Parity effect and phase transitions in quantum Szilard engines

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(Received 1 September 2011; published 17 January 2012)

Quantum Szilard engines with an arbitrary number of identical particles are studied in this paper. Analytical expressions for the total work in the low- and high-temperature limits are obtained. The total work depends on both the particle statistics, the odd-even parity, and the temperature of the system. The parity effect is drastic in fermion systems. An odd number of fermions perform work as if they were a single fermion, and an even number of fermions do not perform any work at all. For bosons, there exists a phase transition at a critical temperature under which work done by the engine is always negative. It is found that only above a certain temperature, bosonic quantum Szilard engine does more work than fermionic one. The possible experimental verification of these effects is discussed.

DOI: 10.1103/PhysRevE.85.011125 PACS number(s): 05.30.Fk, 03.65.Ta, 05.30.Jp, 89.70.Cf

I. INTRODUCTION

The Szilard engine (SZE) is an elaborate form of Maxwell's demon designed to quantitatively analyze how an intelligent being can extract work from a single thermal reservoir along with an increase in entropy that saves the second law of thermodynamics [1,2]. Szilard believed that the entropy, which is widely accepted as $k_B \ln 2$ [3,4], is produced by the measurement process. Bennett and Landauer argued that it is the erasing of the information that generates entropy [5,6]. The profound connection between information and thermodynamics illustrated by the SZE spures the investigation on the information-to-energy conversion and feedback control system [7–16], yet interestingly a solution of the Maxwell demon paradox without invoking the notions of information or entropy was also proposed and analyzed by Scully *et al.* with an experimental scheme [17–19].

Recently Kim *et al*. investigated the "quantum Szilard engine" (QSZE) where the quantum effect of the work substance has to be taken into consideration [20]. As is shown in Fig. 1, N ideal identical molecules (fermions or bosons) are prepared in an isolated box of size L. Naturally, the molecules are in the eigenstates of a potential well of size L and have discrete energy levels. The wall is then isothermally inserted in the box at the position l, after which a measurement is performed to find m molecules in the left room. As long as the molecule numbers of each side are unequal, the QSZE will experience an isothermal expansion which pushes the wall either leftward or rightward to the equilibrium position $l_{\rm eq}^m$. Finally, the wall is removed and the whole thermodynamic cycle is completed. The total work the engines performs during a single cycle holds

$$W_{\rm tot} = -k_B T \sum_{m=0}^{N} f_m \ln\left(\frac{f_m}{f_m^*}\right), \tag{1}$$

where k_B is the Boltzmann constant and T is the temperature of the heat bath. $f_m = Z_m(l)/Z(l)$ gives the probability of having m molecules on the left at the measurement with $Z(l) = \sum_{m=0}^{N} Z_m(l)$, where $Z_m(l)$ represents the partition function for the case in which m molecules are on the left. f_m^* is defined as $f_m^* = Z_m(l_{\rm eq}^m)/Z(l_{\rm eq}^m)$. Using Eq. (1), Kim *et al*. discussed the quantum thermodynamic work (QTW) performed by QSZE containing one and two molecules.

The situation for arbitrary number N, however, remains an important and interesting topic of urgent research. In this work, we studied the QSZE with an arbitrary number of particles. It is found that there is a parity effect in fermionic QSZE, and a phase transition occurs in a bosonic QSZE in which QSZE absorbs work instead of doing work below a critical temperature. It is also found that in contrast to the two particle case, a bosonic QSZE does more work than a fermionic QSZE only above certain temperature.

II. QSZE WITH THREE PARTICLES

For simplicity, we set the insertion position $l = \frac{L}{2}$, and suppose the molecules are in the energy levels of infinite potential, which can be expressed as $E_n(l) = \frac{n^2\pi^2\hbar^2}{2Ml^2}$ where M is the mass of a single molecule and $\hbar$ the reduced Plank constant. Note that $f_0^* = Z_0(l_{\rm eq}^0)/Z(l_{\rm eq}^0) = Z_0(0)/Z_0(0) = 1$ is always satisfied because it corresponds to the case in which the wall moves to the end of the box, and for the same reason $f_N^* = 1$. Besides, the symmetry of the system leads to $f_m = f_{N-m}$, $f_m^* = f_{N-m}^*$. For QSZE containing three molecules, the total work is therefore given by

$$W_{\rm tot} = -2k_B T\left[f_0 \ln f_0 + f_1 \ln\left(\frac{f_1}{f_1^*}\right)\right]. \tag{2}$$

A. Three-fermion case

Let us first consider the case of a three-fermion SZE whose QTW is denoted as $W_{\rm tot}^{3F}$. In the high-temperature limit, theoretical calculations of the partition function give the following results

$$f_0(T\to\infty) = \tfrac{1}{8}, \quad f_1(T\to\infty) = \tfrac{3}{8}, \quad f_1^*(T\to\infty) = \tfrac{4}{9}. \tag{3}$$

By inserting Eq. (3) into Eq. (2), one can easily obtain

$$W_{\rm tot}^{3F}(T\to\infty) = \tfrac{9}{4} k_B T \ln \tfrac{4}{3}. \tag{4}$$

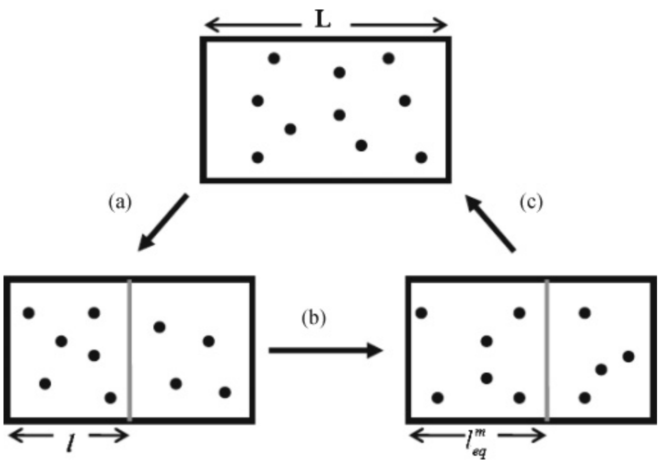

FIG. 1. The working cycle of the classical SZE. (a) A wall is isothermally inserted into the box with N molecules at the position l, then a measurement is performed to find m molecules on the left without any cost of work. (b) The isothermal expansion of the molecules pushes the wall to its equilibrium position $l^m_{\rm eq}$. (c) The wall is removed isothermally at last, and the engine is restored to its original state.

A similar approach can be performed for the low-temperature limit, giving

$$W^{3F}_{\rm tot}(T \to 0) = k_B T \ln 2. \tag{5}$$

B. Three-boson case and the "work phase transition"

The total work performed by the three-boson SZE $W^{3B}_{\rm tot}$ can also be addressed. In the high-temperature limit, we have

$$f_0(T \to \infty) = \tfrac{1}{8}, \quad f_1(T \to \infty) = \tfrac{3}{8}, \quad f_1^*(T \to \infty) = \tfrac{4}{9}, \tag{6}$$

and

$$W^{3B}_{\rm tot}(T \to \infty) = \tfrac{9}{4} k_B T \ln \tfrac{4}{3}. \tag{7}$$

In the low-temperature limit, one reaches

$$\begin{aligned} &f_0(T \to 0) = \tfrac{1}{4}, \quad f_1(T \to 0) = \tfrac{1}{4}, \\ &f_1^*(T \to 0) = e^{-\beta[E_1(l^1_{\rm eq}) - E_1(L - l^1_{\rm eq})]}, \end{aligned} \tag{8}$$

and

$$W^{3B}_{\rm tot}(T \to 0) = 2k_B T \ln 2 - \tfrac{1}{2}\left[E_1\left(l^1_{\rm eq}\right) - E_1\left(L - l^1_{\rm eq}\right)\right], \tag{9}$$

where $\beta = 1/(k_B T)$.

The equilibrium position of the wall $l^m_{\rm eq}$ can be fixed by a procedure described below. The generalized force that acts on the wall is defined as $F = \sum_{n=1}^{\infty} P_n \frac{\partial E_n(l)}{\partial l}$. Note that P_n is the mean occupation number of the nth energy level and can be expressed as $P_n = e^{-\beta E_n(l)}/Z(l)$, where $Z(l)$ is the partition function $Z(l) = \sum_{n=1}^{\infty} e^{-\beta E_n(l)}$. Assume that $E_n(l) = n^2\pi^2\hbar^2/(2Ml^2)$. Obviously, the balance of the wall requires $F^{\rm left} + F^{\rm right} = 0$. Thus, we end up with

$$\begin{aligned} &\sum_{i=1}^{\infty} \frac{m E_i\left(l^m_{\rm eq}\right) e^{-\beta E_i(l^m_{\rm eq})}}{l^m_{\rm eq} Z\left(l^m_{\rm eq}\right)} \\ &= \sum_{j=1}^{\infty} \frac{(N-m) E_j\left(L - l^m_{\rm eq}\right) e^{-\beta E_j(L - l^m_{\rm eq})}}{\left(L - l^m_{\rm eq}\right) Z\left(L - l^m_{\rm eq}\right)}, \end{aligned} \tag{10}$$

where $E_i(l^m_{\rm eq})$ and $E_j(L - l^m_{\rm eq})$ denote the ith and jth energy level of molecules on the left and right, respectively. In Eq. (9), $l^1_{\rm eq}$ is simply the special case for $m = 1$, and is thereby derived as $l^1_{\rm eq} = \frac{L}{1+2^{\frac{1}{3}}}$ by solving Eq. (10). Hence Eq. (9) can be rewritten as

$$\begin{aligned} W^{3B}_{\rm tot}(T \to 0) &= 2k_B T \ln 2 - \frac{1}{2}\left[E_1\left(l^1_{\rm eq}\right) - E_1\left(L - l^1_{\rm eq}\right)\right] \\ &\approx 2k_B T \ln 2 - \frac{0.47\pi^2\hbar^2}{ML^2}. \end{aligned} \tag{11}$$

Equation (11) clearly indicates that there exists a phase transition at which the work done by the SZE $W^{3B}_{\rm tot}$ changes from positive to negative at low temperatures below a critical temperature T_C. We call this phase transition the "work phase transition" (WPT). The critical temperature T_C can thus be naturally obtained. At a temperature above T_C, the bosonic SZE performs positive work. However, as the temperature drops below T_C, one will not extract any work from the bosonic SZE, but will rather do work on it instead because the engine performs negative work, or rather "absorbs" work in a single cycle. This phase transition exists for general multiparticle bosonic SZE's and we will discuss this in more detail in the next section.

III. MANY PARTICLE QSZE

Now we have analyzed the total work performed by a three-molecule QSZE in the high- and low-temperature limits, which, compared with the $N = 1$ and $N = 2$ cases, appears to be more intricate. The second term of $W^{3B}_{\rm tot}$ in Eq. (9) clearly manifests the quantum effect which comes into play at low temperature. It is intriguing and necessary for us to further study the case of QSZE containing an arbitrary number of molecules.

A. Hightemperature limit

One can prove that both fermionic and bosonic N-molecule QSZE satisfy

$$\begin{aligned} f_m(T \to \infty) &= \frac{C^m_N}{2^N}, \\ f^*_m(T \to \infty) &= C^m_N \left(\frac{m}{N}\right)^m \left(1 - \frac{m}{N}\right)^{N-m} \end{aligned} \tag{12}$$

in the high-temperature limits. Then Eq. (1) reduces to

$$\begin{aligned} &W^{NF}_{\rm tot}(T \to \infty) \\ &= W^{NB}_{\rm tot}(T \to \infty) \\ &= \begin{cases} N k_B T \ln 2 - W_O, & N \text{ is odd} \\ N k_B T\left(1 - \frac{C^{\frac{N}{2}}_N}{2^N}\right) \ln 2 - W_E, & N \text{ is even,} \end{cases} \end{aligned} \tag{13}$$

where

$$W_O = \frac{Nk_BT}{2^{N-1}}\sum_{m=1}^{\frac{N-1}{2}} C_N^m H(m),$$
$$W_E = \frac{Nk_BT}{2^{N-1}}\sum_{m=1}^{\frac{N}{2}-1} C_N^m H(m), \quad (14)$$

where $H(m)$ is the Shannon entropy

$$H(m) = -\frac{m}{N}\ln\frac{m}{N} - \frac{N-m}{N}\ln\frac{N-m}{N}.$$

Equation (13) reveals the fact that the fermionic and bosonic SZE perform the same amount of work in the high-temperature limit. This is not surprising because distinguishability occurs at a higher temperature because more states are available for fermions and bosons, which both turn into distinguishable classical molecules when $T \to \infty$.

The low-temperature limit case has to be discussed for the bosonic and fermionic SZE separately. In the following, we discuss them separately.

B. Parity effect in multifermion QSZE

For the N-fermion SZE, when N is odd, we have at low temperature

$$f_m = \begin{cases} \frac{1}{2} & m = \frac{N-1}{2} \text{ or } \frac{N+1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$
$$f^*_{\frac{N-1}{2}} = f^*_{\frac{N+1}{2}} = 1. \quad (15)$$

When N is even

$$f_m = \begin{cases} \frac{1}{2} & m = \frac{N}{2}, \\ 0 & m \neq \frac{N}{2}, \end{cases} \quad f^*_{\frac{N}{2}} = 1. \quad (16)$$

In these cases, Eq. (1) reduces to

$$W_{\rm tot}^{NF}(T \to 0) = \begin{cases} k_BT\ln 2 & N \text{ is odd,} \\ 0 & N \text{ is even.} \end{cases} \quad (17)$$

It is apparent from Eq. (17) that the work done by the fermionic QSZE depends on the parity of the molecule numbers. In the low-temperature limit the QTW of an N-fermion SZE is exclusively dependent on N's parity, but irrelevant to N's value when the parity is fixed. We call it the parity effect, which is directly related to the Pauli exclusion principle. Due to the prohibition to occupy the same state, the more evenly the fermions are allocated over the two rooms the lower the energy of the system. Thus, the total work inevitably reaches zero in the low-temperature limit for the even N-fermionic SZE. It is worth mentioning that this effect can also be comprehended from the aspect of the third law of thermodynamics [21]. The ground state exhibits degeneracies when there is an odd number of fermions, whereas the ground state has no degeneracies when there is an even number of fermions and the entropy of the engine (as well as the work done by the engine) approaches zero as the temperature approaches absolute zero due to the third law of thermodynamics.

C. WPT in multiboson QSZE

On the other hand, at the low-temperature limit, one can prove that for the N-boson SZE

$$f_m(T \to 0) = \frac{1}{N+1} \quad (18)$$

is always satisfied regardless of the parity of N.

When N is odd

$$\begin{aligned} f^*_m(T \to 0) &= f^*_{N-m}(T \to 0) \\ &= \begin{cases} 1 & m = 0, \\ e^{-m\beta[E_1(l^m_{\rm eq}) - E_1(L - l^m_{\rm eq})]} & m = 1, 2, \ldots, \frac{N-1}{2}. \end{cases} \end{aligned} \quad (19)$$

When N is even

$$\begin{aligned} f^*_m(T \to 0) &= f^*_{N-m}(T \to 0) \\ &= \begin{cases} 1 & m = 0, \\ e^{-m\beta[E_1(l^m_{\rm eq}) - E_1(L - l^m_{\rm eq})]} & m = 1, 2, \ldots, \frac{N}{2} - 1, \\ \frac{1}{N+1} & m = \frac{N}{2}. \end{cases} \end{aligned} \quad (20)$$

Then the total work of the N-boson SZE in the low-temperature limit is given by

$$W_{\rm tot}^{NB}(T \to 0) = \begin{cases} k_BT\ln(N+1) - W_o, & N \text{ is odd,} \\ \frac{N}{N+1}k_BT\ln(N+1) - W_e, & N \text{ is even,} \end{cases} \quad (21)$$

where

$$W_o = \frac{2}{N+1}\sum_{m=1}^{\frac{N-1}{2}} m\left[E_1\left(l^m_{\rm eq}\right) - E_1\left(L - l^m_{\rm eq}\right)\right], \quad (22)$$

$$W_e = \frac{2}{N+1}\sum_{m=1}^{\frac{N}{2}-1} m\left[E_1\left(l^m_{\rm eq}\right) - E_1\left(L - l^m_{\rm eq}\right)\right]. \quad (23)$$

Though the expression for the even particle number is different from that for the odd particle number QSZE, there is no parity effect which exists in the fermion QSZE. The difference between Eqs. (22) and (23) is only the number of terms to be summed, which is about half of the particle number.

However, there exists a phase transition in the QTW done by a bosonic QSZE. The bosonic SZE will absorb work rather than perform work in an extremely low temperature. This phenomenon can be viewed as the phase transition of the bosonic SZE from the negative-work phase to positive-work phase with a critical temperature T_C. Note that this phase transition only occurs when $N \geqslant 3$. In Fig. 2, the QTW from a bosonic QSZE is drawn. It is interesting to see that the critical temperature increases as the number of bosons in the QSZE increases. We numerically verified that the W_e and W_o increases as $O(N)$, and this in turn indicates that the critical temperature also increases as $O(N)$. It is helpful to make an estimation of this WPT effect. The system of the trapped cold atoms might be a good candidate. Using appropriate parameters, $M = 10^{-26}$ kg and $L = 10^{-5}$ m, we estimated that the critical temperature of a QSZE with 1000 bosons will be in the order of 10^{-7} K. It is within the reach of present-day technology. It will be interesting to confirm this WPT effect, which is purely quantum mechanical, from experiment.

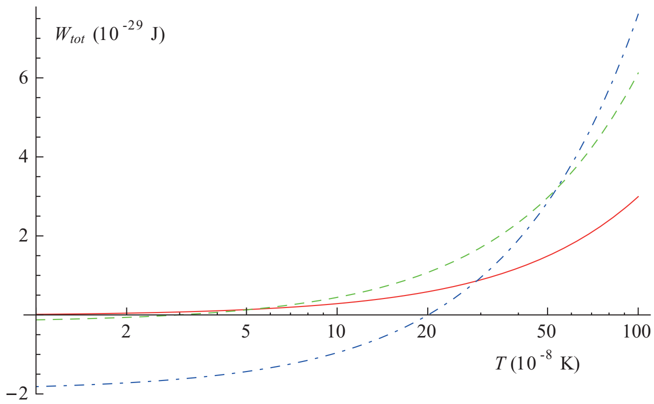

FIG. 2. (Color online) The phase transition of bosonic SZE for $N = 10$ (the red curve), $N = 100$ (the green dashed curve), and $N = 1000$ (the blue dot-dashed curve) cases. The bosons of mass $M = 10^{-26}$ kg are trapped in the infinite potential well of size $L = 10^{-5}$ m. The temperature is given in units of 10^{-8} K, and the total work in units of 10^{-29} J.

IV. SUMMARY

Summarizing the results from Eqs. (13), (17), and (21), Table I presents the generalized QTW of N-molecule QSZE in the high- and low-temperature limits. Not only is it consistent with the special cases of $N = 1, 2, 3$, but also brings about some deep physical insights.

Two purely quantum mechanical effects in QSZE are predicted. At low temperatures, fermionic QSZE exhibits the parity effects. Only QSZE with odd number of fermions can produce work in a evenly separated QSZE, while an even number of fermions does not do any work. The other interesting quantum effect is the WPT effect. At low temperatures, a bosonic QSZE absorbs work rather than performing work. The critical temperature in WPT increases with the number of molecules in the system.

It is also interesting to see that bosonic QSZE does not always do more work that its fermionic counterpart. In the temperature below T_C, more work can be extracted from fermionic SZE as long as N is odd. Nevertheless, the bosonic SZE's ability to perform work improves as the temperature rises, and will finally defeat the fermionic SZE at a certain temperature, which can be calculated as

$$T'_C = \begin{cases} \frac{\ln(N+1)}{\ln(N+1)-\ln 2} T_C & N \text{ is odd} \\ T_C & N \text{ is even} \end{cases} \tag{24}$$

one can numerically verify that W_o and W_e in Eq. (21) approaches to infinity in the order $O(N)$. Consequently, the critical temperature ascends as the molecule number increases, as is revealed in Fig. 2.

From Eq. (13), one can numerically verify that the QTW achieved in the high-temperature limit decreases with the increase of the molecule number in a QSZE and approaches $\frac{1}{2}k_B T$ as the molecule number tends to infinity. Simultaneously, the entropy that is generated in measuring the N molecules' position in the QSZE with $l = \frac{L}{2}$ satisfies $\Delta S = N k_B \ln 2$ [3,22]. Because the work performed by the QSZE never exceeds $N k_B T \ln 2$, the second law of thermodynamics is always satisfied.

In summary, the QSZE containing an arbitrary number of molecules was studied in this paper. The QTW was analytically formulated for the general case of N-molecule QSZE in the high- and low-temperature limits. The results reveal how deeply quantum mechanics, such as indistinguishability and the Pauli exclusion principle, influences the QSZE. We found the parity effect in fermionic QSZE wherein the work performance of the QZSE does not depend on the molecule number, but rather on the parity of the molecule number. In bosonic QSZE, it was predicted that a phase transition effect, the WPT effect, exists. In WPT, bosonic SZE absorbs rather than performs work when below the critical temperature T_C. The N-boson SZE possesses a better work performance than the N-fermion SZE only at high temperatures. The critical temperature rises with the increase in the molecule number, the WPT effect can hopefully be observed in the cold atomic system.

ACKNOWLEDGMENTS

We would like to thank Hui Dong and Changpu Sun for helpful discussions. This work was supported by the National Natural Science Foundation of China (Grant No. 11175094), the National Basic Research Program of China (2009CB929402, 2011CB9216002), and the Specialized Research Fund for the Doctoral Program of Education Ministry of China (2006 0003048).

Note to Appendix. In this Appendix, we present the detailed calculation for the work done by N-molecule SZE and the derivation of Table I. The insertion position of the wall is set to be $l = \frac{L}{2}$ (i.e., the middle of the box). Here we assume the box is an infinite potential well and the energy level can be expressed as $E_n(l) = \frac{n^2\pi^2\hbar^2}{2Ml^2}$.

TABLE I. Total work done by SZE with N identical particles at the low- and high-temperature limits.

	Fermions		Bosons	
Limits	Odd N	Even N	Odd N	Even N
$T \to 0$	$k_B T \ln 2$	0	$k_B T \ln(N+1) - W_o$	$\frac{N}{N+1} k_B T \ln(N+1) - W_e$
$T \to \infty$	$N k_B T \ln 2 - W_O$	$N k_B T \left(1 - \frac{C_N^{\frac{N}{2}}}{2^N}\right) \ln 2 - W_E$	$N k_B T \ln 2 - W_O$	$N k_B T \left(1 - \frac{C_N^{\frac{N}{2}}}{2^N}\right) \ln 2 - W_E$

APPENDIX A: THE FERMIONIC CASE

First we consider the particles to be fermions. Then the probability that m particles are in the left room at a certain temperature T is given by

$$f_m = \frac{Z_m\left(\frac{L}{2}\right)}{\sum_{m=0}^{N} Z_m\left(\frac{L}{2}\right)}, \tag{A1}$$

where

$$Z_m\left(\frac{L}{2}\right) = \sum_{1=i_1<i_2<\cdots<i_m}^{\infty} e^{-\beta\sum_{k=1}^{m} E_{i_k}\left(\frac{L}{2}\right)} \sum_{1=i_1<i_2<\cdots<i_{N-m}}^{\infty} e^{-\beta\sum_{k=1}^{N-m} E_{i_k}\left(\frac{L}{2}\right)}, \tag{A2}$$

with $\beta = \frac{1}{k_B T}$ and $E_{i_k}(\frac{L}{2})$ the i_kth energy level of each room.

In the high-temperature limit Eq. (A1) can be reduced to

$$\lim_{\beta\to 0} f_m = \lim_{k\to\infty} \frac{C_k^m C_k^{N-m}}{C_k^0 C_k^N + C_k^1 C_k^{N-1} + \cdots + C_k^N C_k^0} = \frac{C_N^m}{2^N}, \tag{A3}$$

and also there is $\lim_{\beta\to 0} f_m = \frac{C_N^m}{2^N}$. In the low-temperature limit f_m is given by

$$\lim_{\beta\to\infty} f_m = \lim_{\beta\to\infty} \frac{e^{-\beta\sum_{i=1}^{m} E_i\left(\frac{L}{2}\right)} e^{-\beta\sum_{i=1}^{N-m} E_i\left(\frac{L}{2}\right)}}{2e^{-\beta\sum_{i=1}^{\frac{N+1}{2}} E_i\left(\frac{L}{2}\right)} e^{-\beta\sum_{i=1}^{\frac{N-1}{2}} E_i\left(\frac{L}{2}\right)} + o\left[e^{-\beta\sum_{i=1}^{\frac{N+1}{2}} E_i\left(\frac{L}{2}\right)} e^{-\beta\sum_{i=1}^{\frac{N-1}{2}} E_i\left(\frac{L}{2}\right)}\right]} = \begin{cases} \frac{1}{2} & m = \frac{N-1}{2}, \frac{N+1}{2}, \\ 0 & m \neq \frac{N-1}{2}, \frac{N+1}{2}, \end{cases} \tag{A4}$$

when N is odd, when N is even

$$\lim_{\beta\to\infty} f_m = \lim_{\beta\to\infty} \frac{e^{-\beta\sum_{i=1}^{m} E_i\left(\frac{L}{2}\right)} e^{-\beta\sum_{i=1}^{N-m} E_i\left(\frac{L}{2}\right)}}{e^{-2\beta\sum_{i=1}^{\frac{N}{2}} E_i\left(\frac{L}{2}\right)} + o\left[e^{-2\beta\sum_{i=1}^{\frac{N}{2}} E_i\left(\frac{L}{2}\right)}\right]} = \begin{cases} 1 & m = \frac{N}{2}, \\ 0 & m \neq \frac{N}{2}. \end{cases} \tag{A5}$$

In the high-temperature limit we rewrite Eq. (10) as

$$\lim_{\beta\to 0} \frac{\sum_{n=1}^{\infty} n^2 \exp\left[-\beta c\left(\frac{n}{L-l_{\rm eq}^m}\right)^2\right] \sum_{n=1}^{\infty} \exp\left[-\beta c\left(\frac{n}{l_{\rm eq}^m}\right)^2\right]}{\sum_{n=1}{}^{\infty} n^2 \exp\left[-\beta c\left(\frac{n}{l_{\rm eq}^m}\right)^2\right] \sum_{n=1}{}^{\infty} \exp\left[-\beta c\left(\frac{n}{L-l_{\rm eq}^m}\right)^2\right]} = \frac{m}{N-m}\left(\frac{L-l_{\rm eq}^m}{l_{\rm eq}^m}\right)^3, \tag{A6}$$

with $c = \frac{\pi^2\hbar^2}{2M}$. From Ref. [16], we have

$$\lim_{\beta\to 0} \frac{\sum_{n=1}{}^{\infty} \exp\left[-\beta c\left(\frac{n}{l_{\rm eq}^m}\right)^2\right]}{\sum_{n=1}^{\infty} \exp\left[-\beta c\left(\frac{n}{L-l_{\rm eq}^m}\right)^2\right]} = \lim_{\beta\to 0} \frac{\sqrt{\frac{\pi}{4\beta c}} l_{\rm eq}^m - \frac{1}{2}}{\sqrt{\frac{\pi}{4\beta c}}\left(L - l_{\rm eq}^m\right) - \frac{1}{2}} = \frac{l_{\rm eq}^m}{L - l_{\rm eq}^m}, \tag{A7}$$

thus Eq. (A6) can be further rewritten as

$$\frac{m}{N-m}\left(\frac{L-l_{\rm eq}^m}{l_{\rm eq}^m}\right)^4 = \lim_{\beta\to 0} \frac{\sum_{n=1}^{\infty} n^2 \exp\left[-\beta c\left(\frac{n}{L-l_{\rm eq}^m}\right)^2\right]}{\sum_{n=1}^{\infty} n^2 \exp\left[-\beta c\left(\frac{n}{l_{\rm eq}^m}\right)^2\right]} = \left(\frac{L-l_{\rm eq}^m}{l_{\rm eq}^m}\right)^2 \lim_{\beta\to 0} \frac{\sum_{n=1}^{\infty} \exp\left[-\beta c\left(\frac{n}{L-l_{\rm eq}^m}\right)^2\right]}{\sum_{n=1}^{\infty} \exp\left[-\beta c\left(\frac{n}{l_{\rm eq}^m}\right)^2\right]} = \left(\frac{L-l_{\rm eq}^m}{l_{\rm eq}^m}\right)^3, \tag{A8}$$

and from this, we obtain for the high-temperature limit

$$\frac{l_{\mathrm{eq}}^{m}}{L-l_{\mathrm{eq}}^{m}}=\frac{m}{N-m}. \tag{A9}$$

In the low-temperature limit Eq. (10) becomes

$$\frac{\sum_{i=1}^{m} i^{2}}{\left(l_{\mathrm{eq}}^{m}\right)^{3}}=\frac{\sum_{i=1}^{N-m} i^{2}}{\left(L-l_{\mathrm{eq}}^{m}\right)^{3}} \tag{A10}$$

for fermionic SZE, so the equilibrium position is fixed by

$$\frac{l_{\mathrm{eq}}^{m}}{L-l_{\mathrm{eq}}^{m}}=\left[\frac{m(m+1)(2m+1)}{(N-m)(N-m+1)(2N-2m+1)}\right]^{\frac{1}{3}}, \tag{A11}$$

when $T\rightarrow 0$

By definition f_m^* is expressed as

$$f_{m}^{*}=\frac{Z_{m}\left(l_{\mathrm{eq}}^{m}\right)}{\sum_{m=0}^{N} Z_{m}\left(l_{\mathrm{eq}}^{m}\right)}, \tag{A12}$$

with

$$Z_{m}\left(l_{\mathrm{eq}}^{m}\right)=\sum_{1=i_{1}<i_{2}<\cdots<i_{m}}^{\infty} e^{-\beta \sum_{k=1}^{m} E_{i_{k}}\left(l_{\mathrm{eq}}^{m}\right)} \sum_{1=i_{1}<i_{2}<\cdots<i_{N-m}}^{\infty} e^{-\beta \sum_{k=1}^{N-m} E_{i_{k}}\left(L-l_{\mathrm{eq}}^{m}\right)}. \tag{A13}$$

In the high-temperature limit f_m^* reaches

$$\lim_{\beta\rightarrow 0} f_{m}^{*}=\lim_{k\rightarrow\infty}\frac{C_{mk}^{m} C_{(N-m)k}^{N-m}}{C_{mk}^{0} C_{(N-m)k}^{N}+C_{mk}^{1} C_{(N-m)k}^{N-1}+\cdots+C_{mk}^{N} C_{(N-m)k}^{0}}=C_{N}^{m}\left(\frac{m}{N}\right)^{m}\left(1-\frac{m}{N}\right)^{N-m}, \tag{A14}$$

while in the low-temperature limit only $f_{\frac{N-1}{2}}^*$ needs calculation. We will first show that the allocation pattern $f_{\frac{N-1}{2}}^*$ corresponds to the case that exhibits the lowest energy of all. To prove this we simply have to compare $E_{\frac{N-1}{2}}(l_{\mathrm{eq}}^{\frac{N-1}{2}})$ with $E_{\frac{N+3}{2}}(l_{\mathrm{eq}}^{\frac{N+1}{2}})$. We observe that

$$E_{\frac{N-1}{2}}\left(l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)/E_{\frac{N+3}{2}}\left(l_{\mathrm{eq}}^{\frac{N+1}{2}}\right)=\left(\frac{N-1}{N+3}\right)^{2}\left[\frac{(N+2)(N+3)}{N(N-1)}\right]^{\frac{2}{3}}<1 \tag{A15}$$

holds for all N. Then $f_{\frac{N-1}{2}}^*$ is given as

$$\lim_{\beta\rightarrow\infty} f_{\frac{N-1}{2}}^{*}=\lim_{\beta\rightarrow\infty}\frac{e^{-\beta\sum_{i=1}^{\frac{N-1}{2}} E_{i}\left(l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)} e^{-\beta\sum_{i=1}^{\frac{N+1}{2}} E_{i}\left(L-l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)}}{e^{-\beta\sum_{i=1}^{\frac{N-1}{2}} E_{i}\left(l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)} e^{-\beta\sum_{i=1}^{\frac{N+1}{2}} E_{i}\left(L-l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)}+o\left[e^{-\beta\sum_{i=1}^{\frac{N-1}{2}} E_{i}\left(l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)} e^{-\beta\sum_{i=1}^{\frac{N+1}{2}} E_{i}\left(L-l_{\mathrm{eq}}^{\frac{N-1}{2}}\right)}\right]}=1. \tag{A16}$$

Using the above results the total work of the N-fermion SZE is obtained as

$$W_{\mathrm{tot}}^{NF}(T\rightarrow\infty)=Nk_{B}T\left[\ln 2-\frac{1}{2^{N}}\sum_{m=0}^{N} C_{N}^{m} H(m)\right]=\begin{cases} Nk_{B}T\ln 2-\frac{Nk_{B}T}{2^{N-1}}\sum_{m=1}^{\frac{N-1}{2}} C_{N}^{m} H(m), & N \text{ is odd}\\ Nk_{B}T\left(1-\frac{C_{N}^{\frac{N}{2}}}{2^{N}}\right)\ln 2-\frac{Nk_{B}T}{2^{N-1}}\sum_{m=1}^{\frac{N}{2}-1} C_{N}^{m} H(m), & N \text{ is even}\end{cases} \tag{A17}$$

$$W_{\mathrm{tot}}^{NF}(T\rightarrow 0)=\begin{cases}-k_{B}T\left(\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}\right)=k_{B}T\ln 2, & N \text{ is odd}\\ -k_{B}T\ln 1=0 & N \text{ is even.}\end{cases} \tag{A18}$$

APPENDIX B: THE BOSONIC CASE

Next we shall discuss the bosonic case. The probability f_m still obeys Eq. (A1), where $Z_m(\frac{L}{2})=Z_{\mathrm{left}}\cdot Z_{\mathrm{right}}$ with

$$Z_{\mathrm{left}}=\sum_{1=i_{1}<i_{2}<\cdots<i_{m}}^{\infty} e^{-\beta\sum_{k=1}^{m} E_{i_{k}}\left(\frac{L}{2}\right)}+\sum_{1=i\neq i_{1}<\cdots<i_{m-1}}^{\infty} e^{-\beta\sum_{k=1}^{m-1}\left[2E_{i}\left(\frac{L}{2}\right)+E_{i_{k}}\left(\frac{L}{2}\right)\right]}+\cdots+\sum_{i=1}^{\infty} e^{-m\beta E_{i}\left(\frac{L}{2}\right)} \tag{B1}$$

and

$$Z_{\text{right}} = \sum_{1=i_1<i_2<\cdots<i_{N-m}}^{\infty} e^{-\beta\sum_{k=1}^{N-m} E_{i_k}(\frac{L}{2})} + \sum_{1=i\neq i_1<\cdots<i_{N-m-1}}^{\infty} e^{-\beta\sum_{k=1}^{N-m-1}[2E_i(\frac{L}{2})+E_{i_k}(\frac{L}{2})]} + \cdots + \sum_{i=1}^{\infty} e^{-(N-m)\beta E_i(\frac{L}{2})}. \tag{B2}$$

In the high-temperature limit, f_m is obtained as

$$\lim_{\beta\to 0} f_m = \lim_{k\to\infty} \frac{C_k^m C_k^{N-m} + P(k^{N-1})}{C_k^0 C_k^N + C_k^1 C_k^{N-1} + \cdots + C_k^N C_k^0 + Q(k^{N-1})} = \frac{C_N^m}{2^N}, \tag{B3}$$

where $P(k^{N-1})$ and $Q(k^{N-1})$ are k's polynomials of degree $N-1$. Note that Eq. (A6) also applies to the bosonic case, and so does Eq. (A9) consequently. So f_m^* can be evaluated as

$$\lim_{\beta\to 0} f_m^* = \lim_{k\to\infty} \frac{C_{mk}^m C_{(N-m)k}^{N-m} + R(k^{N-1})}{C_{mk}^0 C_{(N-m)k}^N + C_{mk}^1 C_{(N-m)k}^{N-1} + \cdots + C_{mk}^N C_{(N-m)k}^0 + S(k^{N-1})} = C_N^m \left(\frac{m}{N}\right)^m \left(1-\frac{m}{N}\right)^{N-m}. \tag{B4}$$

In the low-temperature limit f_m is given by

$$\lim_{\beta\to\infty} f_m = \lim_{\beta\to\infty} \frac{e^{-N\beta E_1(\frac{L}{2})} + o\big[e^{-N\beta E_1(\frac{L}{2})}\big]}{(N+1)e^{-N\beta E_1(\frac{L}{2})} + o\big[e^{-N\beta E_1(\frac{L}{2})}\big]} = \frac{1}{N+1} \tag{B5}$$

while f_m^* is given by

$$\lim_{\beta\to\infty} f_m^* = \lim_{\beta\to\infty} \frac{e^{-m\beta E_1(l_{\text{eq}}^m)} e^{-(N-m)\beta E_1(L-l_{\text{eq}}^m)} + o\left[e^{-m\beta E_1(l_{\text{eq}}^m)} e^{-(N-m)\beta E_1(L-l_{\text{eq}}^m)}\right]}{e^{-N\beta E_1(L-l_{\text{eq}}^m)} + o\left[e^{-N\beta E_1(L-l_{\text{eq}}^m)}\right]} \sim e^{-m\beta[E_1(l_{\text{eq}}^m)-E_1(L-l_{\text{eq}}^m)]} \tag{B6}$$

when $0 < m < N-m$,

$$\lim_{\beta\to\infty} f_m^* = \lim_{\beta\to\infty} \frac{e^{-N\beta E_1(\frac{L}{2})} + o\big[e^{-N\beta E_1(\frac{L}{2})}\big]}{(N+1)e^{-N\beta E_1(\frac{L}{2})} + o\big[e^{-N\beta E_1(\frac{L}{2})}\big]} = \frac{1}{N+1} \tag{B7}$$

when $m = \frac{N}{2}$ and $f_0^* = f_N^* = 1$. Finally, we obtain the total work of the bosonic SZE as

$$W_{\text{tot}}^{NB}(T\to\infty) = \begin{cases} Nk_BT\ln 2 - \frac{Nk_BT}{2^{N-1}}\sum_{m=1}^{\frac{N-1}{2}} C_N^m H(m), & N \text{ is odd} \\ Nk_BT\big(1-\frac{C_N^{\frac{N}{2}}}{2^N}\big)\ln 2 - \frac{Nk_BT}{2^{N-1}}\sum_{m=1}^{\frac{N}{2}-1} C_N^m H(m), & N \text{ is even,} \end{cases} \tag{B8}$$

$$W_{\text{tot}}^{NB}(T\to 0) = \begin{cases} k_BT\ln(N+1) - \frac{2}{N+1}\sum_{m=1}^{\frac{N-1}{2}} m\big[E_1\big(l_{\text{eq}}^m\big) - E_1\big(L-l_{\text{eq}}^m\big)\big], & N \text{ is odd} \\ \frac{N}{N+1}k_BT\ln(N+1) - \frac{2}{N+1}\sum_{m=1}^{\frac{N}{2}-1} m\big[E_1\big(l_{\text{eq}}^m\big) - E_1\big(L-l_{\text{eq}}^m\big)\big] & N \text{ is even.} \end{cases} \tag{B9}$$

[1] *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*, edited by H. S. Leff and A. F. Rex (Institute of Physics, Bristol, England, 2003).
[2] L. Szilard, Z. Phys. **53**, 840 (1929).
[3] L. Brillouin, J. Appl. Phys. **22**, 334 (1951).
[4] W. H. Zurek, in *Frontiers of Nonequilibrium Statistical Physics*, edited by G. T. Moore and M. O. Scully (Plenum Press, New York, 1984), pp. 151–161.
[5] C. H. Bennett, Scientific American 257(5):108-116 (November, 1987).
[6] R. Landauer, in *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*, H. S. Leff, A. F. Rex (Institute of Physics Publishing, Bristol, England, 2003), p. 148.
[7] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Nature Phys. **6**, 988 (2010).
[8] H. T. Quan, Yu-xi Liu, C. P. Sun, and Franco Nori, Phys. Rev. E **76**, 031105 (2007).
[9] S. Vaikuntanathan and C. Jarzynski, Phys. Rev. E **83**, 061120 (2011).
[10] H. T. Quan, Y. D. Wang, Y. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. **97**, 180402 (2006).
[11] T. Sagawa and M. Ueda, Phys. Rev. Lett. **100**, 080403 (2008).
[12] K. Jacobs, Phys. Rev. A **80**, 012322 (2009).
[13] J. M. Horowitz and J. M. R. Parrondo, Europhys. Lett. **95**, 10005 (2011).
[14] J. M. R. Parrondo and J. M. Horowitz, Physics **4**, 13 (2011).
[15] K. H. Kim and S. W. Kim, Phys. Rev. E **84**, 012101 (2011).
[16] H. Dong, D. Z. Xu, C. Y. Cai, and C. P. Sun, Phys. Rev. E **83**, 061108 (2011).
[17] M. O. Scully, Y. Rostovtsev, Z.-E. Sariyanni, M. S. Zubairy, Physica E **29**, 29 (2005).
[18] Y. Rostovtsev, Z.-E. Sariyanni, M. S. Zubairy, and M. O. Scully, Physica E **29**, 40 (2005).
[19] Z.-E. Sariyanni, Y. Rostovtsev, M. S. Zubairy, and M. O. Scully, Physica E **29**, 47 (2005).
[20] S. W. Kim, T. Sagawa, S. De Liberato, and M. Ueda, Phys. Rev. Lett. **106**, 070401 (2011).
[21] K. H. Kim and S. W. Kim, e-print arXiv:1108.3644v2.
[22] T. Sagawa and M. Ueda, Phys. Rev. Lett. **102**, 250602 (2009).