Mean-field calculation of critical parameters and log-periodic characterization of an aperiodic-modulated model

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We employ a mean-field approximation to study the Ising model with aperiodic modulation of its interactions in one spatial direction. Two different values for the exchange constant, J_A and J_B, are present, according to the Fibonacci sequence. We calculate the pseudocritical temperatures for finite systems and extrapolate them to the thermodynamic limit. We explicitly obtain the exponents β, δ, and γ and, from the usual scaling relations for anisotropic models at the upper critical dimension (assumed to be 4 for the model we treat), we calculate α, ν, $\nu_\|$, η, and $\eta_\|$. Within the framework of a renormalization-group approach, the Fibonacci sequence is a marginal one and we obtain exponents that depend on the ratio $r \equiv J_B/J_A$, as expected; however, the scaling relation $\gamma = \beta(\delta - 1)$ is obeyed for all values of r we studied. We characterize some thermodynamic functions as log-periodic functions of their arguments, as expected for aperiodic-modulated models, and obtain precise values for the exponents from this characterization.

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I. INTRODUCTION

Nonuniform systems are interesting and important from both the theoretical and experimental points of view. Experimentally, there are already several techniques of surface growth [1–3] that let one control the layout of the layers in order to follow, for example, an aperiodic sequence. Moreover, many theoretical issues may be raised concerning the behavior of systems with random disorder or aperiodic modulations of their interactions; it is the last case that concerns us in this work. More specifically, our interest is to calculate the critical parameters of the Ising model within a mean-field framework and characterize the log-periodic behavior of several thermodynamic quantities.

The interactions of the model we treat can assume one of two different values and are ordered according to the Fibonacci aperiodic sequence. For models that have a continuous transition in its uniform version, the influence of aperiodic modulations on their critical behavior is determined by the Harris-Luck criterion [4] (which seems to hold true for models with a first-order transition as well [5]). According to this criterion, the Fibonacci sequence is a marginal one; several results show that a marginal perturbation leads to a dependence of the critical exponents on the ratio between the two different interactions [6–8]. Using the simplest version of a mean-field approximation, we confirm these results and expand them to include other critical exponents, in order to test scaling relations, and characterize log-periodic oscillations.

The rest of this work is organized as follows. In the following section we present several properties of aperiodic sequences, define the model we treat, and outline the mean-field approximation we use. Our results are shown and discussed in Sec. III. In Sec. IV we summarize our findings.

II. APERIODIC SEQUENCES AND MEAN-FIELD APPROXIMATION

Aperiodic sequences may be used, for example, to model quasicrystals [9]: interactions vary according to the order embodied in the sequences. These are built from substitution rules in such a way that no subset of the sequence is ever repeated. In our case, we define an Ising model on a hypercubic lattice, of coordination number z, given by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j, \tag{1}$$

such that the sum is over nearest-neighbor pairs on the lattice, J_{ij} is the exchange constant between spins S_i and S_j, which can assume the values J_A and J_B in a particular spatial direction according to the respective letter in the aperiodic sequence, and $S_i = \pm 1\ \forall i$.

In this work we are particularly interested in the Fibonacci sequence, which is obtained from the substitution rules

$$A \to s(A) = AB, \quad B \to s(B) = A. \tag{2}$$

This means that from one stage of the construction of the aperiodic sequence to the next, all A are replaced by AB and all B are replaced by A. Starting with the letter A, the first stages of this sequence are $A \to AB \to ABA \to ABAAB \to ABAABABA$. This last finite sequence corresponds to the following sequence of interaction constants: $J_A J_B J_A J_A J_B J_A J_B J_A$. In one of the spatial dimensions of the hypercubic lattice (horizontal, say) the exchange constants follow this sequence, while in the remaining perpendicular hypersurface all interactions assume the same value, which is the same as for the succeeding horizontal bonds. An example of a lattice constructed this way, in two dimensions, is depicted in Fig. 1.

One of the interesting theoretical questions one may pose is about the influence of aperiodic modulations on the critical behavior of the model when compared to its uniform counterpart. For the case of a continuous transition on the uniform model, the Harris-Luck criterion determines whether

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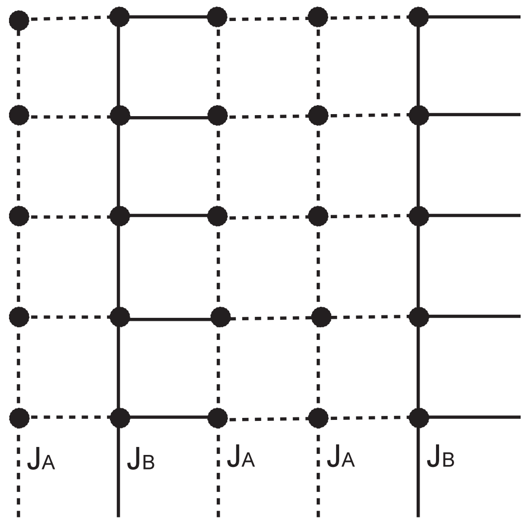

FIG. 1. Example of a lattice with an aperiodic modulation given by the Fibonacci sequence. In the horizontal direction the exchange interactions follow this sequence, while in the vertical planes they assume the same value, equal to the one in the following horizontal bonds. Dashed (solid) lines represent J_A (J_B) interactions.

or not the introduction of a given aperiodic modulation changes the universality class [4]. This change is determined by the crossover exponent Φ, given by

$$\Phi = 1 + d_a \nu(\omega - 1), \tag{3}$$

where ω is the exponent describing the behavior of geometrical fluctuations of the sequence (see below), d_a is the number of dimensions upon which the aperiodic sequence acts ($d_a = 1$ in our case), and ν is the correlation-length critical exponent of the uniform model. When $\Phi > 0$ the introduction of the aperiodic sequence changes the critical exponents from the values assumed for the uniform model (the sequence is said to be relevant in this case) and when $\Phi < 0$ the critical behavior of the aperiodic model is the same as for the uniform one (an irrelevant sequence). For $\Phi = 0$, the sequence is marginal and previous results show that the critical exponents are nonuniversal: they depend on the ratio $r \equiv J_B/J_A$ [6]. In the mean-field framework, $\nu = 1/2$ for the uniform Ising model and the crossover exponent reduces to [see Eq. (3)]

$$\Phi = \tfrac{1}{2}(1 + \omega). \tag{4}$$

Therefore, for $\omega = -1$ the sequence is marginal, which is the case for the Fibonacci sequence, as we will shortly see. This quantity and others properties of two-letter sequences are obtained from their substitution matrix $\mathcal{M}$, which is defined as

$$\mathcal{M} = \begin{pmatrix} n_i^{s(i)} & n_i^{s(j)} \\ n_j^{s(i)} & n_j^{s(j)} \end{pmatrix}, \tag{5}$$

where $n_i^{s(j)}$ is the number of i that are generated by applying the rule $s(j)$. Several features of the sequences are determined by the eigenvalues of $\mathcal{M}$. The greatest eigenvalue λ_1 determines the rate of growth of the total number of letters $\mathcal{N}$ such that $\mathcal{N} \sim \lambda_1^n$, $n \gg 1$, where n is the number of iterations in the construction of the sequence. The second greatest eigenvalue λ_2 determines the *wandering exponent* ω [Eqs. (3) and (4)] through

$$\omega = \frac{\ln|\lambda_2|}{\ln \lambda_1} \tag{6}$$

such that the fluctuation in one of the letters g is given by [8]

$$g \sim \mathcal{N}^{\omega}. \tag{7}$$

For the Fibonacci sequence

$$\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{8}$$

$\lambda_1 = (1+\sqrt{5})/2$, and $\lambda_2 = -\lambda_1^{-1}$ such that $\omega = -1$, as anticipated. Therefore, the aperiodic modulation obtained with the Fibonacci sequence, within the mean-field approximation applied to the Ising model, is a marginal one when the sequence acts on one of the spatial directions.

We study the present model Eq. (1) within the simplest mean-field approximation. It may be obtained either from the Bogoliubov inequality [10] with a single-spin trial Hamiltonian or, in a less rigorous framework, from substituting the magnetization m_i for the spin S_i. Due to the aperiodic modulation, the values of m_i vary along the direction upon which the aperiodic sequence acts (although they are the same for a given hyperplane perpendicular to this direction). The system of equations one has to solve is

$$\begin{aligned} m_i &= \tanh[K_{i-1}m_{i-1} + (z-2)K_i m_i + K_i m_{i+1} + h], \\ i &= 1,\ldots,N, \end{aligned} \tag{9}$$

where $K_l \equiv \beta J_l$, $h \equiv \beta H$, $\beta \equiv 1/k_B T$, H is a uniform magnetic field, k_B is Boltzmann constant, T is the temperature, and N is the number of hyperplanes on the system (or, equivalently, the size of the aperiodic sequence).

III. RESULTS

A. Critical temperatures

The first task is to obtain the critical temperature T_c; our strategy is to calculate pseudocritical temperatures for finite systems and extrapolate the data to the thermodynamic limit. Since the transition is expected to be a continuous one and our goal is to calculate T_c, we can expand Eq. (9) with $H = 0$ up to first order on the magnetizations:

$$\mathbb{K} \cdot \vec{m} = 0, \tag{10}$$

where

$$\mathbb{K} = \begin{pmatrix} \tilde{K}_1 & K_1 & 0 & 0 & \cdots & 0 \\ K_1 & \tilde{K}_2 & K_2 & 0 & \cdots & 0 \\ 0 & K_2 & \tilde{K}_3 & K_3 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & K_{N-1} & \tilde{K}_N \end{pmatrix} \tag{11}$$

and

$$\vec{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_N \end{pmatrix}, \tag{12}$$

with $\tilde{K}_i \equiv (z-2)K_i - 1$ and free boundary conditions $m_{N+1} = m_0 = 0$. The interaction parameters $K_1, K_2, \ldots, K_N$ assume the values K_A or K_B according to the respective letter on the Fibonacci sequence. Since we do not expect the critical exponents to depend on z, we have worked only with $z = 6$ to obtain the critical parameters.

For temperatures greater than the pseudocritical one, the only solution to this system is $\vec{m} = 0$. Thus the matrix $\mathbb{K}$ has an inverse, i.e., $\det(\mathbb{K}) \neq 0$ for this region of temperatures. Therefore, coming from above, the first temperature such that $\det(\mathbb{K}) = 0$ is the pseudocritical temperature. This procedure is applied to systems with different linear sizes L (corresponding to the length of the aperiodic sequence, N) and extrapolated to $L \to \infty$. One has to be sure that the first temperature such that $\det(\mathbb{K}) = 0$ is actually obtained, since many temperatures satisfy this criterion below the first one and they tend to accumulate close to the pseudocritical temperature as L increases.

In order to extrapolate our results to the thermodynamic limit, we have used the so-called Bulirsch-Stoer (BST) extrapolation [11] in two different ways (see below). The errors of our evaluations are obtained as usual for this method of extrapolation [11].

Since we expect log-periodic oscillations on models with aperiodic-modulated interactions, the pseudocritical temperatures do not converge monotonically to the thermodynamic values: on top of an apparent overall convergence, there are oscillations on the values for finite L. Therefore, we have also applied the BST procedure to every other value of the pseudocritical temperatures. Both procedures lead to the same values in the thermodynamic limit. In Table I we show our results for the critical temperatures for several values of the ratio r, extrapolated from pseudocritical temperatures obtained for L up to 121 393 for $r = 0.5$ and 1.3 and up to 196 418 for the other values of r. Note that we show ten decimal figures for $r \neq 1$, which is certainly enough to obtain precise values for the critical exponents. For $r = 1$ we show all figures we are able to obtain since we can compare it to the expected value within the mean-field approximation: there is agreement up to 15 decimal figures.

In Fig. 2 we compare our values for T_c with those obtained in Ref. [6]. The quantity T_c^0 is the critical temperature for a uniform model with the same *mean* value $\bar{J}$ for the interaction constant J for a given r. More precisely, $\bar{J} \equiv J_A(p_A + r p_B)$, where p_A and p_B are the fractions of letters A and B, respectively, on the infinite aperiodic sequence. These fractions are obtained from the entries of the eigenvector corresponding to the greatest eigenvalue of the substitution matrix. We notice the agreement is quite good; the apparent difference for some regions of r comes from the fact that we have few data points and have made an interpolation of our data.

TABLE I. Extrapolated critical temperatures for several values of the ratio $r \equiv J_B/J_A$. For $r = 1$ (the uniform model) we obtain, within error bars, the exact value $k_B T_c/J = z = 6$.

r	$k_B T_c/J_A$
0.5	5.2939768858
0.7	5.4801586902
1.0	6.00000000000038(64)
1.3	6.8300746634
1.5	7.4992699398

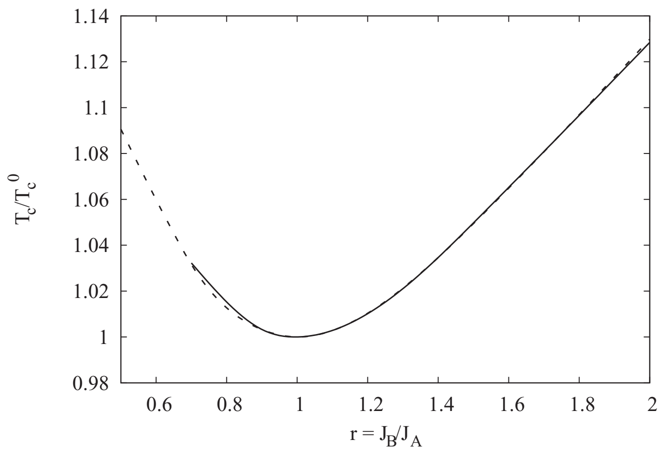

FIG. 2. Comparison between our values for T_c (dashed line) and those obtained in Ref. [6] (solid line). We define the quantity T_c^0 in the text.

B. Magnetization

Having calculated the critical temperatures, we can now obtain, from the original system of equations (9), the magnetization for each plane. The goal is to solve this system for m_i for different values of the reduced temperature $t \equiv (T - T_c)/T_c$ and the reduced magnetic field h $(\equiv \beta H)$. In order to accomplish this we test three procedures: the first one is based on the Newton method [12], the second one uses the secant method [12], and the third one is the so-called fixed-point method [13]. We analyze the convergence time, for large systems and for small values of the reduced temperature, and the accuracy (with respect to known results for small lattices). The first method is the less precise, the secant method is the most efficient for small values of t, and the fixed-point method is the most efficient for large lattices. We choose the last one to be able to go to larger systems.

After a predetermined accuracy is achieved, within the fixed-point method, we stop the iterations and calculate the mean magnetization as

$$m(L) = \frac{1}{L}\sum_{i=1}^{L} m_i. \quad (13)$$

As discussed elsewhere [6,8,14], this quantity may be experimentally accessible. We now have to extrapolate the values obtained for $L \to \infty$. As expected for aperiodic-modulated models, oscillations occur as depicted in Fig. 3; in order to obtain the value of $m \equiv m\,(L \to \infty)$, we use the extrapolation procedure introduced in Ref. [8]. It simply takes the two last pairs of values for $m(L)$ and makes a linear extrapolation with each of them. The values m_1 and m_2 (see Fig. 3), obtained for $1/L = 0$, are then the limits of our estimate for m in the thermodynamic limit. We then take $m = (m_1 + m_2)/2$ and the error $\Delta m = |m_1 - m_2|/2$. From Fig. 3 we clearly see that this procedure gives an interval for the magnetization

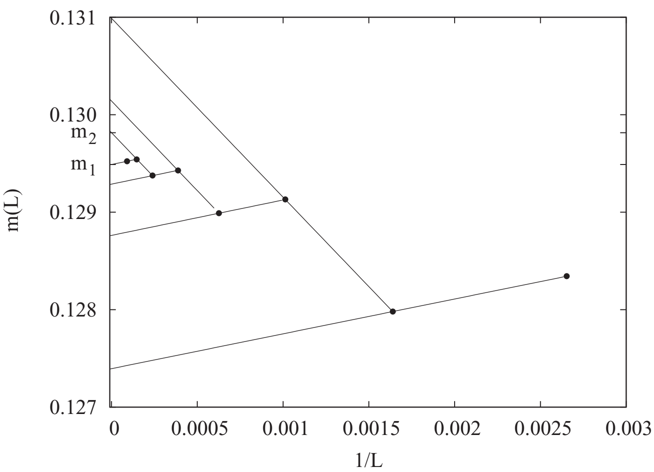

FIG. 3. Typical behavior for the magnetization, for fixed reduced temperature t or fixed reduced magnetic field h, as a function of the linear size of the lattice L. Note the oscillatory convergence to the thermodynamic limit, as expected for aperiodic-modulated models.

that contains the true value in the thermodynamic limit, although it overestimates the error. The same procedure was employed to obtain the magnetization for a nonzero magnetic field, which is necessary to calculate the critical exponents δ and γ (see the following section).

C. Critical exponents

1. Critical exponent β

Our first attempt to estimate the critical exponent β is to fit our data, obtained in the thermodynamic limit, as explained in the preceding section, to a log-periodic function

$$m(t) \sim (-t)^{\beta}\mathcal{P}[\log_{10}(-t)], \tag{14}$$

where we assume the following form for the function $\mathcal{P}[\log_{10}(-t)]$:

$$\mathcal{P}[\log_{10}(-t)] \sim \{1 + B\cos[2\pi C\log_{10}(-t) + \tau\phi]\}. \tag{15}$$

Therefore, for the magnetization we obtain

$$m(t) = A(-t)^{\beta}\{1 + B\cos[2\pi C\log_{10}(-t) + \tau\phi]\}, \tag{16}$$

where 2π and $\tau = (\sqrt{5}+1)/2$ are convenient constants for the fitting.

Our results for β, using Eq. (16), are shown in Table II, column 2. The amplitude of the log-periodic term is roughly 5×10^{-3} for all values of r except $r = 1$ (this term is not present). Two results are worth noting: the exponent for $r = 1$ (the uniform model) is known to be $1/2$; our result, although near this value, is not consistent with it. Also, the χ^2 per degrees of freedom is much greater than 1. This shows that our fitting is not a good one for the aperiodic models.

To improve our estimates for β, we follow another procedure, which consists in calculating the so-called logarithmic derivative, namely,

$$\mathcal{L}(t) \simeq \frac{d\log_{10}[m(t)]}{d\log_{10}(-t)} \sim \beta + \tilde{B}\cos[2\pi C\log_{10}(-t) + \tilde{\phi}], \tag{17}$$

TABLE II. Magnetization critical exponent β as a function of the ratio r, obtained by (a) fitting the data to Eq. (16), (b) using the logarithmic derivative, and (c) fitting the data to Eq. (16), but restricting the interval in $\log_{10}(-t)$. The numbers in parentheses are uncertainties in the last digit.

	β		
r	(a)	(b)	(c)
0.5	0.56872(4)	0.5683(2)	0.56824(4)
0.7	0.5439(3)	0.54558(2)	0.545553(6)
1.0	0.489(1)	0.49989(4)	0.49984(2)
1.3	0.5270(4)	0.53033(5)	0.53041(2)
1.5	0.5465(3)	0.54884(4)	0.54892(2)

where it is assumed that $B \ll 1$ in Eq. (15). This derivative is obtained numerically and the data are fitted to Eq. (17). Examples of the type of behavior we obtain are depicted in Figs. 4(a) and 4(b) for $r = 0.7$ and 1.5, respectively. There is a clear oscillation, as predicted by Eq. (17); the mean value of the fitted curve is the exponent β. Note, however, that for values of $\log_{10}(-t)$ close to -1 the behavior departs from the one predicted. Therefore, this interval is not in the scaling region and should not be used to study the critical behavior. Our fitting is then obtained with the data points in the proper interval.

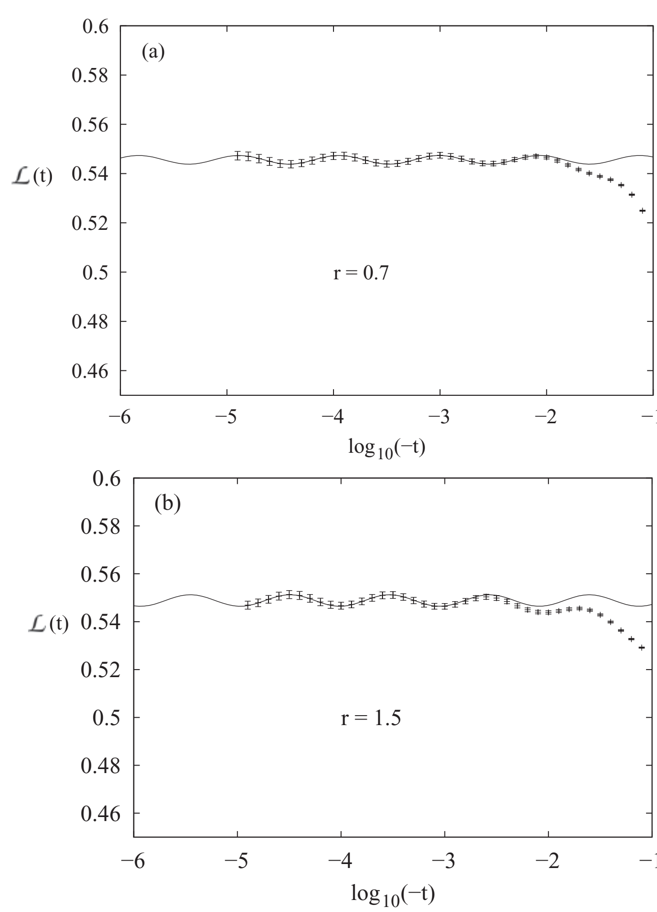

FIG. 4. Logarithmic derivative of the magnetization for (a) $r = 0.7$ and (b) $r = 1.5$. The solid lines are fittings using Eq. (17), while the points are our numerical data.

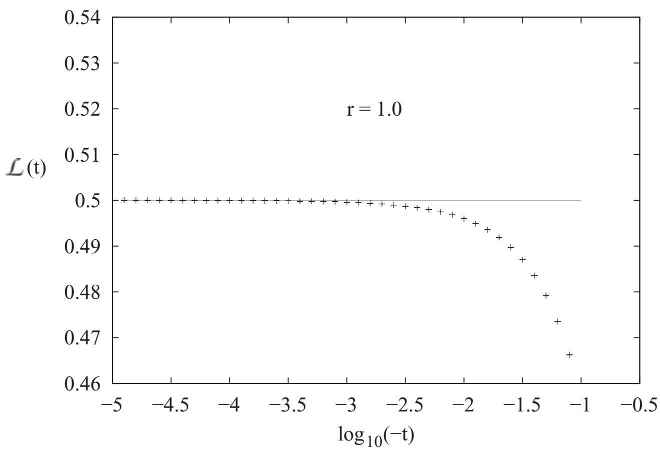

FIG. 5. Logarithmic derivative of the magnetization for $r = 1$. The solid lines are fittings using Eq. (17), while the points are our numerical data. Error bars are approximately the same size as the points.

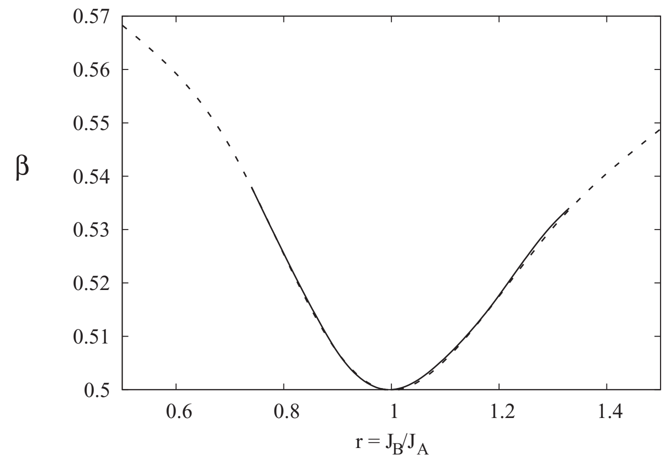

FIG. 6. Exponent β as a function of r from the data obtained in this work (solid line) and from the results of Ref. [6] (dashed line).

For comparison, we show the graph of the log-derivative for $r = 1$ in Fig. 5: no oscillation is present, but the deviation from the expected behavior (in this case, a horizontal line) is obtained for $-t$ big enough. Results for β with this procedure are shown in Table II, column 3. Although the value for $r = 1$ does not include the known value for the mean-field approximation, it is closer to the expected value than for the previous procedure and correct up to the third decimal place. Another improvement with respect to the previous procedure is that the values obtained for χ^2 are orders of magnitude smaller: they range from 10^{-3} to 10^{-1}. The amplitude of the log-periodic term is small, as expected (the maximum value for the values of r we studied is approximately 10^{-2}, for $r = 0.5$) and increases as we move further away from the uniform case, as expected [15].

As a final check, we make fittings using Eq. (16), but now with a restricted interval of the reduced temperature t. We use the interval in which the log-derivative behavior is well described by the data. In Fig. 4(a), for example, this interval is $-4.9 \leqslant \log_{10}(-t) \leqslant -2.5$. The values so obtained of β are shown in Table II, column 4: although the result for $r = 1$ is closer to the expected value within the mean-field approximation than for the first fitting procedure, it is not better than the second one. Also, χ^2 has decreased a great deal compared to the first procedure, but it is still orders of magnitude greater than for the log-derivative fitting. Therefore, we take as our results for β those in Table II, column 3. Finally, we would like to stress the excellent agreement between our results for this exponent and those in Ref. [6] (see Fig. 6).

2. *Critical exponent δ*

In order to calculate the exponent δ, one has to study the dependence of the magnetization on the external uniform magnetic field h. As a log-periodic dependence is expected, we also make all three fitting procedures described above for this case. Again, the best results are obtained for the second one.

More precisely, we assume the dependence of m on H to be [a sgn(H) term is present in the following equation, but we have omitted it for clarity]

$$m(H) = A|H|^{1/\delta}\{1 + B\cos[2\pi C \log_{10}|H| + \tau\phi]\}. \quad (18)$$

Therefore, the log-derivative is given by (again, taking into account that the amplitude of the log-periodic oscillation is small)

$$\mathcal{L}(H) \simeq \frac{d\log_{10}(m)}{d\log_{10}|H|} = \frac{1}{\delta} + \tilde{B}\cos[2\pi C\log_{10}|H| + \tilde{\phi}]. \quad (19)$$

The typical behavior is depicted in Fig. 7: again, log-periodic oscillations are present and the critical exponent δ is obtained from the previous function Eq. (19).

The critical exponents are shown in Table III. The mean-field value for $r = 1$ is $1/3$; our numerical evaluation agrees with this result up to the fourth decimal place. For the uniform model, as expected, no oscillation is present in the log-derivative. Finally, for the values of r quoted in Table III, $\tilde{B}$ [see Eq. (19)] varies from 10^{-3} to 10^{-4} and increases as we move away from the uniform model. The values of χ^2 (not shown) vary from 10^{-3} to 10^{-5} for the aperiodic models and equals 10^{-9} for the uniform case. These results are evidence of good fittings.

3. *Critical exponent γ*

We have calculated the susceptibility $\chi(t)$ using two different methods. First, for each reduced temperature t, we calculate the magnetization for two different (small) magnetic fields and perform a numerical derivative to obtain $\chi(t)$. Alternatively, we can differentiate Eq. (9) with respect to H and obtain a system of equations with $\chi_i(T), i = 1,\ldots,N$, as the variables. Solving for these, we can calculate the susceptibility $\chi(t) \equiv \sum_i \chi_i(T)/N$.

For the first method, we used the first two procedures quoted in Secs. III C 1 and III C 2, namely, fitting the data to the functions

$$\chi(t) = A|t|^{-\gamma}\{1 + B\cos[2\pi C\log_{10}|t| + \tau\phi]\} \quad (20)$$

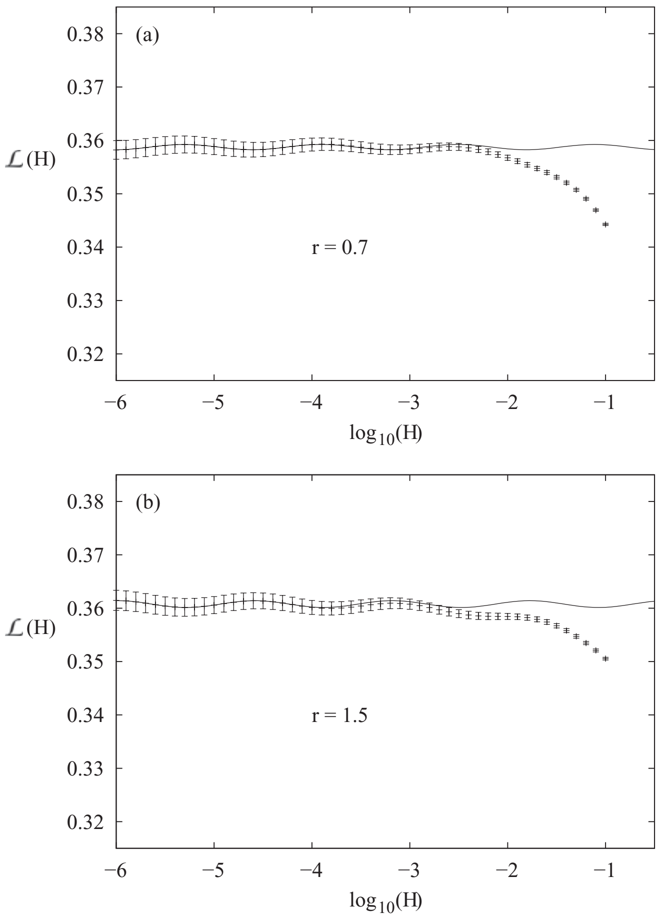

FIG. 7. Field logarithmic derivative of the magnetization for (a) $r = 0.7$ and (b) $r = 1.5$. The solid lines are fittings using Eq. (19), while the points are our numerical data. Error bars are approximately the same size as the points.

and

$$\mathcal{L}(t) = \frac{d \log_{10}[\chi(|t|)]}{d \log_{10}|t|} = -\gamma + \tilde{B}\cos[2\pi C \log_{10}|t| + \tilde{\phi}]. \tag{21}$$

However, contrarily to what happened for the two previous critical exponents, it is not possible to identify a clear log-periodic oscillation for the log-derivative of $\chi(t)$. This may be due to the importance of more than one harmonic in the behavior of this function [15]; we cannot test this hypothesis because we do not have enough data to obtain one period of the log-periodic oscillation.

TABLE III. Critical exponent δ as a function of r for fittings to log-derivative functions Eq. (19). The numbers in parentheses are uncertainties in the last digit.

r	$1/\delta$
0.5	0.37402(2)
0.7	0.358745(3)
1.0	0.33328(1)
1.3	0.34995(2)
1.5	0.360770(5)

TABLE IV. Susceptibility critical exponent as function of r. The term γ stands for the critical exponent calculated using the fitting procedure described in the text; $\gamma_{\rm calc}$ stands for the calculation using the equality between exponents γ, β, and δ; and the last column shows the percentage difference between the two estimates for γ.

r	γ	$\gamma_{\rm calc} = \beta(\delta - 1)$	$\Delta\gamma$ (%)
0.5	0.9557(8)	0.9511(5)	0.5
0.7	0.9812(8)	0.97522(5)	0.6
1.0	1.0010(3)	1.0000(2)	0.1
1.3	0.994(2)	0.9851(2)	0.9
1.5	0.9772(7)	0.9725(1)	0.5

Therefore, for the γ critical exponent we obtain results only from the fitting to a log-periodic function as in Eq. (21). These results, although not as precise as the ones obtained from the log-derivative function, should not differ from the correct values by more than 0.6%, according to the comparison made for the critical exponents β and δ. Our results are shown in Table IV. The mean-field value for the critical exponent of the uniform case is 1; our evaluation is 0.1% off.

We also calculate γ using the usual scaling relation $\gamma = \beta(\delta - 1)$, which still holds true for anisotropic models (see Sec. III C 4), with β and δ taken from the log-derivative fittings. The comparison is in Table IV: note that the discrepancy is 0.9% for the worst case, which confirms our evaluation that the values would not differ by much more than 0.6%.

As mentioned earlier, another possible method to obtain the susceptibility is to perform a field derivative of the system of equations for the magnetization Eq. (9) in order to obtain a system of equations for χ_i. These are given by the solution of this system in the same manner that we did for the magnetization. The results are the same as for the previous method, as expected. In particular, we are not able to characterize the log-periodic oscillations either.

4. *Other critical exponents*

We now turn to the calculation of other critical exponents using the scaling relation for the free energy for anisotropic systems. Due to the presence of the aperiodicity in one dimension, we expect different correlation lengths in the direction of the aperiodic modulation, $\xi_\parallel \sim t^{\nu_\parallel}$, and along the other directions $\xi_\perp \sim t^{\nu}$, with $q \equiv \nu_\parallel/\nu \neq 1$ [16]. Assuming the scaling ansatz for a system in d dimensions (see Ref. [17], where the scaling relation is proposed for two-dimensional models),

$$f_s(t,h,L) = b^{-(d-1+q)} f_s(b^{1/\nu}, b^{y_h}, L/b), \tag{22}$$

where f_s is the singular part of the free energy, b is the rescaling factor, y_h is a scaling exponent, and L is the linear size of the lattice. From Eq. (22) one can show, in the usual way, the following relations between critical exponents [18]:

$$\gamma = \beta(\delta - 1), \quad \alpha + 2\beta + \gamma = 2, \quad \alpha = 2 - \nu(d-1) - \nu_\parallel. \tag{23}$$

Therefore, assuming $\nu = 1/2$ (since the aperiodic sequence we study is a marginal one [6]) and $d = 4$, the exponents α and $\nu_\parallel$ assume the values shown in Table V. Note the good

TABLE V. Critical exponents calculated from scaling relations for anisotropic systems.

r	0.5	0.7	1.0	1.3	1.5
α	−0.0877(9)	−0.06638(9)	0.0002(2)	−0.0458(3)	−0.0701(2)
$\nu_\parallel$	0.5877(9)	0.56638(9)	0.4998(2)	0.5458(3)	0.5701(2)
$\eta_\parallel$	−0.066(4)	−0.0734(4)	0.0004(9)	−0.057(2)	−0.0746(8)

accordance with the mean-field values for the uniform model ($r = 1$) and the expected increase of $\nu_\parallel$ and decrease in α when we move away from $r = 1$.

Assuming a similar scaling form for the two-point correlation function $\Gamma(x,y)$, where x is the distance along the aperiodic direction and y is the distance along the remaining $d-1$ directions,

$$\Gamma(x,y,t) \simeq t^{2\beta}\mathcal{G}(x/|t|^{-\nu_\parallel}, y/|t|^{-\nu}), \qquad (24)$$

one can show that

$$d-2+\eta_\parallel = 2\beta/\nu_\parallel, \quad d-2+\eta = 2\beta/\nu, \qquad (25)$$

where $\Gamma(x,0,0) \sim x^{d-2+\eta_\parallel}$ and $\Gamma(0,y,0) \sim y^{d-2+\eta}$. Therefore, the exponent η assumes the usual mean-field value, namely, $\eta = 0$. The values obtained for the exponent along the aperiodic direction $\eta_\parallel$ are shown in Table V, assuming $d = 4$, as before. As expected, the value for the uniform model is consistent with the known value for the mean-field approximation. However, note that the value for $r = 0.5$ is closer to the uniform results than for $r = 0.7$. Since $\eta_\parallel$ is close to zero and it is obtained from $\nu_\parallel$, which itself is calculated from scaling relations, one expects a higher inaccuracy.

IV. CONCLUSION

We employ a mean-field approximation to treat an Ising model with aperiodic modulation in one spatial direction. The particular aperiodic sequence we use is a marginal one, in the context of the Harris-Luck criterion. We calculate many equilibrium critical exponents, including $\nu_\parallel$ and $\eta_\parallel$, assuming $d = 4$ to be the upper critical dimension of the model and a particular scaling form for the singular part of the free energy per site and for the two-point correlation function, suitable for anisotropic models. As expected, the exponents (with the exception of ν and η) depend on the ratio $r = J_B/J_A$, but obey the usual scaling relations for anisotropic models whenever possible to test these relations. Our results are in accordance with the known values for the mean-field procedure (for the uniform model, $r = 1$) or with previous results for the exponent β and critical temperatures [6].

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