Two-ball Newton's cradle

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Newton's cradle for two balls with Hertzian interactions is considered as a hybrid system, and this makes it possible to derive return maps for the motion between collisions in an exact form despite the fact that the three-halves interaction law cannot be solved in closed form. The return maps depend on a constant whose value can only be determined numerically, but solutions can be written down explicitly in terms of this parameter, and we compare this with the results of simulations. The results are in fact independent of the details of the interaction potential.

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The aim of this Brief Report is to analyze the two-ball Newton's cradle and show that, if the pendulums are linear, an exact return map after collisions can be derived which is independent of the details of the complicated ball-ball interaction during collisions. This return map can be solved explicitly and the motion is either periodic or, more typically, quasiperiodic. The collision interaction contributes a parameter to the solution which could be measured experimentally, so it might be possible to infer the details of the interaction from measurements of this parameter for different initial conditions and pendulum parameters. Finally, since the linear pendulum model (with nonlinear collisions) can be solved, we compare results with simulations of the nonlinear pendulum.

The multiple-ball Newton's cradle is a paradigm for the treatment of Newtonian impulses. It consists of N balls (usually five) fixed to a frame as pendulums so that they can oscillate in one direction and, when hanging in equilibrium under gravity, they are just touching. If one of the end balls is set in motion it strikes the line and an impulse travels through the line and the ball at the other end lifts off, with the others stationary and the cycle continues. This is a standard theoretical narrative, but experiments show that what actually happens is considerably more complicated: all the balls move and the subsequent motion is heavily influenced by the breakup of the line.

A more realistic model of Newton's cradle will involve including at least one of either the interaction of the pendulums via the frame or a more detailed model of the short-time interactions of the balls in collision. Following [1–3] we adopt the latter modification. One of the standard visco-elastic models of this interaction is a Hertzian three-halves power-law force, and the approach below applies to this and indeed any other potential force which satisfies some mild conditions. A second principle when faced with complicated behavior is to consider the simplest case in some detail. Thus we consider the dynamics of a two-ball Newton's cradle. This was analyzed in Ref. [3] by making simplifying assumptions about the interaction terms (essentially that the contact interaction is linear and gravity can be ignored), which makes it possible to see that the likely outcome of the model is slow modulation between a cradle-like dynamics (with each ball approximately at rest during half a cycle) and a more symmetric collision and bounce in which both balls oscillate significantly during a half period.

More complicated collisions between balls have also been considered recently [4], but the aim of this Brief Report is to show that a slightly more sophisticated analysis, considering the collisions and the motion when the balls are not in contact as defining a hybrid system [5], makes it possible to derive explicit return maps for the state of the system immediately after collisions if gravity is considered as a linear potential in the angle of the pendulums (the small-displacement limit). This return map is completely determined by standard parameters of the system except for the appearance of one free parameter which depends on the details of the interaction potential and the initial conditions.

Choose one-dimensional coordinates y_1 and y_2 for the center of mass of the two equal balls labelled by 1 and 2 in the obvious way, with ball one to the left of ball two. In equilibrium the balls are separated by $2R$, where R is the radius of the balls, so it is natural to write $y_1 = x_1$ and $y_2 = 2R + x_2$, where x_i represents the displacement of ball i from its equilibrium position. The distance between the centers of mass is

$$y_2 - y_1 = 2R - (x_1 - x_2),$$

so the balls are in contact, and the interaction potential comes into play, if $x_1 - x_2 > 0$.

If $x_1 - x_2 < 0$ then the balls are not in contact and each behaves as a pendulum, with the linear model

$$\ddot{x}_1 = -\omega^2 x_1, \quad \ddot{x}_2 = -\omega^2 x_2, \tag{1}$$

and $\omega^2 = g/\ell$, where g is the acceleration due to gravity and ℓ is the vertical length of the pendulum wires.

If $x_1 - x_2 > 0$, then the balls are in contact and there is an elastic force (we ignore dissipation until later) in addition to the gravitational force and

$$\ddot{x}_1 = -\omega^2 x - V'(x_1 - x_2), \quad \ddot{x}_2 = -\omega^2 x_2 + V'(x_1 - x_2), \tag{2}$$

where the potential V models the visco-elastic forces (per unit mass) so, for the Hertzian case,

$$V(s) = \frac{K}{1+\alpha} s^{1+\alpha}. \tag{3}$$

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In the simulations below we use the standard Hertzian force, $\alpha = \frac{3}{2}$. In what follows we can treat more general potentials having the properties

$$V'(0) = 0, \quad V'(s) > 0 \quad \text{if } s > 0 \tag{4}$$

and

$$V(s) \to \infty \quad \text{as} \quad s \to \infty. \tag{5}$$

It is natural to work in the center-of-mass frame (times two) and relative position coordinates

$$Q = x_1 + x_2, \quad q = x_1 - x_2, \tag{6}$$

in terms of which

$$\ddot{Q} = -\omega^2 Q, \tag{7}$$

independent of the sign of q and

$$\ddot{q} = \begin{cases} -\omega^2 q & \text{if } q < 0 \\ -\omega^2 q - V'(q) & \text{if } q > 0. \end{cases} \tag{8}$$

Note that, if $q > 0$, the relative position equation is the Hamiltonian with

$$H = \tfrac{1}{2}p^2 + U(q), \quad U(q) = \tfrac{1}{2}\omega^2 q^2 + V(q), \tag{9}$$

and U satisfies the same conditions (4) and (5) as V.

To describe solutions of (7) and (8) we will assume that the values of Q and q are known immediately after the nth collision, together with the corresponding velocities, and derive a recursion equation for their values immediately after the following collision. Suppose that immediately after the nth collision, $t = t_n$ and

$$Q = Q_n, \quad \dot{Q} = P_n, \quad q = 0, \quad \dot{q} = p_n < 0 \tag{10}$$

(noting that, at the beginning and end of a collision interaction, $q = 0$, with $\dot{q} > 0$ at the beginning of the collision and $\dot{q} < 0$ at the end of the collision). Since $\dot{q} < 0$, q begins to decrease and, whilst $q < 0$, the evolution is defined by (7) and the first of equations (8), and so

$$\begin{aligned} Q &= Q_n \cos\omega(t - t_n) + \frac{P_n}{\omega}\sin\omega(t - t_n), \\ P &= -\omega Q_n \sin\omega(t - t_n) + P_n \cos\omega(t - t_n), \\ q &= \frac{p_n}{\omega}\sin\omega(t - t_n), \\ p &= p_n \cos\omega(t - t_n), \end{aligned} \tag{11}$$

and these remain valid until the first time $t_n' > t_n$ such that $q(t_n') = 0$. Due to the simple form of q this implies

$$t_n' = t_n + \frac{\pi}{\omega},$$

at which value the cosine is -1 and so $p(t_n') = -p_n > 0$ and the corresponding values of Q and P are $-Q_n$ and $-P_n$, respectively. Since $p > 0$, q increases and the (q,p) dynamics is determined by the Hamiltonian system with Hamiltonian (9). The potential U satisfies conditions (4) and (5) so the solutions are symmetric under reflections $p \to -p$ and cannot tend to a stationary point in $q > 0$ [as $U'(q) > 0$] nor to infinity [as $U(q) \to \infty$ as $q \to \infty$]; see, for example, Ref. [6]. Hence there exists a time $\tau > 0$ such that $q = 0$ again for the first time and $p = -(-p_n) = p_n < 0$. Thus, after this time τ, $t = t_{n+1} = t_n + \frac{\pi}{\omega} + \tau$ and

$$\begin{aligned} Q &= Q_{n+1} = -Q_n \cos\omega\tau - \frac{P_n}{\omega}\sin\omega\tau, \\ P &= P_{n+1} = \omega Q_n \sin\omega\tau - P_n\cos\omega\tau, \\ q &= 0, \\ p &= p_{n+1} = p_n < 0. \end{aligned} \tag{12}$$

Since p does not change from one collision to another, the same τ is used in each collision, and this is determined by both p_n and the details of the potential U, but once fixed it does not change from collision to collision.

It is not hard to solve this difference equation. In terms of the complex variable $Z_n = \omega Q_n + i P_n$, the first two equations of (12) are $Z_{n+1} = -Z_n e^{-i\omega\tau}$ and so

$$Z_n = (-1)^n Z_0 e^{-in\omega\tau}. \tag{13}$$

This exact solution shows that the dynamics is equivalent to a solid rotation of phase in the complex plane with a sign oscillation that could also be included in the phase, and so the qualitative behavior depends only on $\omega\tau$. If $\omega\tau = 2\pi k/\ell$, where k and ℓ are coprime integers, then all solutions are periodic with marginal stability, whilst if $\omega\tau$ is not a rational multiple of 2π, as is typically the case, then solutions are quasiperiodic.

In terms of the variables Q_n, P_n, and p_n, the exact solution (13) is

$$\begin{aligned} Q_n &= (-1)^n Q_0 \cos(n\omega\tau) + (-1)^n \frac{P_0}{\omega}\sin(n\omega\tau), \\ P_n &= -(-1)^n \omega Q_0 \sin(n\omega\tau) + (-1)^n P_0 \cos(n\omega\tau), \\ p_n &= p_0 < 0, \end{aligned} \tag{14}$$

or, in terms of the original variables x_1 and x_2 using (6) and the fact that $q = 0$ just after a collision,

$$\begin{aligned} x_1(t_n) &= x_2(t_n) = \tfrac{1}{2}Q_n, \\ \dot{x}_1(t_n) &= \tfrac{1}{2}(P_n + p_0), \\ \dot{x}_2(t_n) &= \tfrac{1}{2}(P_n - p_0). \end{aligned} \tag{15}$$

Equations (13)–(15) provide a general solution to the two-ball Newton's cradle, but the classic Newton's cradle solution would correspond to initial conditions after the first collision (having set ball one in motion first) of

$$x_1 = x_2 = \dot{x}_2 = 0, \quad \dot{x}_2 = v > 0,$$

and this translates to

$$Q_0 = q_0 = 0, \quad P_0 = v, \quad p_0 = -v. \tag{16}$$

Substituting these values into (14) and (15) gives

$$\begin{aligned} x_1(t_n) &= x_2(t_n) = \frac{v}{2\omega}(-1)^n \sin(n\omega\tau), \\ \dot{x}_1(t_n) &= -\frac{v}{2}[1 - (-1)^n]\cos(n\omega\tau), \\ \dot{x}_2(t_n) &= \frac{v}{2}[1 + (-1)^n]\cos(n\omega\tau). \end{aligned} \tag{17}$$

On the reasonable assumption that $\omega\tau$ is small (as the contact time τ is small), these solutions have an interesting interpretation. If $n\omega\tau \approx k\pi$ then the behavior is like the classically described Newton cradle: there is negligible oscillation from the vertical of the collision point, and immediately after the collision one ball is at rest and the other moves off, with the balls interchanging roles at each collision. On the other hand, if $n\omega\tau \approx (2k+1)\frac{\pi}{2}$ then the position of the collision oscillates and after the collision both balls recoil back and swing with approximately equal initial speeds.

Thus there is a slow periodic oscillation having approximately $\pi/(\omega\tau)$ collisions [since there is no discernible difference between the cases $\cos(n\omega\tau) \approx 1$ and $\cos(n\omega\tau) \approx -1$ apart from an odd-even n exchange, we consider the period of these oscillations to be half the full period of oscillation of the trigonometric functions] each taking time $\frac{\pi}{\omega} + \tau$, so the total time of the full period is

$$\frac{\pi}{\omega^2\tau}(\pi + \omega\tau),$$

representing a modulation of frequency

$$\frac{2\omega^2\tau}{\pi + \omega\tau}.$$

This is the same expression as derived by [3], but crucially their derivation relies on an assumption of constant small contact time (which is proven above in the general case) with the collision being modelled by a potential $\frac{1}{2}kq^2$ in $q > 0$, which also means that they associate a frequency and spring constant to the collision interaction.

To investigate this behavior numerically we have chosen units with

$$\omega^2 = 1, \quad K = 5000, \quad \alpha = \tfrac{3}{2},$$

where the value of K [the Hertzian constant, cf. (3)] is chosen so that the contact time is small enough to make the slow drift described above observable. The equations were integrated using a fixed step ($h = 0.000\,06$) third-order Verlet method that preserves the symplectic structure of solutions. A sample trajectory projected onto the (x_1,x_2) plane is shown in Fig. 1, which has initial conditions $x_1(0) = -2$, with $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$, integrated for time equal to 60 units. A classic Newton's cradle solution would move close to the x_1 axis in $x_1 < 0$ and then up the x_2 axis and return. As shown in Fig. 1, the actual behavior is a drift out to a region with both x_1 and x_2 large, and if the solution had been extended it would have returned close to the ideal Newton's cradle solution. The noncradle motion [$\cos(n\omega\tau)$ close to zero in the terminology of (17)] moves between a collision in $x_1 > 0$ through x_2 large and x_1 large and negative, back to a collision in $x_1 < 0$.

Of course, the model described here is for small displacement; for larger displacements both the nonlinear nature of the gravitational force on the angle of displacement and the effect of geometry due to the balls no longer striking each other symmetrically because of the offset at the hanging points would need to be modelled. However, the advantage of having a simple model that can be accurately analyzed makes the approach here worth taking into account. The addition of nonlinearity in the pendulum does change the dynamics when the angle is large. Figures 2 and 3 shows the results of simulations of the equations (1) and (2) with the small-amplitude $\omega^2 x$ approximations replaced by $\omega^2 \sin x$ and the same values of the other parameters. Figure 2 has initial conditions $x_1 = -0.1$ with $\dot{x}_1 = x_2 = \dot{x}_2 = 0$, and the result is similar in nature to the simulation of the linear system, as would be expected from the small amplitude of the initial condition. Interestingly, Fig. 3 shows a solution with initial condition $x_1 = -2$ and the other variables all zero. In this case the solution is closer to the ideal Newton's cradle solution than the linear approximation studied above. A fuller investigation of this case might prove worthwhile.

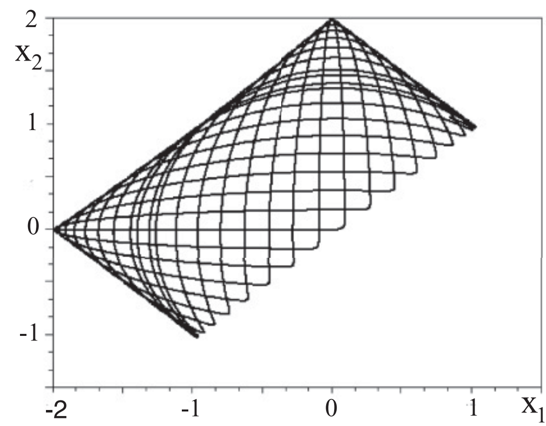

FIG. 1. Trajectory of a solution projected onto the (x_1,x_2) plane. Initial conditions and parameters are given in the text.

Another feature of the dynamics of the two-ball Newton's cradle that can be described using this framework is the effect of dissipation. Assuming that most of the dissipation is in the collisions, so that these are inelastic rather than elastic, then the natural model would be to replace the conclusion that p is simply reversed during a collision to a reversal with a reduction in the modulus. In this case, the inelastic collisions would lead eventually to $p = 0$ with $q = 0$, and hence the asymptotic dynamics has both balls touching, with the pendulums swinging in phase. This of course would then decay due to friction.

Using a commercial cradle (e.g., by Zeon Tech, endorsed by the Science Museum, UK) it is easy to confirm the asymptotic in-phase oscillation and also to observe solutions

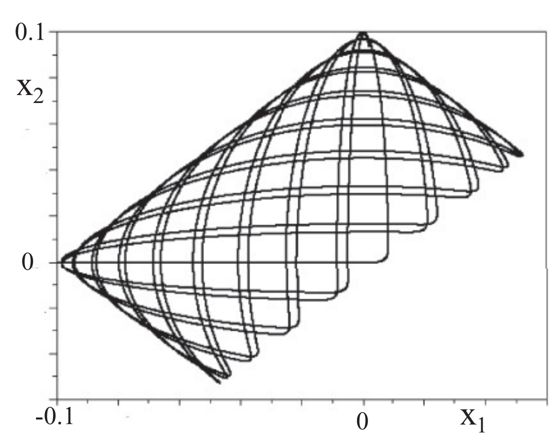

FIG. 2. Projection onto (x_1,x_2) plane of trajectory of solution to model with nonlinear term $\omega^2 \sin x$ replacing the linearization ω^{2x}. The initial condition has small amplitude ($x_1(0) = -0.1$; see text) and parameters are otherwise the same as in Fig. 1.

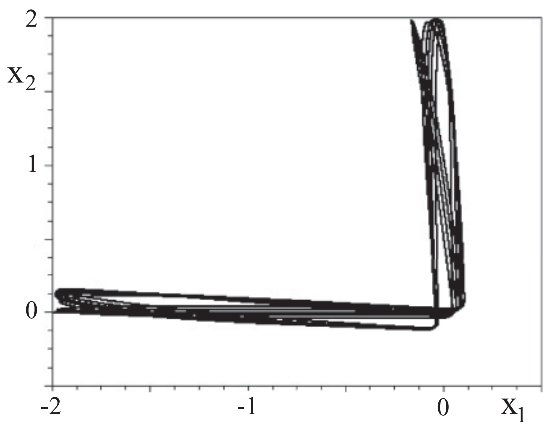

FIG. 3. Same as Fig. 2 except that initial conditions and parameters are the same as in Fig. 1.

corresponding to the motion described by Figs. 1 and 2 provided the base is kept clamped (a heavy hand will do!). In the latter case the key observation is that the every other collision moves from being in one phase of the swing to the other on a slow timescale, passing close to an ideal cradle solution in the process. In other words the even collisions will occur in $x_1 > 0$ for a while, and then in $x_1 < 0$, and so on; this can be observed by viewing a film of the interaction in slow motion. However, the system does not spend long close to the ideal cradle motion, but passes through it fairly rapidly and so it is hard to detect. This would, of course, also be true of the motion depicted in the figures.

The analysis reported here uses the hybrid nature of the collisions to derive return maps that can be solved explicitly and, through this, a very accurate description of the motion is possible. The analysis is not hard, but it is revealing. Although this deals with a situation that is considerably simpler than the many-ball Newton's cradle, it shows that in the two ball case the finer details of the interactions between the ball are unimportant to the outcome of any experiment except insofar as they determine τ for a given p_0. This suggests that an experiment could be made to determine the dependence of τ and p_0, and then this in turn could be fit to different powers of α in the Hertzian model as a means of assessing the effective α independent of the classical Hertzian-force arguments.

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