Directed random walks on hierarchical trees with continuous branching: A renormalization group approach

David B. Saakian*

Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China, Yerevan Physics Institute, 2 Alikhanian Brothers St., Yerevan 375036, Armenia, and National Center for Theoretical Sciences: Physics Division, National Taiwan University, Taipei 10617, Taiwan, Republic of China

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We investigate the directed random walk on hierarchic trees. Two cases are investigated: random variables on deterministic trees with a continuous branching, and random variables on the trees constructed through the random branching process. We derive renormalization group (partial differential) equations for the branching models with binomial, Poisson, and compound Poisson distributions of random variables on the links of a tree. These renormalization group equations are a new class of reaction-diffusion equations in one dimension.

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I. INTRODUCTION

Models on hierarchic trees are rather popular in statistical physics [1], and those models with random quenched disorder [2–6] are especially popular objects of research. They are connected with the random energy model (REM). Due to special geometry, these models can be solved through recursive equations [1]. Taking into account random walks (directed polymers) on trees, the recursive equations for these models have been approximated with the Kolmogorov-Petrovsky-Piscounov (KPP) equation.

Statistical physics models with quenched disorder on hierarchic trees with continuous branching have been the subject of extensive studies. The models with continuous branching are especially popular in the financial market literature [7,8] as well as in studies on turbulence. In Ref. [9], hierarchic trees with continuous branching were considered when there are random variables with normal distribution on the tree. Later investigations were performed in Refs. [10–12], in which the models were applied to the problems of disordered systems and string theory. To solve hierarchic models with continuous branching, a renormalization group equation was used in Ref. [9]. This equation was first obtained in Ref. [13] and is similar to the KPP equation; see, Ref. [12] for discussion. In this paper, we will derive a renormalization group equation for the general case of the distribution of random variables on the branches of a hierarchic tree.

We are going to investigate the random directed walks on hierarchic trees described through the partition function:

$$Z = \sum_i \exp[-\beta y_i], \tag{1}$$

where y_i is defined on the end points of the tree and β is the inverse temperature.

The model is defined by considering a tree with branching number q and K levels of hierarchy, therefore there are $q^K \equiv e^L$ end points of the tree. At any level of hierarchy, there are q links from every node, and on every link a random variable ϵ_l is defined. The variable y_i entering the definition of the partition function Eq. (1) is associated with a unique path going from the origin O to the ith tree end point, and it amounts to summing up variables ϵ_l along this path,

$$y_i = \sum_l \epsilon_l. \tag{2}$$

One can calculate the mean free energy $\ln Z$ using the identity [4]

$$\begin{aligned}\langle \ln Z\rangle &= \int_0^\infty \left(\frac{e^{-p}}{p} - \frac{\langle e^{-pZ}\rangle}{p}\right)dp \\ &= \Gamma'(1) + \int_0^\infty \ln p\, d\langle e^{-tZ}\rangle.\end{aligned} \tag{3}$$

Thus we need to calculate

$$g(p,\beta) \equiv G_K(x) = \left\langle \exp\left[-p\sum_i e^{\beta y_i}\right]\right\rangle\Big|_{\epsilon_l}, \quad p = e^{-\beta x} \tag{4}$$

for the average $\langle \exp[-p\sum_i e^{-\beta y_i}]\rangle|_{\epsilon_l}$ over the configurations on the K-level hierarchic tree.

Having the expression for $g(p,\beta)$, we can calculate the moments $\langle Z^{-n}\rangle$:

$$\langle Z^{-n}\rangle = \int_0^\infty dp\, p^{n-1}/\Gamma(n) g(p,\beta). \tag{5}$$

In Ref. [2], an equivalent problem has been considered, and recursive relations have been introduced. Following [5], we define

$$G_0(x) = \exp[-e^{-\beta x}] \tag{6}$$

and other K functions $G_l(x) = \langle \exp[-e^{-\beta x}\sum_i e^{-\beta y_i}]\rangle|_{\epsilon_l}$ for the models on the trees with an l level of hierarchy, $0 \leqslant l \leqslant K$. Then we identify $g(p,\beta) \equiv G_K(-\ln p/\beta)$.

Consider the l-level hierarchic tree. It could be fractured into q trees, each with a hierarchy level $l-1$. The y_i at the end points of the l-level tree can be derived from the y_i of the $(l-1)$ level tree adding a random variable ϵ. A simple consideration gives a recursive relation [2,5]

$$G_l(x) = \left[\int d\epsilon\; G_{l-1}(x+\epsilon)\rho(\epsilon)\right]^q. \tag{7}$$

*saakian@yerphi.am

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Considering recursive relations for $1 \leqslant l \leqslant K$ with the boundary condition in Eq. (6), we can calculate G_K.

Equation (7), derived in Refs. [2–5], is an exact recursive relation. Similar recursive equations have been considered in the research of the real-space renormalization approach to quantum disorder in d-dimensional space; see Eq. (32) in Ref. [14]. The latter is some approximation, while Eq. (7) is an exact relation. There is a more serious difference, i.e., our random variables ϵ have independent distributions for different hierarchy levels l, while at the quantum disorder case [14] there is some correlation of noise at different hierarchy levels. To obtain some analytical estimates, Ref. [14] considered the limit of a large branching number q (K in the notation of [14]), similar to those used in Ref. [5], and a general distribution of random variables was used.

The KPP-like equation has been applied for investigations of the quenched disorder model. In the next section, we will consider the $q \to 1$ limit of Eq. (7), deriving a renormalization group (KPP-like) partial differential equation for general distribution of ϵ.

II. RANDOM WALKS ON A HIERARCHIC TREE

A. Renormalization group equation for hierarchic trees with continuous branching

While Eq. (7) is derived for integer q, we can consider the equation for any positive value of q. Let us consider the model in which the branching number is close to 1. We can now derive an exact differential equation instead of the iteration equation (7). We have $K = 1/\Delta$ levels of branching on our tree with $q = e^{L\Delta v}$, where $L\Delta v \ll 1$. We identify

$$G_l(x) \equiv G(x,v), \quad v = \frac{l}{K}, \quad g(p,\beta) = G\left(-\frac{\ln p}{\beta}\right). \tag{8}$$

To define a distribution for random variable ϵ_l, let us start with some random distribution $\hat{\rho}(x)$. In the next step, we calculate the distribution for the sum of L random variables with such distributions. For our purposes, it is convenient to use the following representation for $\hat{\rho}$:

$$\hat{\rho}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dh \exp[\phi(ih) - ihx]. \tag{9}$$

Then for ϵ, a sum of L random variables x, we get the composed distribution $\rho(L,\epsilon)$ just by multiplying ϕ in the exponent of Eq. (9) by L:

$$\rho(L,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dh \exp[L\phi(ih) - ih\epsilon]. \tag{10}$$

To ensure the probability balance condition, there is a constraint

$$\phi(0) = 0. \tag{11}$$

We consider the Taylor expansion

$$\phi(ik) = \sum_{l\geqslant 1} b_l(ik)^l. \tag{12}$$

According to our notations, $\hat{\rho}(x) \equiv \rho(1,x)$. A concrete case of distribution by Eq. (10) with $\phi(ih) = \ln\cosh(h)$ has been used while considering the diluted REM [15]. Here we will consider another distribution that is popular in turbulence and financial theory.

We take $\rho(L,y_i)$ as a distribution of y_i. The L in the exponent of Eq. (10) gives correct scaling for the $\langle e^{\beta y_i}\rangle$,

$$\langle e^{y_i}\rangle = \exp[L\phi(\beta)], \tag{13}$$

where e^L is the number of end points of the hierarchic tree.

As y_i is a sum of $1/\Delta v$ random variables ϵ on our tree, we define the following distribution for the distribution of random variables on the links $\rho(\epsilon)$:

$$\rho(\epsilon) \equiv \rho(L\Delta v,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dh \exp[\phi(ih)L\Delta v - ih\epsilon]. \tag{14}$$

For $L\Delta v \ll 1$, we expand in the exponent the terms with $b_l, l > 1$ and derive

$$\begin{aligned}
&\int d\epsilon\, G(x+\epsilon,v)\rho(\epsilon) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dh \exp[iLb_1\Delta v - ih\epsilon] \\
&\quad \times \left(1 + Ldv\sum_l b_l(ih)^l\right) G(x+\epsilon,v) \\
&= \int d\epsilon \Bigg[\delta(\epsilon - b_1 L\Delta v) + L\Delta v \sum_{l\geqslant 2} b_l(ih)^l \frac{1}{2\pi} \\
&\quad \times \int_{-\infty}^{\infty} dh \exp(-ih\epsilon)\Bigg] G(x+\epsilon,v).
\end{aligned} \tag{15}$$

We assume a smooth behavior of $G(x,v)$ at the infinity $x \to \pm\infty$, and $\frac{\partial^n G(v,x)}{(\partial x)^n}|_{x=\pm\infty} = 0$ for $n \geqslant 1$.

Integrating by parts, we derive

$$\int d\epsilon\, G(x+\epsilon,v)\rho(\epsilon) = G(x,v) + L\Delta v\phi\left(\frac{\partial}{\partial x}\right)G(x,v). \tag{16}$$

Thus we obtain

$$\begin{aligned}
G(x,v) &\to [G + L\Delta v\, \phi(\partial x)G(x,v)]^{1+L\Delta v} \\
&\approx G(x,v) + \Delta v\, L[\phi(\partial x)G(x,v) \\
&\quad + G(x,v)\ln G(x,v)]
\end{aligned} \tag{17}$$

or

$$\frac{\partial G(x,v)}{\partial v} = \phi(\partial x)G(x,v) + G(x,v)\ln G(x,v), \tag{18}$$

where $0 \leqslant v \leqslant 1$ plays the role of time from the reaction-diffusion equation, and we should solve the equation with the initial distribution

$$G(x,0) = \exp[-e^{-\beta x}]. \tag{19}$$

Due to the proper choice of scaling in the exponent of Eq. (14), there is no L dependence in Eq. (18).

Contrary to the KPP equation for the models in Ref. [2], Eq. (18) is an exact equation. Solving Eq. (18) for the initial distribution by Eq. (19), we can calculate $\rho(Z)$ using the inverse Fourier transformation.

B. Different distributions

Consider different distributions, popular in cascade processes.

1. Normal distribution

Taking

$$\phi(k) = k^2/2, \tag{20}$$

we obtain

$$\frac{\partial G(x,v)}{\partial v} = \frac{\partial^2 G(x,v)}{\partial x^2} + G\ln G. \tag{21}$$

We derived Eq. (21) in Ref. [9] solving the continuous branching model. This equation was derived for the first time in Ref. [13] to investigate nonlinear diffusion.

2. Binomial distribution

We have

$$\hat{\rho}(\epsilon) = \frac{\alpha_2\delta(\epsilon-(1-\alpha_1))}{\alpha_1+\alpha_2} + \frac{\alpha_1\delta(\epsilon-(1+\alpha_2))}{\alpha_1+\alpha_2}. \tag{22}$$

We get for $\phi(k)$ an expression

$$\begin{aligned}\phi(k) &= k(1-\alpha_1) + \ln\left[\frac{\alpha_2}{\alpha_1+\alpha_2}\right] + \ln\left[1+\frac{\alpha_2}{\alpha_1}e^{-k(\alpha_1+\alpha_2)}\right]\\ &= (1-\alpha_1) + \ln\left[\frac{\alpha_2}{\alpha_1+\alpha_2}\right] + \sum_{n\geqslant 1}\frac{(-1)^n}{n!}\left[\frac{\alpha_2}{\alpha_1}e^{-k(\alpha_1+\alpha_2)}\right]^n.\end{aligned} \tag{23}$$

Eventually we get the following partial differential equation (PDE):

$$\begin{aligned}\frac{\partial G(x,v)}{\partial v} &= G\ln G + (1-\alpha_1)\frac{\partial G(x,v)}{\partial x} + \ln\left[\frac{\alpha_2}{\alpha_1+\alpha_2}\right]\\ &\quad + \sum_{n\geqslant 1}\frac{(-1)^n}{n!}\left[\frac{\alpha_2}{\alpha_1}G(v,x+n(\alpha_1+\alpha_2))\right]^n.\end{aligned} \tag{24}$$

We are seeking solutions in which $G(x,v)\to 0$ for $x\to\infty$, therefore our equation is well-defined.

3. Gamma distribution

Now we consider the following distribution for positive ϵ:

$$\hat{\rho}(\epsilon) = \frac{\gamma^\epsilon}{\Gamma(\gamma)}e^{-\gamma\epsilon}. \tag{25}$$

We have the following expression for $\phi(k)$:

$$\phi(k) = \gamma\ln(1+k/\gamma). \tag{26}$$

While we can formulate formally a PDE in this case,

$$\frac{\partial G(x,v)}{\partial v} = \gamma\ln\left[1+\frac{1}{\gamma}\frac{\partial}{\partial x}\right]G(x,v) + G\ln G, \tag{27}$$

the equation can be better formulated in Fourier space.

4. Poisson distribution

This distribution is popular in the financial mathematics literature. We have integer values of ϵ,

$$\hat{\rho}(\epsilon) = e^{-\gamma}\gamma^\epsilon/\epsilon!. \tag{28}$$

Equation (9) gives $\phi(k)=\gamma(-1+e^k)$. Then from Eq. (18) we derive

$$\frac{\partial G(x,v)}{\partial v} = \gamma[-G+G(x+1)] + G\ln G. \tag{29}$$

This renormalization group equation differs from Eq. (21) in that the second-order derivative is replaced by a finite difference.

5. Compound Poisson distribution

Cascade processes with this type of distribution are also rather popular in the literature. Now we have a Poisson distribution for integers n given by Eq. (28), i.e., $p_1(n) = e^{-\gamma}\gamma^n/n!$, and we define ϵ as a sum of n random variables x_l with some random distribution $p(x)$, defined through the representation

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dh\exp[\alpha(ih) - ih\epsilon]. \tag{30}$$

Thus

$$\alpha(k) = \ln\int_{-\infty}^{\infty} p(y)dy\exp[ky]. \tag{31}$$

A simple calculation gives the function $\phi(k)$ for the distribution of ϵ,

$$\phi(k) = \gamma(e^{\alpha(k)}-1). \tag{32}$$

Thus we get the following PDE for $G(v,x)$:

$$\frac{\partial G(x,v)}{\partial v} = \gamma\left[e^{\alpha(\frac{\partial}{\partial x})}-1\right]G(x)] + G\ln G \tag{33}$$

or, using Eq. (31),

$$\frac{\partial G(x,v)}{\partial v} = \gamma\left[\int dy\, p(y)G(x+y,v) - G(x,v)\right] + G\ln G. \tag{34}$$

C. Phase-transition point

1. General case of distribution

The free energy of the directed polymer model Eq. (7) for any q is equivalent to the free energy of the corresponding REM with the same number of energy configurations and the same distribution of y_i. This issue is well discussed in the REM literature [2,3,5,6].

We have $M=e^L$ end points of the tree, and any y_i has a distribution by $\rho(L,y)$; see Eq. (10). Let us consider the REM with an M energy level and an independent distribution of energy by Eq. (10).

In the high-temperature phase, assuming $\langle\ln Z\rangle = \ln\langle Z\rangle$, we have [3]

$$\langle\ln Z\rangle = L[1+\phi(\beta)]. \tag{35}$$

The phase transition is at the point where the entropy of free energy by Eq. (35) disappears [3],

$$1+\phi(\beta_c) - \beta_c\phi'(\beta_c) = 0. \tag{36}$$

2. Binomial distribution

Now we have the following expression for the free energy in the high-temperature phase:

$$\frac{\ln Z}{L} = \ln\left[\frac{\alpha_2}{\alpha_1+\alpha_2}e^{\beta(1-\alpha_1)} + \frac{\alpha_1}{\alpha_1+\alpha_2}e^{\beta(1+\alpha_2)}\right]. \quad (37)$$

Equation (36) has no solution in this case, therefore the system is always in the high-temperature phase.

3. Gamma distribution

Now we have the following expression for the free energy in the high-temperature phase:

$$\frac{\ln Z}{L} = 1 + \gamma \ln\left(1+\frac{\beta}{\gamma}\right). \quad (38)$$

Equation (36) has no solution in this case, therefore the system is always in the high-temperature phase.

4. Poisson distribution

For this case, we derive the following for the free energy of the high-temperature phase and transition point:

$$\frac{\ln Z}{L} = 1 + \gamma(e^{\beta}-1), \quad 1+\gamma[e^{\beta_c}(1-\beta_c)-1] = 0. \quad (39)$$

In the next sections, we will use the traveling-wave analysis to investigate the paramagnetic phase solution of our equations and to identify the transition point. To investigate the spin-glass-like solutions, one needs to use the advanced mathematics of [16].

5. Compound Poisson distribution

Now we have the following expression for the free energy in the high-temperature phase:

$$\frac{\ln Z}{L} = 1 + \gamma(e^{\alpha(\beta)}-1) = 1 + \gamma\left(\int dx\, p(x)e^{\beta x} - 1\right). \quad (40)$$

For the critical point, we get the equation

$$1 + \gamma\left[\int dx\, p(x)e^{\beta_c x} - 1 - \beta\int dx\, p(x)e^{\beta_c x}x\right] = 0. \quad (41)$$

D. KPP equation versus new renormalization group equation

Let us try to describe our hierarchic tree model using the KPP equation,

$$\frac{\partial Q(x,v)}{\partial v} = \frac{\partial^2 Q(x,v)}{\partial x^2} + Q - Q^2. \quad (42)$$

In Ref. [2], Eq. (42) was derived as an approximation of Eq. (7) for the special case of Eq. (20). Later, Eq. (42) was derived in Ref. [17] as an approximate renormalization group equation for the models of random disorder with logarithmic correlations. While [2,5] already considered the general distribution, the KPP equation has been derived only for the normal distribution. Although the authors of [14] used a general distribution of random variables, they did not use the KPP equation. The reason is clear: for the general distribution of random variables on the tree, the KPP equation gives incorrect results even for the transition point, contrary to our new renormalization group equation, which is exact.

To compare, we take the case of Eq. (28) with $\gamma = 1$. The KPP version, having the same variance of distribution $\rho(\epsilon)$, gives

$$\frac{\partial G(x,v)}{\partial v} = \frac{\partial G(x,v)}{\partial x} + \frac{\partial^2 G(x,v)}{\partial x^2} + G - G^2. \quad (43)$$

Using the mapping

$$G(x,v) = Q(x+t,v), \quad (44)$$

we return to the KPP equation by Eq. (42). This equation is well investigated in the literature. It describes a transition between the spin-glass and paramagnetic phases at

$$\beta_c = \sqrt{2}, \quad (45)$$

and the free energy in the paramagnetic phase is

$$\langle \ln Z\rangle = L(1+\beta+\beta^2). \quad (46)$$

The correct transition points of the model and free energy are:

$$\beta_c = 1, \quad \langle \ln Z\rangle = Le^{\beta}. \quad (47)$$

Thus the KPP equation incorrectly describes the model, and we need a new exact renormalization group equation for the general form of distribution of random variables on the tree.

Let us investigate Eq. (29). $G(v,x)$ is a monotonic function of x according to the definition by Eqs. (4) and (7),

$$\begin{aligned} G(x,v) &\to 1 \quad \text{for} \quad x\to\infty,\\ G(x,v) &\to 0 \quad \text{for} \quad x\to -\infty. \end{aligned} \quad (48)$$

Consider the case of small p in Eqs. (4) and (7). The direct calculations give

$$G(x,v) = \exp\{-p\exp\beta x + v[1+\gamma(e^{\beta}-1)]\}. \quad (49)$$

The latter expression is not a solution of Eq. (43). Nevertheless, it is the solution of Eq. (29) when

$$p\exp\{\beta x + v[1+\gamma(e^{\beta}-1)]\} \ll 1. \quad (50)$$

Let us investigate the behavior of the asymptotic solution for the situation that is more general than the case of Eq. (50).

Following [2,5,17], we assume a traveling-wave-like asymptotic solution

$$G(x,v) = g(x+cv). \quad (51)$$

We get the ODE:

$$cg'(x) = [-g(x)+g(x+1)] + g(x)\ln g(x). \quad (52)$$

Equation (52) describes a traveling wave with a front at

$$x = -cv. \quad (53)$$

We assume that for large negative x, there is a solution

$$g(x) = \exp[-e^{\beta x}]. \quad (54)$$

Putting the latter expression into Eq. (52), we get the following equation for c:

$$c(\beta) = \frac{[e^{\beta}-1]\gamma+1}{\beta}. \quad (55)$$

Equations (51), (54), and (55) give the free-energy expression by Eq. (46).

We can identify the phase-transition point as the value of β giving the maximum velocity, which yields the equation

$$c'(\beta_c) = 0. \tag{56}$$

Equation (56) gives the result of Eq. (47), $\beta_c = 1$, for the case $\gamma = 1$.

At high values of β there is another solution for $c(\beta)$, but for its derivation one needs to use the advanced mathematical approach of [16]. Thus the investigation of our renormalization group equation, using the qualitative approach of [2,5,17], gives an exact expression for the free energy.

III. RANDOM BRANCHING MODEL

In the previous section, we derived a new renormalization group equation using continuous deterministic branching, later considering the $q \to 1$ limit. Let us derive a similar renormalization group equation using a less abstract model of random branching [2].

We again have a tree. It starts at some point O and grows down, where the vertical coordinate measures the time v. We put a random variable ϵ_l with a distribution from Eq. (8) on the links (between two adjacent nodes of the tree), replacing Δv by the difference of the time coordinate between two nodes of the branch. During the period of time dv, there is a branching with a probability dt.

Consider the dynamics of the partition sum $Z(\beta,v)$ defined as

$$Z(\beta,v) = \sum_j e^{\beta y_j}. \tag{57}$$

Here the index j numerates the end points of our tree at time v.

$Z(\beta,v)$ has a deterministic behavior at $v = 0$:

$$Z(\beta,0) = 1. \tag{58}$$

$Z(\beta,v)$ is a random variable at $v > 0$ due to both the randomness of branching and the randomness of ϵ_l.

Then one has the following recursive equation for the random partition function [2]:

$$Z(\beta,v+dv) = Z(\beta,v)e^{-\beta\epsilon}, \tag{59}$$

with probability $1 - dv$, and

$$Z(\beta,v+dv) = [Z^1(\beta,v) + Z^2(\beta,v)]e^{-\beta\epsilon}, \tag{60}$$

with probability dv. Here $Z^1(\beta,v)$ and $Z^2(\beta,v)$ are random independent variables with the same probability distribution as $Z(\beta,v)$.

We identify $Z(0,v)$ with the number of end points of the tree, and Eqs. (59) and (60) give the intuitive result

$$\langle Z(0,v)\rangle = \exp(t), \tag{61}$$

confirming the self-consistency of the choice Eqs. (59)–(61). Actually, the random process $Z(t)$ is defined completely through Eqs. (58)–(60), and we can work with these equations without any reference to the branching trees.

We define now, following [2],

$$G(x,v) = \langle \exp[-e^{-\beta x} Z(v)]\rangle. \tag{62}$$

Equations (59) and (60) give

$$G(x,v+dv) = (1-dv)\int d\epsilon\, \rho(\epsilon)G(v,x+\epsilon) + dv\, G(v,x)^2. \tag{63}$$

Eventually, we get the following equation:

$$\frac{\partial G(x,v)}{\partial v} = \phi(\partial x)G(x,v) - G(x,v)[1 - G(x,v)]. \tag{64}$$

For the binomial distribution, we get

$$\frac{\partial G(x,v)}{\partial v} = -G(1-G) + (1-\alpha_1)\frac{\partial G(x,v)}{\partial x} + \ln\left[\frac{\alpha_2}{\alpha_1+\alpha_2}\right] + \sum_n \frac{(-1)^n}{n!}\left[\frac{\alpha_2}{\alpha_1}G(x+n(\alpha_1+\alpha_2),v)\right]^n. \tag{65}$$

For the Poisson distribution, we get

$$\frac{\partial G(x,v)}{\partial v} = \gamma[-G(x,v) + G(x+1,v)] + G(x,v)^2 - G(x,v). \tag{66}$$

We can repeat the derivations of Sec. II D, and again get the same solutions for the free energy of the paramagnetic phase and the critical temperature.

For the compound Poisson distribution, we get

$$\frac{\partial G(x,v)}{\partial v} = \gamma\left[\int dy\, p(y)G(x+y,v) - G(x,v)\right] + G(G-1). \tag{67}$$

IV. CONCLUSIONS

We provided an exact (renormalization group) partial differential equation for the analysis of models on a hierarchic tree with continuous branching and general distribution of random variables. We derived a new class of reaction-diffusion equation (18), which sometimes has a finite difference, Eqs. (24), (29), and (34). While recursive equations for hierarchic models with a general distribution of random variables on the tree have been well known [2,14], our paper is the first to replace these recursive equations with the correct PDE.

Our derivation of the renormalization group equation is quite rigorous. The renormalization group equation is exact, including finite-size corrections, while the renormalization group equation in Ref. [17] is an approximation and could be applied to calculate only bulk characteristics of corresponding models.

The KPP equation, used in Refs. [2,5,17], correctly (exact in the thermodynamic limit) describes the bulk free energy and critical temperature for the hierarchic tree models with a normal distribution of random variable [2,5], as well as for the finite-dimensional models of disorder with logarithmic correlations [17]. However, as per our analysis, the KPP equation failed to describe correctly the hierarchic models with non-normal distributions. That is why the general case of the hierarchic model, given by the recursive equation (7) derived in Ref. [2], was not investigated using the KPP equation in Refs. [2,5]. To verify our new renormalization group equations, we used the idea of the traveling wave and the methods of [5,17]. Analyzing our equations, we found the correct expression for the free energy in the paramagnetic phase and the phase-transition temperature.

It remains an open problem to construct the finite-dimensional models of disorder, which could be described by our new equations, as the models of [17] are described by the KPP equation. We hope to succeed using dynamic stochastic processes in the one-dimensional case.

Our approach (continuous branching) can be applied to investigate quantum disorder in d-dimensional space. In the case of continuous branching, we derived exact PDE equations that can be solved numerically, while in the alternative approach of [14], approximate renormalization group equations have been derived and there are errors $O(1)$ in the expression of free energy (bulk term $\sim L$).

The normal distribution version Eq. (21) has a deep mathematical meaning: in some sense, it describes the p-adic space with $p = 1$ [18]. We hope that the investigation of these new reaction-diffusion equations will be further encouraged.

Our renormalization group equations can also be applied to calculate the complicated correlation function, as has been done in Ref. [11] for the string case.

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