6
(1,2 + 2,4)		1	7
(1,2 + 2,5)			5
(2,3 + 3,4)			1
(2,3 + 3,5)			1
(1,3 + 2,4)			4
(1,3 + 2,5)			3
(1,2 + 2,3 + 3,4)			1
(1,2 + 2,3 + 3,5)			1
(1,2 + 2,4 + 3,5)			1
(1,2/3 + 2,3/4)			1
(1,2/3 + 2,4/5)			1

the total number of blocks is obtained by summing triangle numbers from 1 to $(N-3)$,

$$n_{\text{4-blocks}} = \sum_{k=1}^{N-3} \frac{1}{2}\, k(k+1) = \frac{1}{6}\,(N-3)(N-2)(N-1). \tag{32}$$

TABLE VI. All possible 4-blocks (see explanation in main text) underlying the reduction type (1,2), for $N = 4$, $N = 5$, and $N = 6$. Note the systematic repetition and expansion of the 4-block set as N increases.

Correlation	$N = 4$	$N = 5$	$N = 6$
_1_3	2143	2143	2143
		2153	2153
			2163
		4153	4153
			4163
			5163
_2_4		3254	3254
			3264
			5264
_3_5			4365

The second step in determining the number cases with the reduction type (1,2) consists of finding the number of ways to place the $(N-4)$ label values that do not belong to the 4-block. These label values can be placed either in front of or behind the 4-block and freely exchanged between the available sites, without any influence on the reduction of the 4-block. The number of ways to place these remaining sites is $(N-3)$ [and not $(N-4)$]: All may be placed behind the 4-block, one may be placed in front of the 4-block, and the others behind it,......, all may be placed in front of the 4-block. Thus, there are

$$n_{\text{outside 4-block}} = (N-3)(N-4)! \tag{33}$$

ways to distribute the label values outside each 4-block. The number of (1,2) cases is given by $n_{(1,2)} = n_{\text{4-blocks}} \cdot n_{\text{outside 4-block}}$; therefore,

$$n_{(1,2)} = \tfrac{1}{6}(N-3)(N-1)!. \tag{34}$$

A similar calculation for (1,3), which contains _1_4, _2_5, etc., gives the result

$$n_{(1,3)} = \tfrac{1}{6}(N+1)(N-3)(N-4)(N-3)!. \tag{35}$$

The mismatch between the values from Eq. (34) and Table V for $N = 6$ is due to a case where two (1,2) blocks overlap and form a new reduction type (see permutation 2 14365 in Table IV).

The most interesting aspect of the results (34) and (35) is that they are close to, but slightly less than $N!$. In Sec. V D we show that the average boarding time follows a power law $\langle t \rangle \propto N^{\alpha}$ with an exponent α lower than the first-order result $\alpha = 1$. This implies that the number of reductions must increase faster than $N!$ [see Eq. (31)]. This is not achieved by the most frequent reduction types in Eqs. (34) and (35) alone, the other reduction types indicated in Table V are necessary components.

Several trends in Table V can be understood with the calculations above as a background. For the simplest Extended cases, there are 5-blocks of sites with restrictions, and these types are less frequent than Simple, uncompressed and Simple, compressed, which are determined by 4-blocks. A similar pattern applies within each group and subgroup. Under Simple, uncompressed, (1,2) cases are determined by 4-blocks, (2,3), by 5-blocks, and (3,4) by 6-blocks, and the frequency falls off rapidly.

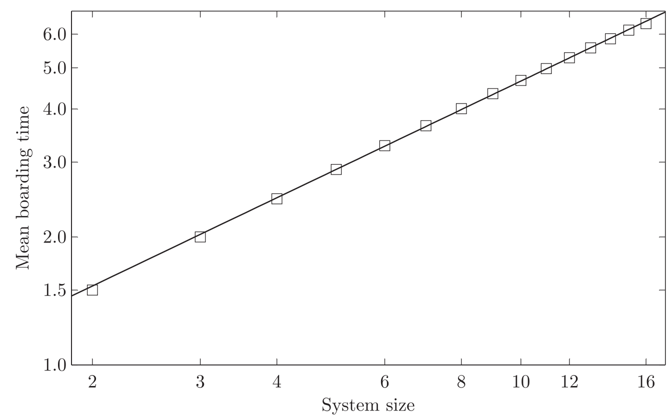

FIG. 3. Average boarding time as a function of N, on log scales. The line represents Eq. (36), with parameters as given in (37).

D. A power law

For N up to 14 all $N!$ passenger permutations were generated, the number of reduction cases was recorded, and the average boarding time was determined. Since all permutations were covered, the results are exact. For $N = 15$ and 16 several runs with nonoverlapping subsets of permutations have been used (no sampling). By this procedure 40% of the permutations were included for $N = 15$, for $N = 16$ merely 2% were included.

The data is shown in Fig. 3 as a log-log plot. The results are consistent with a power law of the form

$$\langle t(N) \rangle = cN^{\alpha}. \tag{36}$$

Using linear regression on the log-scale data we find

$$\alpha = 0.69 \pm 0.01 \quad \text{and} \quad c = 0.95 \pm 0.02. \tag{37}$$

The uncertainties reflect variations as different subsets of the data are used. Note that there is no noise in the data for $N < 15$.

The lowering of the exponent α from the value $\alpha = 1$ in first order emphasizes the importance of including correlations. It implies that the number of reductions is very large for large N, since it clearly must increase much faster than $N!$ if the exponent value $\alpha = 1$ is to be avoided. Boarding is much more efficient than predicted by Eq. (1).

In addition to the average value of the boarding time distribution we have also determined the standard deviation σ for different N. The standard deviation scales differently from the average,

$$\sigma(N) \propto N^{\beta}, \tag{38}$$

with $\beta \simeq 0.32 \pm 0.02$.

VI. CONCLUDING REMARKS

We have formulated a simple model for the airline boarding process. In particular, the N dependence of the average boarding time is of interest, when all permutations of the N incoming passengers occur with equal probability.

The main mechanism that determines the boarding time for a given permutation is associated with the sequences of increasing passenger labels along the queue direction: All passengers in one sequence take their seats simultaneously. This perspective leads to the simple expression (1) for the average boarding time. We have been able to find an analytic expression for the full distribution of the number of these sequences. For increasing N the normal distribution (18) is approached.

However, we show that due to certain correlations between the queue ordering and the seat arrangement in the airplane, additional passengers may be seated. For N up to 14 the corresponding reductions in boarding time are determined exactly for the different permutations, for $N = 15$ and 16 in good approximation. The average boarding time now behaves as N^{α}, with $\alpha \simeq 0.69$, for increasing N, instead of $\alpha = 1$ to first order, see (1).

A variety of geometrical structures generates the power law. With increasing N, the number and types of these structures increase rapidly, as for many other complex systems. The present model is one-dimensional, and the microscopic structures in terms of integer series is easily systematized.

An interesting question is how sensitive the N dependence of the boarding time is to model details. Bachmat and coworkers report $N^{0.5}$ [12,15] (as compared with our result $N^{0.69}$), but the N dependence was not a main concern in their research.

We have previously studied a model that resembles the present one: N distinguishable particles with unidirectional motion in one dimension [17]. In that model as well bottlenecks formed by structures in the permutations lead to significant slowing down. However, there is no interaction with the substrate in the previous model. The time needed to have the particles transported some given distance behaves like $N/\ln N$, which increases faster with N than $N^{0.69}$, the result for the boarding model above. This difference reflects a stronger bottleneck mechanism in the previous model.

Efficient boarding is obtained when many positions along the aisle are occupied by passengers loading carry-on luggage and getting seated, rather than by passengers waiting in line. Several airlines use groups to improve efficiency, for example, boarding first all passengers for rows 17–32, thereafter rows 1–16. This is referred to as a back-to-front strategy. For our model, however, boarding with two *disconnected* groups leads to an average boarding time equal to $2\,c\,(N/2)^{\alpha} = 1.24\,c\,N^{\alpha}$, that is, *less* efficient boarding.

One might wonder whether this unexpected result is due to the simplicity of our model. In a real aircraft there are several seats in each row, and boarding delays arise due to row obstructions as well as to aisle obstructions. If our model is modified to include more than one seat in each row, higher values of the boarding time are expected. At this point we are not able to make quantitative statements on the combined effects of several seats per row and of grouping. There are, however, in the literature several studies where both these effects are investigated using simulation models, with results that do not clearly support the conventional wisdom that grouping procedures are efficient [5,7,11,15]. Reference [15] states that "Somewhat surprisingly, these studies have found that back-to-front policies are not necessarily effective, and may even be detrimental" (compared with random boarding).

Finally, we remark that several studies suggest that boarding is most effective by use of more exotic boarding strategies. These are combinations of a back-to-front procedure and a window-to-aisle policy in which a window seat in a row is filled first and the aisle seat last [6–8,11]. On the other hand, sophisticated and efficient boarding designs tend to split, during the boarding process, groups of people traveling together.

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