Breatherlike electromagnetic wave propagation in an antiferromagnetic medium with Dzyaloshinsky-Moriya interaction

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We investigate the nature of propagation of electromagnetic waves (EMWs) in an antiferromagnetic medium with Dzyaloshinsky-Moriya (DM) interaction environment. The interplay of bilinear and DM exchange spin coupling with the magnetic field component of the EMW has been studied by solving Maxwell's equations coupled with a nonlinear spin equation for the magnetization of the medium. We made a nonuniform expansion of the magnetization and magnetic field along the direction of propagation of EMW, in the framework of reductive perturbation method, and the dynamics of the system is found to be governed by a generalized derivative nonlinear Schrödinger (DNLS) equation. We employ the Jacobi-elliptic function method to solve the DNLS equation, and the electromagnetic wave propagation in an antiferromagnetic medium is governed by the breatherlike spatially and temporally coherent localized modes under the influence of DM interaction parameter.

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I. INTRODUCTION

In the past few years the nonlinear behavior of electromagnetic waves in nonlinear magnetic materials has risen to some prominence in the literature because of the progress in developing the technologically important magneto-optic recording and high density data storage devices [1–3]. Many interesting results have been obtained, which not only exposed the interaction between light and materials but also provided new ideas to develop new devices [4,5]. In a similar context, propagation of electromagnetic waves (EMWs) through ordered magnetic medium has also assumed a lot of importance in recent times [6,7]. In this case, the magnetic field component of the electromagnetic field is found to excite the magnetization of a ferromagnetic medium in the form of solitons and also the small amplitude plane electromagnetic wave propagates in the form of EM solitons [8]. Nakata [9,10] and Leblond and co-workers [11,12], showed that the EMW propagates in the form of soliton in a ferromagnetic medium using the reductive perturbation method, however, by neglecting the spin-spin exchange energy. Recently, Veerakumar and Daniel [13] investigated in this direction by taking into account the basic magnetic interactions, namely the spin-spin exchange interaction in isotropic and anisotropic ferromagnetic and antiferromagnetic media [14,15]. In addition to the dominant magnetic interactions, such as exchange, anisotropy, etc., which involve integrable spin models with soliton spin excitations, there exist certain magnetic interactions that are less spoken about in the literature of nonlinear dynamics due to the mathematical complexity of their representations in the Hamiltonian and in the governing dynamical equations [16–19]. Notable among them is the Dzyaloshinsky-Moriya (DM) interaction, which has been reexamined by several authors in recent times [20,21]. This interaction is essentially an antisymmetric spin coupling interaction that occurs when the symmetry around the magnetic ions is not high enough, thus leading to the mechanism of weak ferromagnetism, which is due to the combined effect of spin-orbit coupling and spin-spin exchange interactions. Weak ferromagnets play an important role in describing insulators, spin glasses, low-temperature phases of copper oxide superconductors, phase transitions, etc. [22,23].

The DM interaction that is often present in the models of many low-dimensional magnetic materials is known to generate many spectacular features [24,25]. Though ferromagnetic spin systems have been studied extensively, the study of nonlinear dynamics of antiferromagnetic systems is still in its infant stage. In antiferromagnets, the adjacent spins are aligned antiparallel to each other and hence the dynamics is governed by highly nontrivial coupled nonlinear partial differential equations. However, some progress has been made by identifying the problem as a two-sublattice model and formulating the governing equations of motion.

The study of breathers has attracted growing attention in the past several years in a wide variety of physical systems. Breathers are characterized by spatially and temporally localized excitations in nonlinear and periodic systems. Rigorous proof has been given for the existence of breathers [26], and their properties have been studied extensively [27,28]. However, their experimental observation has been reported only recently. Recent evidence points to the important role of breathers in an impressive variety of contexts, including low-dimensional materials [29], macroscopic-mechanical systems [30], spin lattices [31], spin waves in antiferromagnets [32], Josephson arrays [33] and Josephson ladders [34], molecular chains [35], Bose-Einstein

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condensates [36], dispersion managed optical fibers [37], and finally optical breathers in photonic-crystal waveguides [38]. Recent experiments on the quasi-one-dimensional antiferromagnetic Heisenberg spin chain (AFHC) Cu benzoate $Cu(C_6H_5COO)_2 \cdot 3H_2O$, in the presence of magnetic field revealed the existence of breather mode excitations directly by an electron spin resonance investigation [39].

In view of the above, in the present paper we explore the breatherlike nature of EMW propagation in an antiferromagnetic medium with antisymmetric DM interaction, in the presence of crystal field anisotropy. The outline of the paper is as follows. In Sec. II, we present the mathematical model for the weak antiferromagnetic system and construct the equations of motion. In Sec. III, we employ the reductive perturbation technique on the equations of motion and formulate a nonlinear evolution equation that describes EMW propagation in the form of solitons. In Sec. IV, by invoking the Jacobi elliptic function method, we attempt to construct a set of solitary and breatherlike wave solutions and explore the influence of DM parameters on the same. The results are concluded in Sec. V.

II. WEAK ANTIFERROMAGNETIC SPIN DYNAMICS AND EMW EQUATION

The Heisenberg Hamiltonian describing the anisotropic antiferromagnetic spin chain can be written as

$$H = -\sum_n \left[J(\mathbf{S}_n \cdot \mathbf{S}_{n+1}) + \mathbf{D}_n \cdot (\mathbf{S}_n \times \mathbf{S}_{n+1}) - A\left(S_n^z\right)^2 + \gamma \mathbf{S}_n \cdot \mathbf{H}\right], \quad J < 0, \tag{1}$$

where $\mathbf{S}_n$ denotes the classical spin vector at the lattice site n and J represents the exchange integral between the nearest neighbors. The first term in Eq. (1) is associated with the bilinear spin-spin exchange interaction and in order to have an appreciable overlapping of wave function we need to consider the nearest neighboring spins. The second term is the DM interaction term and is proportional to the vector product of interacting spins and is allowed by symmetry in noncentric crystal structures. The antisymmetric spin coupling was first suggested by Dzyaloshinsky [40] to explain the mechanism of weak ferromagnetism of antiferromagnetic crystals from a purely symmetry ground state and was later derived theoretically by Moriya [41,42]. This DM interaction is of interest in its own right and is known to be the cause of weak ferromagnetism in certain materials such as Hematite $\alpha - Fe_2O_3$ [42]. This interaction is also found to enhance the fluctuation of the spin components in the plane perpendicular to **D**. The vector **D** denotes the intensity of DM interaction imposed along the chain. To understand what is going on, we first note that for two spins $\mathbf{S}_1$ and $\mathbf{S}_2$, interacting via isotropic exchange and the DM term, the interaction energy is minimized at $-\sqrt{J^2 + D^2S^2}$ when both spins $\mathbf{S}_1$ and $\mathbf{S}_2$ are perpendicular to **D** in the absence of an external magnetic field. As shown by Moriya, the cross-product term $\mathbf{D}_n \cdot (\mathbf{S}_n \times \mathbf{S}_{n+1})$ originates from spin-flop hopping, which made the possible existence of spin-orbit interactions resulting in a canted spin system. The parameter A characterizes the strength of the crystal field anisotropy along the easy axis of magnetization. The external magnetic field $\mathbf{H}(\mathbf{r},t)$ here is the magnetic field component of the propagating EMW, $\gamma = g\mu_B$, where g is the gyromagnetic ratio and μ_B represents the Bohr magneton. In antiferromagnets the neighboring spin vectors will have a strong tendency to align antiparallel to each other due to energetic considerations and hence the problem can be conveniently studied by dividing the lattice into two sublattices corresponding to the "up" ($\mathbf{S}_{u,n}$) and "down" ($\mathbf{S}_{d,n}$) spins. As a result, we have two lattices with a ferromagnetic state with spin up and a ferromagnetic state with spin down. The equation of motion corresponding to the spin Hamiltonian (1) can be constructed from

$$\frac{d\mathbf{S}_n}{dt} = \{\mathbf{S}_n, H\}_{PB}. \tag{2}$$

The Poisson bracket on the right-hand side of Eq. (2) for any two arbitrary functions F and G of spins is defined as

$$\{F,G\}_{PB} = \sum_{i=1}^{N} \sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \frac{\partial F}{\partial S_i^\alpha} \frac{\partial G}{\partial S_i^\beta} S_i^\gamma, \tag{3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the complete antisymmetric Levi-Civita tensor. The above spin Poisson bracket satisfies the same algebraic relations as that of the usual canonical Poisson bracket. On using our spin Hamiltonian Eq. (1) in Eq. (2), we obtain two equations of motion for the spin vectors $\mathbf{S}_{u,n}$ and $\mathbf{S}_{d,n-1}$ on up and down lattices as

$$\frac{d\mathbf{S}_{u,n}}{dt} = \mathbf{S}_{u,n} \times \left[J(\mathbf{S}_{d,n+1} + \mathbf{S}_{d,n-1}) + D_n^z\left[\left(S_{d,n+1}^y - S_{d,n-1}^y\right)\hat{i} - \left(S_{d,n+1}^x - S_{d,n-1}^x\right)\hat{j}\right] - 2A\left(S_{u,n}^z\right)\hat{n} + \gamma\mathbf{H}\right], \tag{4}$$

$$\frac{d\mathbf{S}_{d,n-1}}{dt} = \mathbf{S}_{d,n-1} \times \left[J(\mathbf{S}_{u,n} + \mathbf{S}_{u,n-2}) + D_n^z\left[\left(S_{u,n}^y - S_{u,n-2}^y\right)\hat{i} - \left(S_{u,n}^x - S_{u,n-2}^x\right)\hat{j}\right] + 2A\left(S_{d,n-1}^z\right)\hat{n} + \gamma\mathbf{H}\right]. \tag{5}$$

The structure of Eqs. (4) and (5) demand that the length of the spin vector does not change with time and hence all the spins are assumed to have unit length ($\mathbf{S}_n^2 = 1$). Now, in order to understand the associated antiferromagnetic spin dynamics, we have to solve the discrete spin equations. However, in the low temperature and long wavelength limit one can go to the continuum limit by assuming that the lattice constant is very small compared with the length of the lattice. Therefore it is appropriate to make a continuum approximation and by making use of the Taylor series expansion with $z = n\lambda$, where λ is the lattice parameter, we get

$$\frac{\partial \mathbf{S}_u}{\partial t} = \mathbf{S}_u \times \left\{ 2J\left(\mathbf{S}_d + \lambda\frac{\partial \mathbf{S}_d}{\partial z}\right) + D^z\left[2\lambda\left(\frac{\partial S_d^y}{\partial z}\hat{i} - \frac{\partial S_d^x}{\partial z}\hat{j}\right)\right] - 2AS_u^z\hat{n} + \gamma\mathbf{H}\right\}, \tag{6}$$

and

$$\frac{\partial \mathbf{S}_d}{\partial t} = \mathbf{S}_d \times \left\{ 2J\left(\mathbf{S}_u + \lambda\frac{\partial \mathbf{S}_u}{\partial z}\right) + D^z 2\lambda\left[\left(\frac{\partial S_u^y}{\partial z}\hat{i} - \frac{\partial S_u^x}{\partial z}\hat{j}\right)\right] + 2AS_d^z\hat{n} + \gamma\mathbf{H}\right\}. \tag{7}$$

The above set of equations describes the dynamics of weak antiferromagnetic canted spin system of two sublattices up and down in the classical continuum limit, which is analogous to the Landau-Lifshitz equations for ferromagnets. Addition of Eqs. (6) and (7) yields

$$\frac{\partial(\mathbf{S}_u+\mathbf{S}_d)}{\partial t} = 2J\lambda\left[\mathbf{S}_u\times\frac{\partial\mathbf{S}_d}{\partial z}+\frac{\partial\mathbf{S}_u}{\partial z}\times\mathbf{S}_d\right]+2\lambda D^z\left[\mathbf{S}_u\times\frac{\partial\left(S_d^y\hat{i}-S_d^x\hat{j}\right)}{\partial z}+\mathbf{S}_d\times\frac{\partial\left(S_u^y\hat{i}-S_u^x\hat{j}\right)}{\partial z}\right]$$
$$+2A\left[\left(\mathbf{S}_d\times S_d^z\hat{n}\right)-\left(\mathbf{S}_u\times S_u^z\hat{n}\right)\right]+\gamma(\mathbf{S}_u+\mathbf{S}_d)\times\mathbf{H} \tag{8}$$

and subtraction of Eq. (7) from Eq. (6) gives

$$\frac{\partial(\mathbf{S}_u-\mathbf{S}_d)}{\partial t} = 4J(\mathbf{S}_u\times\mathbf{S}_d)+2J\lambda\left[\mathbf{S}_u\times\frac{\partial\mathbf{S}_d}{\partial z}-\frac{\partial\mathbf{S}_u}{\partial z}\times\mathbf{S}_d\right]+2\lambda D^z\left[\mathbf{S}_u\times\frac{\partial S_d^y}{\partial z}\hat{i}-\mathbf{S}_d\times\frac{\partial S_u^y}{\partial z}\hat{i}\right]$$
$$+2\lambda D^z\left[\mathbf{S}_d\times\frac{\partial S_u^x}{\partial z}\hat{j}-\mathbf{S}_u\times\frac{\partial S_d^x}{\partial z}\hat{j}\right]-2A\left[\left(\mathbf{S}_d\times S_d^z\hat{n}\right)+\left(\mathbf{S}_u\times S_u^z\hat{n}\right)\right]+\gamma(\mathbf{S}_u-\mathbf{S}_d)\times\mathbf{H}. \tag{9}$$

In order to make the above two equations in a more standard form, we define [43]

$$(\mathbf{S}_u-\mathbf{S}_d)=2\sqrt{(1-\epsilon^2)}S\mathbf{M},\quad(\mathbf{S}_u+\mathbf{S}_d)=2\epsilon S\mathbf{M}', \tag{10}$$

where $\mathbf{M}$ and $\mathbf{M}'$ are unit vectors with $\mathbf{M}\cdot\mathbf{M}'=0$, $\epsilon^2=\frac{1}{2}(1+\frac{\mathbf{S}_u\cdot\mathbf{S}_d}{S^2})$ and S is the magnitude of the spin. Further, substitution of Eq. (10) in Eq. (9) and calculating individually the terms in the right hand side of the above Eq. (9) yields

$$\mathbf{S}_u\times\mathbf{S}_d=2\epsilon\sqrt{1-\epsilon^2}S^2(\mathbf{M}\times\mathbf{M}'),\ \mathbf{S}_u\times\frac{\partial\mathbf{S}_d}{\partial z}-\frac{\partial\mathbf{S}_u}{\partial z}\times\mathbf{S}_d=S^2\left[2\epsilon^2\left(\mathbf{M}'\times\frac{\partial\mathbf{M}'}{\partial z}\right)-2(1-\epsilon^2)\left(\mathbf{M}\times\frac{\partial\mathbf{M}}{\partial z}\right)\right], \tag{11}$$

$$\mathbf{S_u}=S(\epsilon\mathbf{M}'+\sqrt{1-\epsilon^2}\mathbf{M}),\ S_u^x\hat{i}+S_u^y\hat{j}+S_u^z\hat{k}=S\{\epsilon(M'^x\hat{i}+M'^y\hat{j}+M'^z\hat{k})+\sqrt{1-\epsilon^2}(M^x\hat{i}+M^y\hat{j}+M^z\hat{k})\}, \tag{12}$$

$$\mathbf{S_d}=S(\epsilon\mathbf{M}'-\sqrt{1-\epsilon^2}\mathbf{M}),\ S_d^x\hat{i}+S_d^y\hat{j}+S_d^z\hat{k}=S\{\epsilon(M'^x\hat{i}+M'^y\hat{j}+M'^z\hat{k})-\sqrt{1-\epsilon^2}(M^x\hat{i}+M^y\hat{j}+M^z\hat{k})\}. \tag{13}$$

Substituting the above set of equations (11)–(13) in Eq. (9) and at low energy configurations $|\mathbf{S_u}-\mathbf{S_d}|\approx 2S$ and $|\mathbf{S_u}+\mathbf{S_d}|\simeq 0$ corresponding to $\epsilon\ll 1$ yields

$$2S\frac{\partial\mathbf{M}}{\partial t}=2J\lambda S^2\left[-2\left(\mathbf{M}\times\frac{\partial\mathbf{M}}{\partial z}\right)\right]+2\lambda D^zS^2\left\{-\left(\mathbf{M}\times\frac{\partial M^y}{\partial z}\hat{i}\right)-\left(\mathbf{M}\times\frac{\partial M^x}{\partial z}\hat{j}\right)-\left(\mathbf{M}\times\frac{\partial M^y}{\partial z}\hat{i}\right)-\left(\mathbf{M}\times\frac{\partial M^x}{\partial z}\hat{j}\right)\right\}$$
$$-4AS^2(\mathbf{M}\times M^z\hat{n})+2\gamma S(\mathbf{M}\times\mathbf{H}). \tag{14}$$

After cumbersome algebraic calculations and suitable rescaling of $t\to 2St$ with redefinition of parameters, we obtain the magnetization density $\mathbf{M}$, which is governed by the torque equation given as

$$\frac{\partial\mathbf{M}}{\partial t}=\mathbf{M}\times\left[\hat{\gamma}\mathbf{H}-J\lambda\frac{\partial\mathbf{M}}{\partial z}-AM^z\hat{n}-\lambda D^z\frac{\partial(M^y\hat{i}-M^x\hat{j})}{\partial z}\right], \tag{15}$$

where $\hat{\gamma}=\gamma/2S$ and Eq. (15) has been written under the assumption of the low energy configurations in the presence of weak ferromagnetism when $\epsilon\ll 1$. Equation (15) is analogous to the Landau-Lifshitz equation for ferromagnets [44]. However, the contribution of spin-spin exchange interaction to the effective field in ferromagnets appears to be $\frac{\partial^2\mathbf{M}}{\partial z^2}$, whereas in our case of a weak antiferromagnet the effective field from bilinear interaction appears to be $\frac{\partial\mathbf{M}}{\partial z}$, since the continuum model approximation is treated for the two sublattices individually.

We consider the propagation of electromagnetic waves in a magnetic material medium in the presence of an external magnetic field. The governing Maxwell's equations are the following [45]:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{16}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}. \tag{17}$$

In Eqs. (16) and (17), $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$ are, respectively, the electric field, the electric induction, the magnetic field, and the magnetic induction. The constitutive equation for $\mathbf{E}$ and $\mathbf{D}$ is

$$\mathbf{D}=\epsilon_0\mathbf{E}, \tag{18}$$

where we shall assume that ϵ_0 is the scalar permittivity of the magnetic medium, whereas the constitutive equation for $\mathbf{H}$ and $\mathbf{B}$ is

$$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}), \tag{19}$$

where μ_0 is the magnetic permeability of the medium and $\mathbf{M}$ is the magnetization density in the magnetic medium of propagation. Eliminating $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{B}$, from Eqs. (16)–(18), we have

$$\frac{\partial^2}{\partial t^2}[\mathbf{H}+\mathbf{M}] = c^2\left[\frac{\partial^2 \mathbf{H}}{\partial z^2} - \frac{\partial^2 H^z}{\partial z^2}\hat{n}\right], \tag{20}$$

where $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ is the velocity of the EMWs propagating in the magnetic medium with $\hat{n} = (0,0,1)$. The set of coupled equations (15) and (20) completely describe the propagation of EMWs in an anisotropic ferromagnetic medium with DM interaction especially when the adjacent antiparallel spin pairs are locked under the low energy configurations.

III. PERTURBATION SCHEME AND THE GOVERNING DYNAMICAL EQUATION

Having derived the equations of motion, the task now lies in solving them to understand the underlying spin excitations. We find Eqs. (15) and (20) as a set of highly nontrivial nonlinear coupled partial differential equations which are not amenable to exact analysis in general. In this section, we attempt to solve the coupled Landau-Lifshitz equations for magnetization and the Maxwell's equations for electromagnetic field within the framework of reductive perturbation method along the lines of Taniuti and Yajima [46]. This technique adopts the nonlinear modulation of the slowly varying envelopes of EM plane waves of small but finite amplitude in the antiferromagnetic medium. In order to carry out this perturbation, the magnetization of the medium and the magnetic induction of the EM field have to be expanded nonuniformly in the anisotropic weak antiferromagnetic medium. Since the easy axis of magnetization of the anisotropic medium lies parallel to the direction of propagation (z direction), we assume that at the lowest order of expansion, the magnetization of the medium and the magnetic induction lie parallel to the propagation axis and turn around to the $(x-y)$ plane at higher orders. Therefore writing $\mathbf{M} = (M_x, M_y, M_z)$ and $\mathbf{H} = (H_x, H_y, H_z)$ and expressing the Fourier components of $\mathbf{M}$ and $\mathbf{H}$ in powers of a small parameter ϵ, we write

$$\begin{aligned} M^x &= \epsilon^{1/2}\left[M_1^x + \epsilon M_2^x + \cdots\right],\\ M^y &= \epsilon^{1/2}\left[M_1^y + \epsilon M_2^y + \cdots\right],\\ M^z &= M_0 + \epsilon M_1^z + \epsilon^2 M_2^z + \cdots, \end{aligned} \tag{21}$$

and

$$\begin{aligned} H^x &= \epsilon^{1/2}\left[H_1^x + \epsilon H_2^x + \cdots\right],\\ H^y &= \epsilon^{1/2}\left[H_1^y + \epsilon H_2^y + \cdots\right],\\ H^z &= H_0 + \epsilon H_1^z + \epsilon^2 H_2^z + \cdots. \end{aligned} \tag{22}$$

Thus the magnetization and magnetic field intensity have been expanded about uniform values M_0 and H_0, respectively, along the direction of propagation of the EMWs.

We now introduce the stretching variables $\xi = \epsilon(z - vt)$ and $\tau = \epsilon^3 t$, which characterize the shape of the pulse propagating at the speed v and time variable accounts for the evolution of the propagating pulse. We now substitute the expansions for $\mathbf{M}$ and $\mathbf{H}$ as given in Eqs. (21) and (22) in the component form of Eqs. (15) and (20). After collecting and solving the coefficients at different orders of ϵ, we get the following:

At the order ϵ^0

$$H_1^x = kM_1^x, \tag{23a}$$

$$H_1^y = kM_1^y \tag{23b}$$

where

$$k \equiv \frac{H_0}{M_0} = \frac{1}{\epsilon_0(c^2 - v^2)}. \tag{23c}$$

At the order ϵ^1

$$\frac{\partial}{\partial \xi}\left[H_2^x - kM_2^x\right] = -\frac{\partial H_1^x}{\partial \tau}, \tag{24a}$$

$$\frac{\partial}{\partial \xi}\left[H_2^y - kM_2^y\right] = -\frac{\partial H_1^y}{\partial \tau}, \tag{24b}$$

also

$$\frac{\partial M_1^z}{\partial \xi} = \frac{J\lambda}{v}\left[M_1^x\frac{\partial M_1^y}{\partial \xi} - M_1^y\frac{\partial M_1^x}{\partial \xi}\right] + \frac{1}{v}\left[M_1^x\int_{-\infty}^{\xi}\frac{\partial M_1^y}{\partial \tau}d\xi' - M_1^y\int_{-\infty}^{\xi}\frac{\partial M_1^x}{\partial \tau}d\xi'\right] + \lambda D^z\left[M_1^x\frac{\partial M_1^x}{\partial \xi} - M_1^y\frac{\partial M_1^y}{\partial \xi}\right], \tag{25a}$$

$$\frac{\partial M_1^x}{\partial \xi} = \frac{\hat{\gamma}k}{v}(M_1^z M_1^y) - \frac{J\lambda}{v}\left(M_0\frac{\partial M_1^y}{\partial \xi}\right) + \frac{A}{v}(M_1^y M_0) - \frac{\lambda D^z}{v}\left(M_0^2\frac{\partial M_1^y}{\partial \xi}\right) + \frac{M_0}{v}\int_{-\infty}^{\xi}\frac{\partial M_1^y}{\partial \tau}d\xi', \tag{25b}$$

$$\frac{\partial M_1^y}{\partial \xi} = -\frac{\hat{\gamma}k}{v}(M_1^z M_1^x) + \frac{J\lambda}{v}\left(M_0\frac{\partial M_1^x}{\partial \xi}\right) - \frac{A}{v}(M_1^x M_0) + \frac{\lambda D^z}{v}\left(M_0^2\frac{\partial M_1^x}{\partial \xi}\right) - \frac{M_0}{v}\int_{-\infty}^{\xi}\frac{\partial M_1^x}{\partial \tau}d\xi'. \tag{25c}$$

Without loss of generality, we assume a new complex field q in order to identify Eqs. (25b) and (25c) with a more standard nonlinear evolution equation by defining

$$q = \left(M_1^x - iM_1^y\right); \tag{26}$$

using q in the relation for the conservation of length of the magnetization vector

$$M_1^z = -H_1^z = \frac{1}{2M_0}\left[\left(M_1^x\right)^2 + \left(M_1^y\right)^2\right], \tag{27}$$

we obtain

$$|q|^2 = -2M_0 M_1^z. \tag{28}$$

After a single differentiation of Eqs. (25b) and (25c) and using the new complex field as given in Eqs. (26) and (28) with the transformation of $Z = \xi + A\tau$, we obtain the resultant equation after some lengthy algebra as

$$i\frac{\partial q}{\partial \tau} + \eta\frac{\partial^2 q}{\partial Z^2} + i\beta\frac{\partial}{\partial Z}|q|^2 q = 0, \tag{29a}$$

where

$$\eta = (1 - i\beta' D^z), \tag{29b}$$

$$\beta = -\hat{\gamma} k \left[M_0 \left(iJ\lambda + \frac{v}{M_0} \right) \right]^{-1/2}, \tag{29c}$$

$$\beta' = -i\lambda[(iJ\lambda M_0 + v)]^{-1/2}, \tag{29d}$$

while writing Eq. (29a), we have rescaled $Z \to (-iJ\lambda - \frac{v}{M_0})^{1/2} Z$. Equation (29a) resembles the well known completely integrable derivative nonlinear Schrödinger (DNLS) equation, which has been solved for soliton solutions by Kaup and Newell [47] using the inverse scattering transformation (IST) method in the absence of DM interaction ($D^z = 0$).

IV. PROPAGATING ELECTROMAGNETIC SOLITON

In this section, we make our attempt to construct a set of exact propagating electromagnetic solitons for the derivative NLS Eq. (29a) in the presence of DM interaction. The completely integrable DNLS equation was first given by Rogister [48] for the nonlinear evolution of parallel Alfvén waves in plasmas and later encountered in many different contexts by other authors, emerging as one of the canonical nonlinear equations in physics. Kundu demonstrated the explicit auto-Bäcklund relation for the DNLS equation through gauge transformation in the absence of weak ferromagnetism [49] when the velocity of the propagating magnetization pulse $v = \hat{\gamma}^2 k^2 - iM_0 J\lambda$ and find the one-soliton solution as

$$q = \pm 4\triangle \sin\gamma' \left\{ \frac{\exp[(2\eta' - 2i\xi)x - i\mu^+]}{\exp(4\eta' x) + \exp(\pm i\gamma')} \right\}, \tag{30}$$

for $\epsilon = -1$, $\eta' = \triangle^2 \sin\gamma'$; $\xi = \mp\triangle^2 \cos\gamma'$.

More recently the DNLS equation, which governs the propagation of the femtosecond optical pulse in a monomodal optical fiber, is analytically studied [50] and breather as well as double-pole solutions are derived from the two-soliton solution with the choices of certain physical parameters. The periodic wave solutions and the solitary wave solutions in terms of Jacobi elliptic functions for the nonlinear partial differential equations attract considerable interest [51–54], because of the elegant properties of elliptic functions. We employ the Jacobi elliptic function method aided with the symbolic computation to find a series of exact solutions governing the electromagnetic solitons and will explore the role of DM interaction parameter on the propagation of EMWs. Symbolic computation, as a new branch of artificial intelligence, has been playing an important role in dealing with a large amount of complicated and tedious algebraic calculations. Let us assume for the sake of simplicity $Z \to z$ and $\tau \to t$, so that Eq. (29a) becomes

$$i\frac{\partial q}{\partial t} + \eta \frac{\partial^2 q}{\partial z^2} + i\beta \frac{\partial}{\partial z}|q|^2 q = 0. \tag{31}$$

We seek the traveling wave solution of the form

$$q(z,t) = u(\xi')\exp[i(rz + st)], \tag{32}$$

where $\xi' = k_1 z + \omega t$, k_1 and ω represent the wave number and wave speed, respectively, whereas r and s are arbitrary constants. Using the Jacobi elliptic function method, $u(\xi')$ can be expressed as a finite series of Jacobi elliptic functions $\mathrm{sn}(\xi'|m)$ or $\mathrm{cn}(\xi'|m)$ using the following ansatz:

$$u = u(\xi') = \sum_{j=0}^{n} a_j \mathrm{cn}^j(\xi'|m), \tag{33}$$

or

$$u = u(\xi') = \sum_{j=0}^{n} a_j \mathrm{sn}^j(\xi'|m), \tag{34}$$

where $\mathrm{sn}(\xi'|m)$ and $\mathrm{cn}(\xi'|m)$ are the Jacobi elliptic sine and cosine functions, respectively. The parameter m is the modulus $(0 < m < 1)$ and when $m \to 1, \mathrm{sn}(\xi'|m) \to \tanh(\xi'), \mathrm{cn}(\xi'|m) \to \mathrm{sech}(\xi')$. Upon substituting Eq. (32) in Eq. (31) the real and imaginary parts of the equation can be expressed as follows:

$$-su(\xi') + k_1^2 \frac{d^2u(\xi')}{d\xi'^2} - r^2 u(\xi') + 2\beta' D^z r k_1 \frac{du(\xi')}{d\xi'} - \beta r u(\xi')^3 = 0, \tag{35a}$$

$$\omega \frac{du(\xi')}{d\xi'} + 2rk_1 \frac{du(\xi')}{d\xi'} - \beta' D^z k_1^2 \frac{d^2u(\xi')}{d\xi'^2} + \beta' D^z r^2 u(\xi') + 3\beta k_1 u(\xi')^2 \frac{du(\xi')}{d\xi'} = 0. \tag{35b}$$

Here the highest degree of $\frac{d^p u}{d\xi'^p}$ is taken as

$$O\left(\frac{d^p u}{d\xi'^p}\right) = n + p, O\left(u^q \frac{d^p u}{d\xi'^p}\right) = (q+1)n + p, \tag{36}$$

where $p = 1,2,3,\ldots$ and $q = 0,1,2,3,\ldots$. Balancing the higher order linear term with the nonlinear term in Eqs. (35a) and (35b) will yield the value of $n = 1$ in Eq. (36). Upon substituting Eq. (33) in Eqs. (35a) and (35b) with $n = 1$, we obtain a system of algebraic equations as presented in Appendix A. Now solving the set of equations with the aid of symbolic computation yields

$$\omega = \omega, \quad a_0 = a_0, \quad s = -r^2 + k_1^2 m^2 \mathrm{sn}(\xi'|m)^2 - k^2 \mathrm{dn}(\xi'|m)^2 - 3\beta r a_0^2,$$

$$a_1 = \frac{1}{2} \frac{a_0\left(-k_1^2 m^2 \mathrm{sn}(\xi'|m)^2 \mathrm{dn}(\xi'|m)^2 + 2\beta r a_0^2\right)}{\beta' D^z r k_1 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m)}.$$

Thus the exact soliton solution of Eq. (31) is given by

$$q(z,t) = \left(a_0 + \frac{1}{2} \frac{a_0\left[-k_1^2 m^2 \mathrm{sn}(\xi'|m)^2 \mathrm{dn}(\xi'|m)^2 + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m)} \right) \times \exp[i(rz + st)]. \tag{37}$$

As $m \to 1$, the Jacobi solution as represented in Eq. (37) degenerates into the solitary wave solution given as

$$q(z,t) = \left(a_0 + \frac{1}{2} \frac{a_0\left[-k_1^2 \tanh^2(\xi') + k_1^2 \mathrm{sech}^2(\xi') + 2\beta r a_0\right]^2}{\beta' D^z r k_1 \mathrm{sech}(\xi')\tanh(\xi')} \right) \times \exp[i(rz + st)]. \tag{38}$$

In Fig. 1, we have plotted $q(z,t)$ represented as Eq. (38) as a function of the DM interaction parameter D^z for the

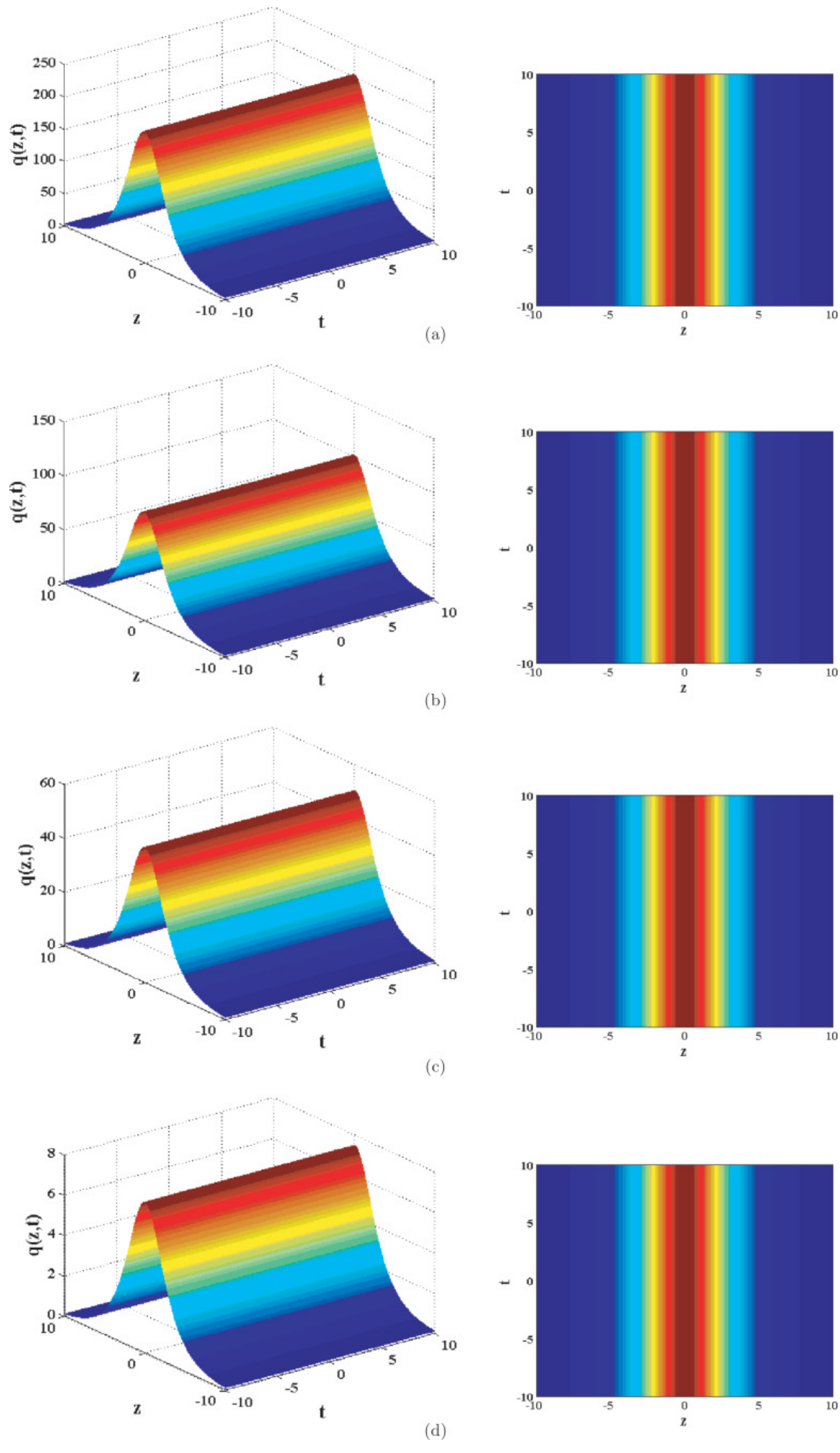

FIG. 1. (Color online) Snapshots of EM soliton Eq. (38) propagating in the antiferromagnetic medium with parametric values $a_0 = 0.000\,05$, $k_1 = 0.01$, $\beta = 1.5$, $\beta' = 0.5$, $r = 1.5$, $\omega = 0.5$; (a) $D^z = 0.1$, (b) $D^z = 0.5$, (c) $D^z = 1.0$, and (d) $D^z = 2.5$ depict the modulation of soliton amplitude in accordance with the DM interaction parameter.

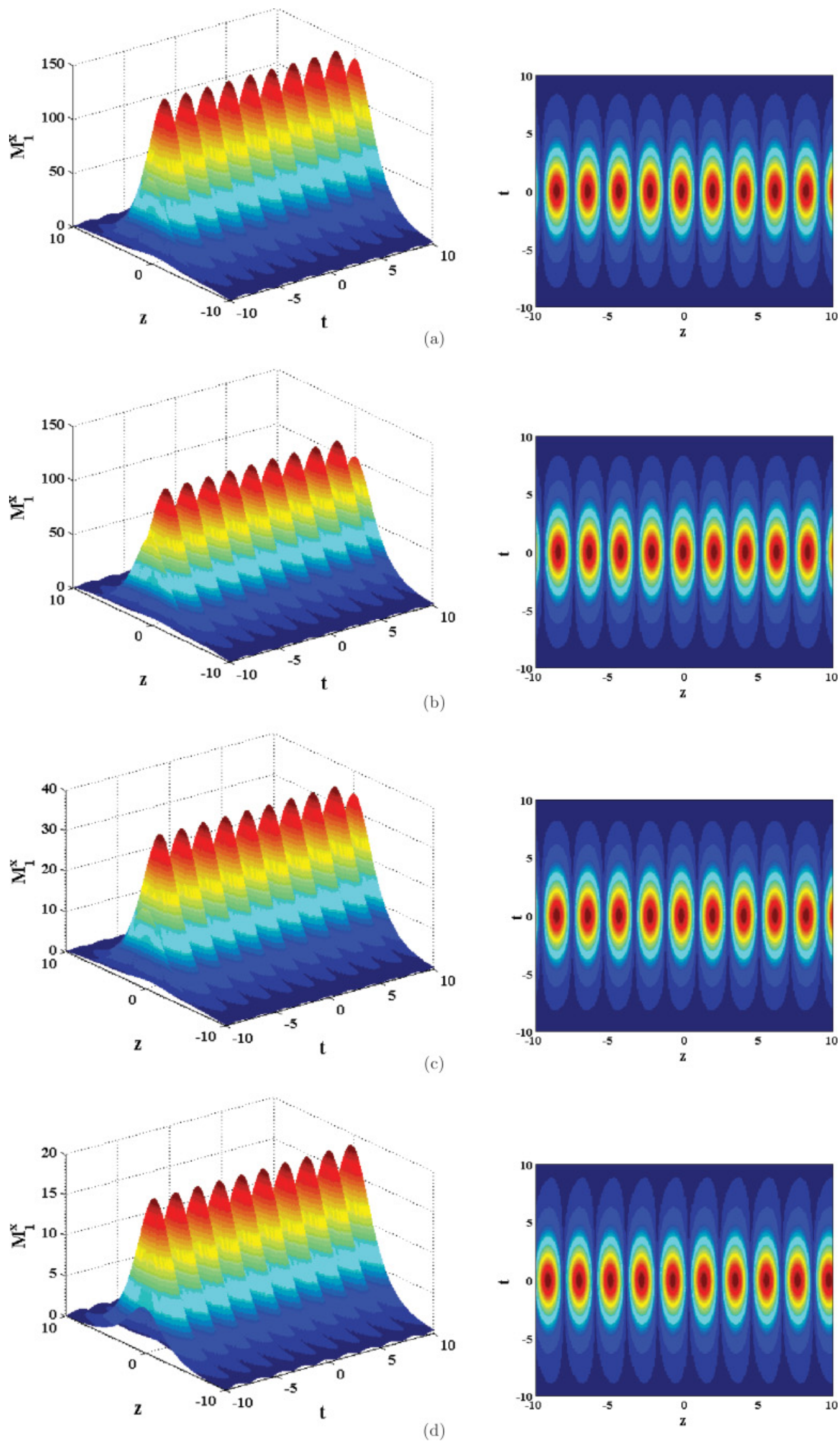

FIG. 2. (Color online) Snapshots of breathing localized modes (BLMs) of M_1^x [Eq. (40)] with parametric values $a_0 = 0.001$, $k_1 = 0.01$, $\beta = 1.0$, $\beta' = 1.5$, $r = 1.5$, $\omega = 0.5$; (a) $D^z = 0.5$, (b) $D^z = 1.5$, (c) $D^z = 2.0$, and (d) $D^z = 2.5$ portray the significant effect of spin canting on the amplitude of breathing modes.

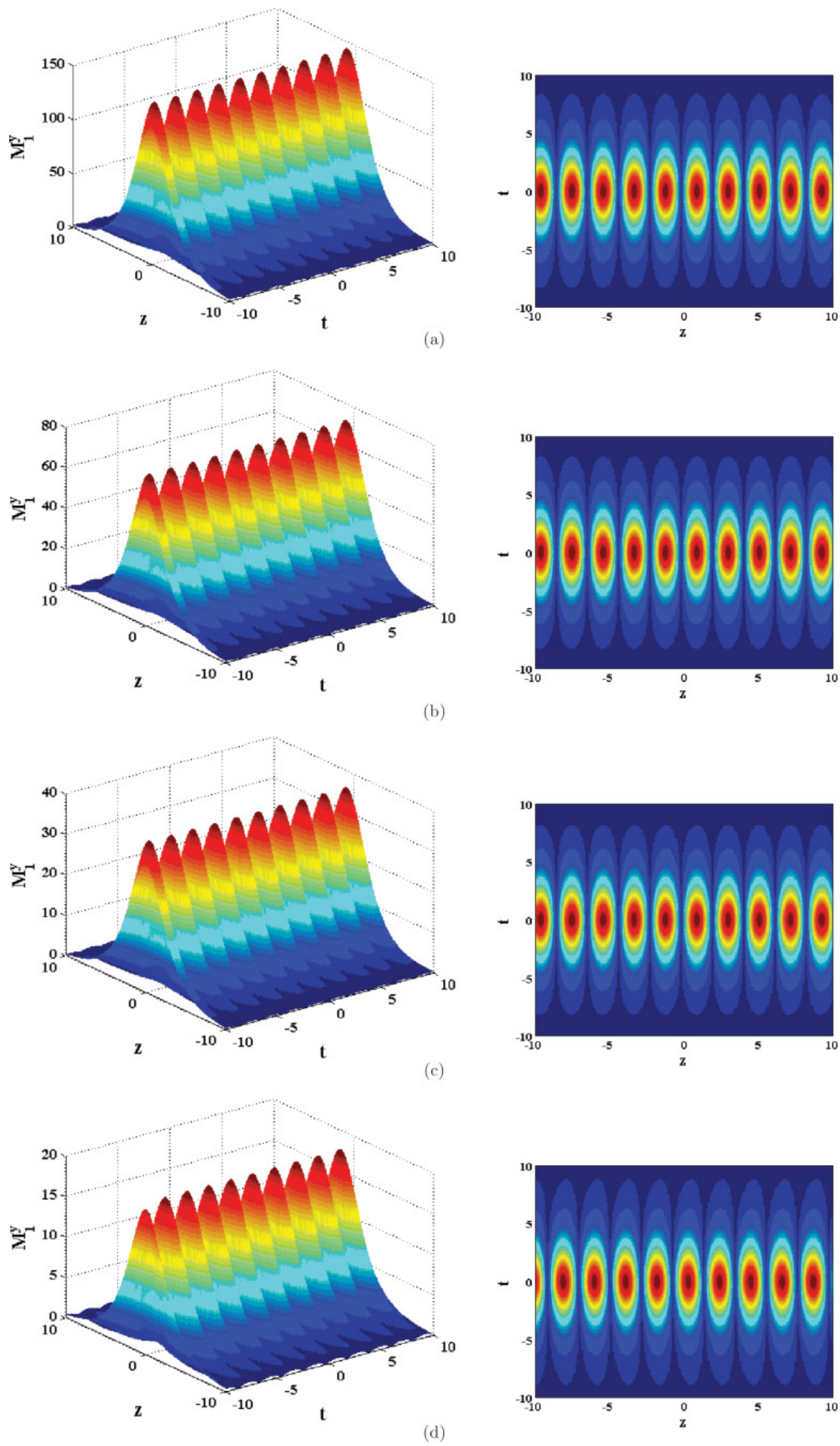

FIG. 3. (Color online) Snapshots of breathing localized modes of M_1^y [Eq. (41)] with parametric values $a_0 = 0.001$, $k_1 = 0.01$, $\beta = 1.0$, $\beta' = 1.5$, $r = 1.5$, $\omega = 0.5$; (a) $D^z = 0.5$, (b) $D^z = 1.0$, (c) $D^z = 2.0$, and (d) $D^z = 2.5$ confirm the inverse relationship between the amplitude of BLMs and the degree of spin canting.

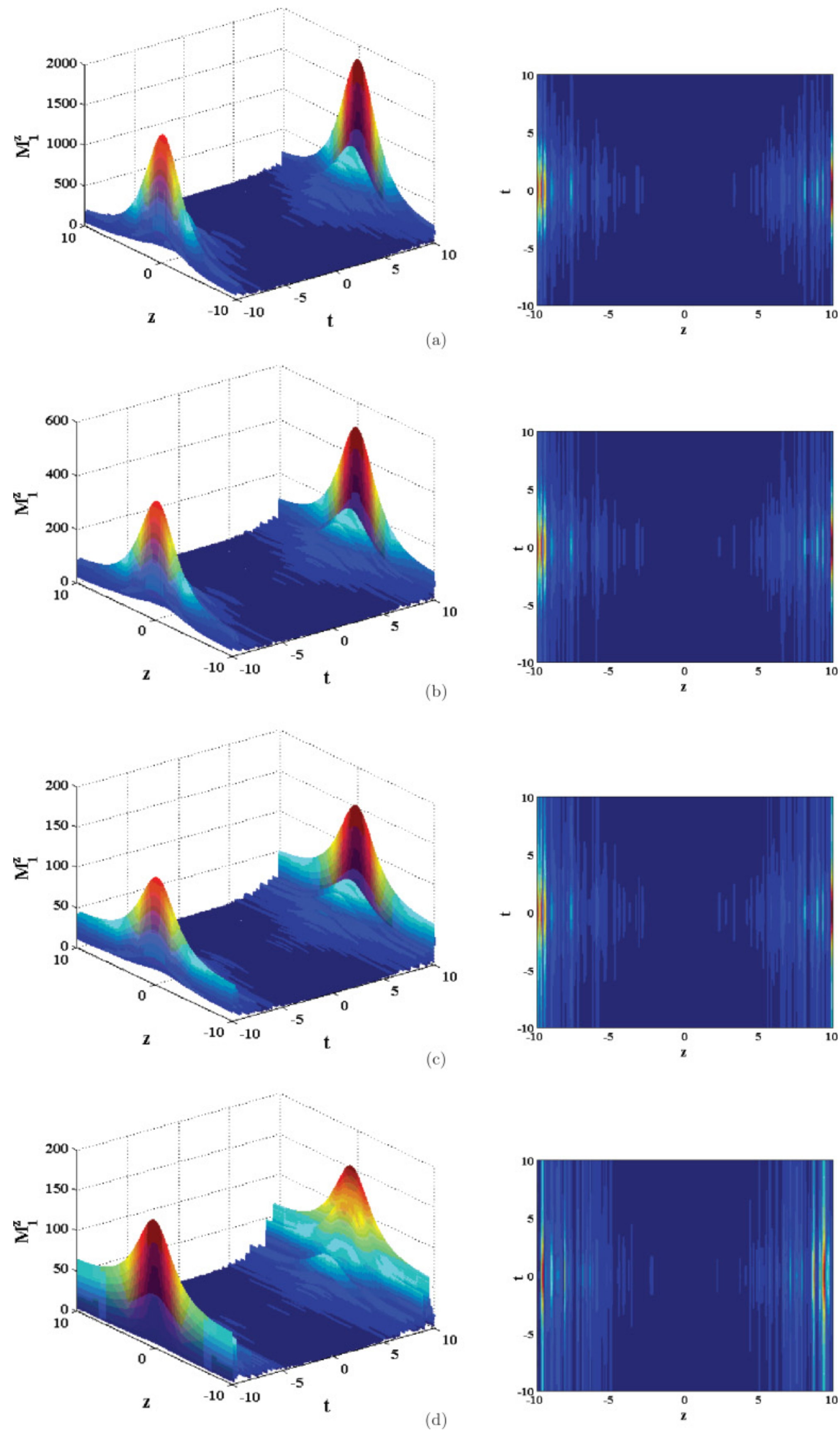

FIG. 4. (Color online) Evolution of antisymmetric BLMs of M_1^z for the choices of parameters $a_0 = 0.001$, $k_1 = 0.01$, $\beta = 1.5$, $\beta' = 2.0$, $r = 1.5$, $\omega = 0.5$, $M_0 = 0.03$; (a) $D^z = 0.5$, (b) $D^z = 1.0$, (c) $D^z = 2.0$, and (d) $D^z = 2.5$ show an appreciable effect of DM interaction on the amplitude.

parameteric values $a_0 = 0.000\,05$, $k_1 = 0.01$, $\beta = 1.5$, $\beta' = 0.5$, $r = 1.5$, and $\omega = 0.5$. At the outset it is interesting to observe from the snapshots that the EMW soliton propagates in the weak antiferromagnetic medium without any change in its shape. However, when the DM interaction parameter D^z is slowly enhanced from 0.1 to 2.5, the amplitude of the EMW soliton remarkably decreases anyhow keeping the bell shape intact. It is also evident that the amplitude of EMW soliton is inversely proportional to the strength of the DM interaction. The stronger the DM interaction is, the shorter the EMW soliton appears. Thus physically it is realized that in the antiferromagnetic medium, the spin canting as a result of weak ferromagnetism modulates the amplitude of the EMW soliton without distorting the robust nature of the same.

Further, using the Jacobi sine function method we construct the other type of soliton solution as (for details see Appendix B)

$$q(z,t) = \left(a_0 - \frac{1}{2}\frac{a_0\left[2k_1^2\mathrm{sech}^2(\xi') + 2\beta r a_0\right]^2}{\beta' D^z r k_1 \mathrm{sech}^2(\xi')} \right) \exp[i(rz+st)]. \tag{39}$$

The solution Eq. (39) also exhibit the localized solitary nature of the EMW propagation in the weak antiferromagnetic medium. Upon substituting the solution Eq. (38) in Eq. (26), we can write down the one-soliton solution for the x,y, and z components of the magnetization ($\mathbf{M}_1$) or in another way the magnetic field components of EM field ($\mathbf{H}_1$) from Eqs. (26) and (27).

$$M_1^x = \left(a_0 + \frac{1}{2}\frac{a_0\left[-k_1^2\tanh^2(\xi) + k_1^2\mathrm{sech}^2(\xi) + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{sech}(\xi)\tanh(\xi)} \right) \times\cos(rz+st), \tag{40}$$

$$M_1^y = \left(a_0 + \frac{1}{2}\frac{a_0\left[-k_1^2\tanh^2(\xi) + k_1^2\mathrm{sech}^2(\xi) + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{sech}(\xi)\tanh(\xi)} \right) \times\sin(rz+st), \tag{41}$$

$$M_1^z = -\frac{1}{2M_0}\left(a_0 + \frac{1}{2} \times \frac{a_0\left[-k_1^2\tanh^2(\xi) + k_1^2\mathrm{sech}^2(\xi) + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{sech}(\xi)\tanh(\xi)} \right)^2. \tag{42}$$

In Fig. 2, we have exhibited the snapshots of the evolution of M_1^x for the choices of parameters $a_0 = 0.001$, $k_1 = 0.01$, $\beta = 1.0$, $\beta' = 1.5$, $r = 1.5$, and $\omega = 0.5$. We vary the value of the DM interaction parameter and thus physically the degree of canting of spins in the antiferromagnetic medium. It is surprising to observe that as we vary the DM parameter, the soliton amplitude evolves periodically along the longitudinal direction and subsequently develops into breatherlike collective periodically oscillating localized modes. Figure 3 corresponds to the evolution of M_1^y, which also exhibits the same type of breatherlike spatially and temporally localized modes. From the figures it has been realized that the intrinsic antisymmetric coupling present in the antiferromagnetic medium can significantly influence the properties of nonlinear excitations. Therefore under the influence of antisymmetric DM spin coupling, the magnetization of the magnetic medium generates breathing modes of excitations and also the propagating plane EMW is modulated in the form of breatherlike stable excitations. The z component of the magnetization of the antiferromagnetic medium displays a similar but antisymmetric breathing mode with a lesser frequency rather than x and y components as shown in Fig. 4. In Cu benzoate AFHC, besides the dynamical incommensurability expected in high fields, an unexpected energy gap in the magnetic excitation spectrum of Cu benzoate was observed to develop as a function of the applied magnetic field and Dender *et al.* [50] suggested that the unexpected gap could be due to the staggered magnetic field generated by the alternating g tensor in Cu benzoate. Later Affleck and Oshikawa [55] and subsequently Essler and Tsvelik claimed that these effects were caused by the staggered fields generated by the Dzyaloshinski-Moriya interaction acting between neighboring spins in Cu benzoate antiferromagnetic spin chain. The experimental proof of the existence of breatherlike excitations in a Cu benzoate antiferromagnetic spin chain with DM interaction in the presence of magnetic field has been observed by Asano *et al.* [39]. Thus experimentally the well defined breatherlike excitation modes were observed directly by an electron spin resonance investigation performed on the Cu benzoate AFHC exposed to a staggered field arising from DM interaction. To conclude, our theoretical investigations bring out a similar breatherlike nonlinear excitation in an antiferromagnetic spin chain in the presence of DM interaction and in the presence of magnetic field which is a component of electromagnetic wave propagating in the magnetic medium.

V. CONCLUSIONS

We performed a systematic analysis on the EMW propagation in an anisotropic antiferromagnetic medium with Dzyaloshinskii-Moriya (DM) interaction. The coupled evolution equations governing the dynamics of EMWs and the magnetization of the antiferromagnetic medium have been mapped onto a generalized derivative nonlinear Schrödinger (GDNLS) equation using reductive perturbation method. We solve the GDNLS equation by invoking the Jacobi elliptic function method, and it is found that the electromagnetic wave propagation is supported by the collective coherent breatherlike solitary excitations. We observed that the spin-orbit induced Dzyloshinsky-Moriya interaction has a profound effect on the nonlinear excitations of the magnetic field component of EMWs and the magnetization of the antiferromagnetic medium in agreement with the earlier experimental observation. Obviously, the antisymmetric nature of the canted antiferromagnetic medium supports breathing stable solitary modes of EMW propagation.

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APPENDIX A

In this appendix, we derive explicitly the soliton solution of Eq. (31) using the Jacobi elliptic function method as explained briefly in Sec. IV. We assume the following ansatz for $u(\xi')$:

$$u(\xi') = a_0 + a_1 \mathrm{cn}(\xi'|m),$$

and substituting this ansatz in Eq. (31) will yield a system of algebraic equations by collecting various powers of $\mathrm{cn}(\xi'|m)$ as follows:

$$\begin{aligned}
&\mathrm{cn}(\xi'|m)^3 : -\beta r a_1^3 = 0,\\
&\mathrm{cn}(\xi'|m)^2 : -3\beta r a_0 a_1^2 = 0,\\
&\mathrm{cn}(\xi'|m)^1 : -r^2 a_1 - s a_1 + k_1^2[a_1 m^2 \mathrm{sn}(\xi'|m)^2\\
&-a_1 \mathrm{dn}(\xi'|m)^2] - 3\beta r a_0^2 a_1 = 0,\\
&\mathrm{cn}(\xi'|m)^0 : -s a_0 - 2\beta' D^z r k_1 a_1 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m)\\
&-\beta r a_0^3 - r^2 a_0 = 0,
\end{aligned}$$

and,

$$\begin{aligned}
&\mathrm{cn}(\xi'|m)^2 : -3\beta k_1 a_1^3 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m) = 0,\\
&\mathrm{cn}(\xi'|m)^1 : -\beta' D^z k_1^2[a_1 m^2 \mathrm{sn}(\xi'|m)^2 - a_1 \mathrm{dn}(\xi'|m)^2]\\
&+\beta' D^z r^2 a_1 - 6\beta k_1 a_0 a_1^2 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m) = 0,\\
&\mathrm{cn}(\xi'|m)^0 : \beta' D^z r^2 a_0 - \left(\omega a_1 + 2 r k_1 a_1 + 3\beta k_1 a_0^2 a_1\right)\\
&\mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m) = 0.
\end{aligned}$$

On solving the above system of equations, we find

$$\begin{aligned}
&\omega = \omega,\, a_0 = a_0,\, s = -r^2 + k_1^2 m^2 \mathrm{sn}(\xi'|m)^2\\
&-k^2 \mathrm{dn}(\xi'|m)^2 - 3\beta r a_0^2,\\
&a_1 = \frac{1}{2}\frac{a_0\left[-k_1^2 m^2 \mathrm{sn}(\xi'|m)^2 \mathrm{dn}(\xi'|m)^2 + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{dn}(\xi'|m)\mathrm{sn}(\xi'|m)},
\end{aligned}$$

and consequently we construct the exact soliton solution as presented in Eq. (38).

APPENDIX B

In a similar way, we construct another type of solution for the Jacobi sine function by assuming the ansatz for $u(\xi')$ as $u(\xi') = a_0 + a_1 \mathrm{sn}(\xi')$ and performing a similar analysis as above will yield a system of equations corresponding to Eq. (31) as follows:

$$\begin{aligned}
&\mathrm{sn}(\xi'|m)^3 : -\beta r a_1^3 = 0,\\
&\mathrm{sn}(\xi'|m)^2 : -3\beta r a_0 a_1^2 = 0,\\
&\mathrm{sn}(\xi'|m)^1 : -r^2 a_1 - s a_1 + k_1^2[-a_1 \mathrm{dn}(\xi'|m)^2\\
&-a_1 \mathrm{cn}(\xi'|m)^2] - 3\beta r a_0^2 a_1 = 0,\\
&\mathrm{sn}(\xi'|m)^0 : -s a_0 + 2\beta' D^z r k_1 a_1 \mathrm{cn}(\xi|m)\mathrm{dn}(\xi|m)\\
&-\beta r a_0^3 - r^2 a_0 = 0,
\end{aligned}$$

also,

$$\begin{aligned}
&\mathrm{sn}(\xi'|m)^2 : 3\beta k_1 a_1^3 \mathrm{cn}(\xi'|m)\mathrm{dn}(\xi'|m) = 0,\\
&\mathrm{sn}(\xi'|m)^1 : -\beta' D^z k_1^2[-a_1 \mathrm{dn}(\xi'|m)^2 - a_1 \mathrm{cn}(\xi'|m)^2 m^2]\\
&+\beta' D^z r^2 a_1 + 6\beta k_1 a_0 a_1^2 \mathrm{cn}(\xi'|m)\mathrm{dn}(\xi'|m) = 0,\\
&\mathrm{sn}(\xi'|m)^0 : \left(\omega a_1 + 2 r k_1 a_1 + 3\beta k_1 a_0^2 a_1\right)\mathrm{cn}(\xi'|m)\mathrm{dn}(\xi'|m)\\
&+\beta' D^z r^2 a_0 = 0.
\end{aligned}$$

Further, we solve the above system of equations using symbolic computation and we find

$$\begin{aligned}
&\omega = \omega,\, a_0 = a_0,\, s = -r^2 - k_1^2 \mathrm{dn}(\xi'|m)^2 - k_1^2 \mathrm{cn}(\xi'|m)^2 m^2\\
&-3\beta r a_0^2,\\
&a_1 = -\frac{1}{2}\frac{a_0\left[k_1^2 \mathrm{dn}(\xi'|m)^2 + k_1^2 \mathrm{cn}(\xi'|m)^2 m^2 + 2\beta r a_0^2\right]}{\beta' D^z r k_1 \mathrm{dn}(\xi'|m)\mathrm{cn}(\xi'|m)},
\end{aligned}$$

and substituting these values we obtain the other exact soliton solution as presented in Eq. (39).

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