Type of spiral wave with trapped ions

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Pattern formation in ultracold quantum systems has recently received a great deal of attention. In this work, we investigate a two-dimensional model system simulating the dynamics of trapped ions. We find a spiral wave that is rigidly rotating, but with a peculiar core region in which adjacent ions oscillate in antiphase. The formation of this spiral wave is ascribed to the excitability previously reported by Lee and Cross. The breakup of the spiral wave is probed and, especially, an extraordinary scenario of the disappearance of the spiral wave, caused by spontaneous expansion of the antiphase core, is unveiled.

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I. INTRODUCTION

Spiral waves are the most frequently encountered pattern formation in two-dimensional systems far away from equilibrium. They are usually thought to be responsible for the patterns in a wide range of systems (nonlinear optics [1], magnetic films [2], new chemical systems [3], subcellular biology [4], and complex plasma [5]). Spiral waves can display a number of distinct behaviors, some of which are quite complex. The simplest transition in spiral waves is Hopf bifurcation [6], which turns a rigidly rotating spiral wave into a quasiperiodic meandering one. Being another type of common transition, the breakup of spiral waves derives from two different ways: the core breakup due to the Doppler effects [7,8] and the far-field breakup assisted by the absolute Eckhaus one [9,10].

As one type of wave propagation, spiral waves are always explored in systems where the spatial variable is continuous. However, spiral waves also present themselves in systems of coupled oscillators where spatial variables are discrete. In the context of coupled oscillators, the states between adjacent oscillators are not required to be continuous, which may give rise to some impressive phenomena in the dynamics of spiral waves. Kuramoto *et al.* studied two-dimensional nonlocally coupled oscillators and found the existence of spiral wave chimera [11,12]. In a spiral wave chimera, the oscillators in the core region of spiral wave are desynchronized, while those around the periphery of the core are in synchronization. Martens *et al.* analyzed the spiral wave chimera [13]. Yang *et al.* studied two-dimensional locally coupled Rössler oscillators [14]. They found a sandwiched spiral wave in which any two adjacent oscillators are in antiphase and they attributed the presence of the sandwiched spiral wave to the shortwave instability of the homogeneous oscillation of the model [15,16].

Recently, pattern formation in certain ultracold quantum systems that shows the nature of a coupled system has received a great deal of attention. Lee and Cross considered a chain of ions [17], where dissipation is provided by laser heating and cooling and nonlinearity comes from trap anharmonicity and beam shaping. When the nonlinearity and interaction are small perturbations relative to the harmonic motion of ions, they derived an amplitude equation for the ions,

$$\frac{dA_n}{dt} = ib(A_{n-1} + A_{n+1} - 2A_n) + A_n - (1 + ic)|A_n|^2 A_n, \tag{1}$$

where i is the imaginary unit, b is the coupling between adjacent trapped ions, and c denotes how an ion's amplitude affects its harmonic frequency. The amplitude equation is similar to the complex Ginzburg-Landau equation (CGLE). Different from the CGLE, which includes both reactive and dissipative interactions, the above amplitude equation contains only reactive interaction, since the adjacent ions interact through reactive Coulomb force. Lee and Cross considered the pattern formation and found that the homogeneous oscillation of all ions are expected for $bc > 0$, while short-wavelength waves are expected for $bc < 0$. The most surprising discovery is that, when the homogeneous oscillation for $bc > 0$ is perturbed by a localized pulse of antiphase oscillation, the perturbation would probably travel across the system for a long time before it dies off. This excitability in an oscillatory medium is ascribed to the nature of the reactive interaction.

In a two-dimensional coupled system, the phase singularity (the rotation center) of a spiral wave may be off lattice, and adjacent oscillators on the opposite sides of the phase singularity are always in antiphase, which indicates that there should exist a persistent source for antiphase perturbation. Then it will be of particular interest to delve into how the type of excitability reported by Lee and Cross influences the dynamics of spiral waves on a two-dimensional lattice.

II. MODEL AND NUMERICAL RESULTS

We use a square lattice where ions are trapped to the nodes. The system is described as

$$\begin{aligned}\frac{dA_{i,j}}{dt} &= ib(-4A_{i,j} + A_{i-1,j} + A_{i+1,j} + A_{i,j-1} + A_{i,j+1}) \\ &\quad + A_{i,j} - (1 + ic)|A_{i,j}|^2 A_{i,j},\end{aligned} \tag{2}$$

with $i,j = 1,\ldots,N$. Open boundary conditions as in Ref. [17] are imposed upon the system.

In the following numerical simulation, we apply the fourth-order Runge-Kutta method with $dt = 0.001$. A spiral wave occurs with the initial conditions as follows: the oscillators at

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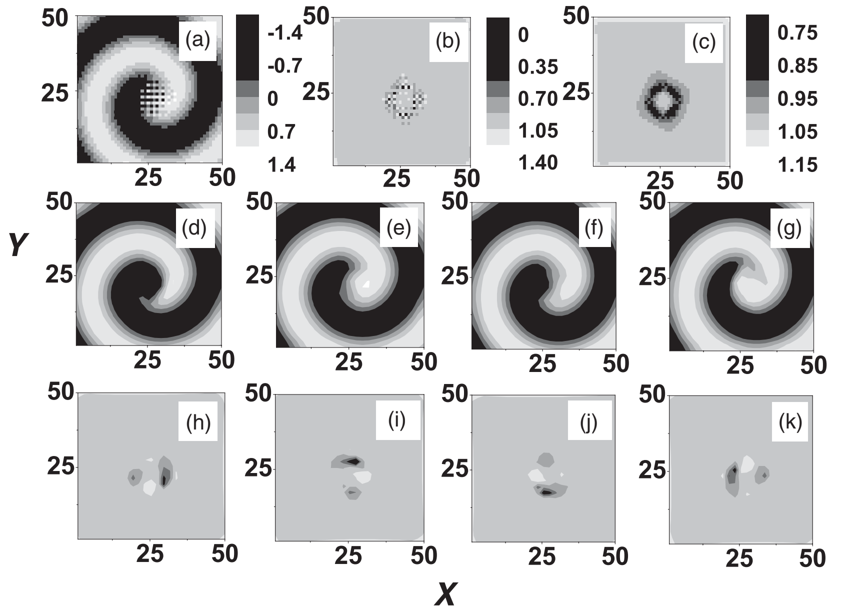

FIG. 1. $b = 1$ and $c = 0.2$. (a) The wave pattern of SWAPC for $\mathrm{Re}(A_{i,j})$. (b) The snapshot of $|A_{i,j}|$. (c) The snapshot of $\langle |A_{i,j}| \rangle_t$. The middle four panels show the snapshots of $\mathrm{Re}(A_{i,j})$ for different subsets and the bottom four show the snapshots of $|A_{i,j}|$ for different subsets. (d) and (h) for the subset in which oscillators locate at positions with $(2i,2j)$. (e) and (i) The subset with $(2i,2j+1)$. (f) and (j) The subset with $(2i+1,2j)$. (g) and (k) The subset with $(2i+1,2j+1)$. The color scale used by (a) [or (b)] are for all plots of $\mathrm{Re}(A_{ij})$ (or $|A_{ij}|$) in this work. $N = 50$.

the boundary have their complex amplitude to be unit one in magnitude and to be from 0 to 2π in phase along the boundary; the amplitudes of all the other oscillators are set to be 0. We first let $b = 1$, $c = 0.2$, and $N = 50$. Stunningly, a different type of spiral wave, a rigidly rotating spiral wave with a core where adjacent oscillators are in antiphase, shows up.

For this different type of spiral wave, two striking features can be revealed by the snapshots of the wave patterns of the real part and the module of $A_{i,j}$ in Fig. 1. First, the spiral wave possesses an odd core where a pattern with the shortest wavelength appears [see Fig. 1(a)]; second, there exists a transition annulus where the oscillation may be weaker (lower $|A_{i,j}|$). It is obvious that the transition annulus divides the wave pattern into the core region and the arm region [see Fig. 1(b)] and, particularly, $|A_{i,j}|$ in the core region is as strong as that in the arm region. In Fig. 1(c), we exhibit the pattern for the time average of $|A_{i,j}|$. Together with Fig. 1(b), Fig. 1(c) illustrates that this type of spiral wave is stable, which is confirmed by the unchanged location of the core. To be mentioned, the dimension of the core of the spiral wave may be defined as that of the transition annulus. Furthermore, we pick up two pairs of adjacent oscillators: one is located in the core region and the other in the arm region. The time evolutions of these four oscillators are monitored. As shown in Figs. 2(a) and 2(b), all of them behave periodically. Nevertheless, it is interesting to find that the phase difference between adjacent oscillators within the core region is around π, which is responsible for the pattern with the shortest wavelength in the core region. In contrast, there is only a minor phase shift between adjacent oscillators within the arm region, which is induced by the spiral wave propagation. In other words, oscillators in the arm region perform an in-phase oscillation. In addition, Figs. 2(a) and 2(b) show that the antiphase oscillation is much faster than that of the homogeneous one, which is also seen in Ref. [17]. Combining the above observations together, we draw the conclusion that the spiral wave in Fig. 1(a) is a rigidly rotating spiral wave with an antiphase spiral core (we denote it as SWAPC) and this spiral wave possesses two domains, each of which has its own temporal and spatial periodicities.

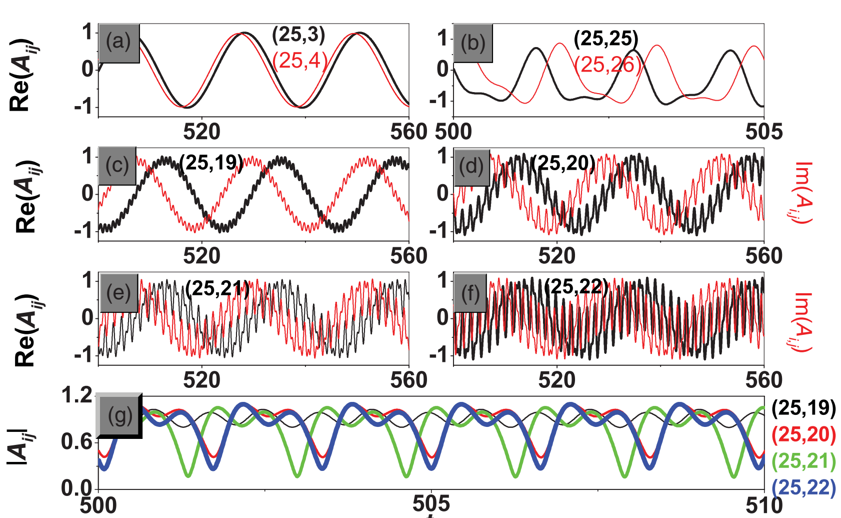

FIG. 2. (Color online) Time evolutions of different oscillators locating at a line that crosses the core region of a SWAPC. $b = 1$ and $c = 0.2$. The subscripts (i,j) in these plots denote the positions of the oscillators. (a) A pair of oscillators in the arm region. (b) A pair of oscillators in the core region. (c)–(f) The successive oscillators in the transition annulus. The black (bold) curves are for $\mathrm{Re}(A_{i,j})$ and the red (thin) ones for $\mathrm{Im}(A_{i,j})$. (g) The evolutions of $|A_{i,j}|$ for the oscillators in (c)–(f). The width of the curve increases from the node (25,19) to the node (25,22).

For a CGLE, a rigidly rotating spiral wave always takes the following form: $A(r,t) = F(r)\exp\{i[m\theta + \psi(r) - \omega t]\}$ and $F(r) = 0$ at the rotation center (the phase singularity) of the spiral wave [18]. In other systems such as reaction-diffusion systems, a similar formulation can be found provided that the spiral wave is rigidly rotating, whereas the spiral wave presented in Figs. 1(a) and 1(b) does not follow this formulation in the core region. Particularly, the rotation center of this spiral wave is replaced by an antiphase spiral core and the phase singularity of a normal spiral wave is lost. The statement is supported by the patterns of both $\mathrm{Re}(A_{i,j})$ and $|A_{i,j}|$ for four subsets of the system: the subset consisting of all oscillators with location $(2i,2j)$ $(i,j = 1,2,\ldots,N/2)$, the subset with $(2i+1,2j)$, the subset with $(2i,2j+1)$, and the subset with $(2i+1,2j+1)$. As shown in Figs. 1(d)–1(g), each subset displays a clear spiral wave pattern, which shows that the antiphase oscillation of $\mathrm{Re}(A_{i,j})$ in the core region is modulated by the in-phase oscillation in the arm region. Nevertheless, $|A_{i,j}|$ in Figs. 1(h)–1(k) for each subset substantiates that the phase singularity for a normal spiral wave is lost. Any subset itself possesses no phase singularity characterized by $|A_{i,j}| = 0$ at the center of the spiral core, in spite of the unique singularity a normal rigidly rotating spiral wave has. For each subset, two patches with low $|A_{i,j}|$, which situate opposite to the center of the spiral core, show up. It is nothing but these patches in all subsets that contribute to the formation of the transition annulus.

Furthermore, we regard how the in-phase oscillation in the arm region passes through the transition annulus to the antiphase oscillation in the core region along a line extending transversely. We present the time evolutions of successive oscillators on a line down the crossover area. The results in Figs. 2(c)–2(f) show distinctly two periodic components in each oscillator. When the oscillator is close to the arm region, it is dominated by the in-phase oscillation, which is superimposed with a weak antiphase oscillation. As the oscillator approaches the core region, the component of the antiphase oscillation grows stronger and stronger and finally predominates the core region. Recalling that the phase singularity of a normal spiral wave is actually manifested in the in-phase oscillation, it is the replacement of the in-phase oscillation by the antiphase one in the core region that causes the traditional phase singularity to be lost for SWAPC. To be mentioned, low $|A_{i,j}|$ ($|A_{i,j}| \simeq 0$) in the transition annulus is not related to the phase singularity of a spiral wave. As seen from Fig. 2(g), low $|A_{i,j}|$ in the transition annulus just occurs when the antiphase oscillation becomes comparable to the in-phase one. Especially, Fig. 2(g) shows that $|A_{i,j}| \simeq 0$ in the transition annulus shows up with an equivalent period as that of the antiphase one, which demonstrates that the existence of low $|A_{i,j}|$ in the transition annulus roots in the antiphase oscillation and is foreign to the phase singularity. In short, the phase singularity in a normal spiral wave vanishes for SWAPC and, instead, an antiphase core region plays the role of a rotor to support the spiral wave propagation.

Emphatically, though the wave pattern in Fig. 1(a) looks similar to the spiral wave chimera, they are essentially different. First, the phases of adjacent oscillators in the core region in a spiral wave chimera state are unrelated, while the phase difference between adjacent oscillators in SWAPC is kept around π. Secondly, as discussed in Ref. [13], the form $A(r,t) = F(r)\exp\{i[m\theta + \psi(r) - \omega t]\}$ is recovered for a spiral wave chimera provided that $A(r,t)$ is replaced by an order parameter $R(r,t)$. That is, there exists a well-defined phase singularity in the spiral wave chimera. However, it is quite different for SWAPC, since the ordinary phase singularity has been substituted by an antiphase core region where oscillation amplitude is as strong as that in the arm region.

The SWAPC is also different from the sandwiched spiral wave proposed in Ref. [14]. Behind the sandwiched spiral wave is the short wavelength instability to the homogeneous oscillation. However, the formation of SWAPC is due to the excitability of an antiphase perturbation to the homogeneous oscillation and is not related to the short wavelength instability. Especially, in most of the parameter regimes for SWAPC, homogeneous oscillation is stable and there is no instability yielding the short wavelength pattern.

To get more insight into how the excitability of antiphase perturbation to a homogeneous oscillation leads to the formation of SWAPC, we monitor the evolution of $\mathrm{Re}(A_{i,j})$ and $|A_{i,j}|$ of a chain of oscillators crossing the spiral wave core from the very initial stage when the spiral wave begins to build. Both spatiotemporal plots of $\mathrm{Re}(A_{i,j})$ and $|A_{i,j}|$ in Fig. 3 show a general scenario leading to SWAPC. An ordinary spiral wave is generated at first; then, near the phase singularity of the normal spiral wave, an antiphase perturbation comes into being; ultimately, an antiphase region with unchanged size appears and the spiral wave is established. The elaboration above explicitly demonstrates that the excitability of the antiphase perturbation from the in-phase oscillation plays a crucial role in the formation of SWAPC.

SWAPC could exist for a large parameter range of b and c. The variation of b or c affects the dynamics of SWAPC distinctly. Augmenting b always leads to the expansion of the antiphase core and longer wavelength in the arm region. In contrast, given b fixed, neither the size of the antiphase core nor the wavelength in the arm region depend on c. In addition, either an outwardly or an inwardly propagating SWAPC wave [19] can always be observed in suitable range of b and c.

To complete the dynamics of SWAPC, we investigate how it breaks up, which is always an interesting topic in the field of pattern formation. We consider a large system with $N = 100$. The results are insensitive to the system size (we have verified

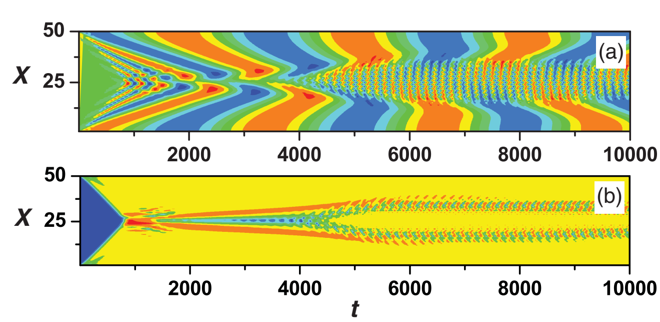

FIG. 3. (Color online) Spatial-temporal plots of $\mathrm{Re}(A_{i,j})$ in (a) and $|A_{i,j}|$ in (b) for the development of SWAPC. $b = 1$ and $c = 0.2$.

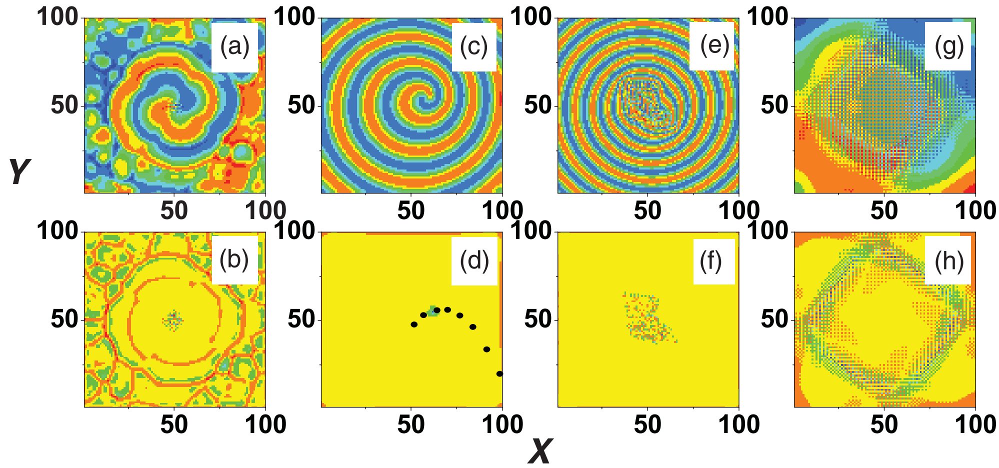

FIG. 4. (Color online) Different types of instability of SWAPC. The top panels show the snapshots of Re($A_{i,j}$) and the bottom $|A_{i,j}|$. (a) and (b) are the far-away breakup of SWAPC owing to Eckhaus instability. $b = 1$ and $c = -0.33$. (c) and (d) are the evanesce of SWAPC resulting from spiral wave drifting. The black dots in (d) designate the trajectory of the successive phase singularities. $b = 1$ and $c = 0.8$. (e) and (f) are the near-core breakup of SWAPC due to the spontaneous expansion of the antiphase core. $b = 0.25$ and $c = 0.2$. (g) and (h) are the disappearance of SWAPC on account of the larger antiphase core. $b = 5$ and $c = 0.2$. The size of the system $N = 100$.

this prediction for different system sizes). In a normal CGLE, the spiral wave always breaks up through Eckhaus instability. However, depending on how the parameters vary, SWAPC in the system Eq. (2) may display rich scenes. In this work, several simple situations are taken into account. First, we prescribe $b = 1$. When c decreases beyond a critical value, the Ekhaus instability steps in and the normal far-away breakup of the spiral wave appears. A snapshot is shown in Fig. 4(a), where a spiral wavelet with an antiphase core is surrounded by a chaotic sea. The snapshot of the corresponding $|A_{i,j}|$ presented in Fig. 4(b) shows that there are many defects outside the spiral wavelet. As time goes on, the spiral wavelet will shrink further and finally die off when the antiphase core is swallowed by the defect sea. To be mentioned, the final state after the disappearance of the spiral wavelet is portrayed by a complex antiphase pattern. On the other hand, though increasing c may also cause SWAPC to vanish, it does not lead to the spiral wave breakup. Actually, when c is above a critical value, SWAPC is replaced by a normal spiral wave in the beginning; subsequently, to suppress the growth of the antiphase perturbation brought about by the phase singularity, the normal spiral wave has to drift spontaneously. Figure 4(c) shows a snapshot of a drifting spiral wave with normal phase singularity. The trajectory of the drifting spiral wave is presented in Fig. 4(d), where the phase singularities at successive times are identified. Clearly, the drifting spiral wave dies away in the end due to the collision between its phase singularity and the boundary. Then, we change b with c fixed. For small b, an instability of SWAPC arises, where the antiphase core dilates spontaneously and persistently until it overspreads the system and the arm region eventually drifts away. The snapshots of Re($A_{i,j}$) and $|A_{i,j}|$ in Figs. 4(e) and 4(f) exhibit SWAPC with a bulk of antiphase core. Different from aforementioned circumstances, there is no instability of SWAPC observed by increasing b. Nevertheless, there exists no SWAPC for sufficiently large b in any finite-size system owing to the ever swelling antiphase core.

It is necessary to mention that the parameters used in the numerical simulations are physically meaningful in real ultracold atom systems. As expressed in Ref. [17],

$$b = \frac{2k_e e^2}{\nu m d^3 \omega_0^2}, \quad c = \frac{3\alpha_0 l^2}{\nu \omega_0^2}, \quad \nu = \frac{8\hbar k^2 \gamma^3 \Delta\omega I_B / I_S}{M\omega_0^2(\gamma^2 + 4\Delta^2\omega)^2}, \tag{3}$$

where k_e is the Coulomb constant, e elementary charge, α_0 the coefficient of the anharmonic quadratic term in the trap potential, ω_0 the harmonic trap frequency, d the distance between the trap centers, and l the displacement of ions. The two-level atom of mass m allows for the dipole transition of wavelength $\lambda = 2\pi/k$ and linewidth γ. Each ion is heated and cooled by near-resonant laser beams depending on the blue or red detuning $\Delta\omega$. I_R and I_B are the intensities of the blue and red beams, respectively. I_S is the laser saturation intensity. Lee and Cross considered the ion $^{24}Mg^+$ with an $S_{\frac{1}{2}}$-$P_{\frac{3}{2}}$ dipole transition and estimated $b = 1$ and $c = 1.1$ under suitable circumstances [17]. The entire parameter space of b and c space can be explored by tuning the parameters I_B and α_0.

III. CONCLUSION

In summary, we find a different type of spiral wave when studying the dynamics of trapped ions using the model system Eq. (2): a rigidly rotating spiral wave with an antiphase core (SWAPC). The formation of SWAPC is mainly due to the excitability discovered by Lee and Cross. We also investigate the dynamics of SWAPC and find some interesting breakup scenarios. Some problems deserve further investigation on SWAPC in the model system Eq. (3), such as the dependence of the size of the antiphase core on parameters, the parameter regimes for inwardly and outwardly propagating SWAPC, and the comprehensive description on the breakup scenarios of SWAPC. Another fascinating topic is whether SWAPC exists in other cold atom quantum systems. In addition, SWAPC could exist only in discrete space systems, since the antiphase oscillation is conceptually impossible for a continuous space system. However, whether the observed phenomena in this work only exist in other discrete space systems such as those with dissipative coupling is also an open problem.

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