Numerical study of the correspondence between the dissipative and fixed-energy Abelian sandpile models

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We consider the Abelian sandpile model (ASM) on the square lattice with a single dissipative site (sink). Particles are added one by one per unit time at random sites and the resulting density of particles is calculated as a function of time. We observe different scenarios of evolution depending on the value of initial uniform density (height) h_0. During the first stage of the evolution, the density of particles increases linearly. Reaching a critical density $\rho_c(h_0)$, the system changes its behavior and relaxes exponentially to the stationary state of the ASM with density ρ_s. Considering initial heights $-1 \leqslant h_0 \leqslant 4$, we observe a dramatic decrease of the difference $\rho_c(h_0) - \rho_s$ when h_0 is zero or negative. In parallel with the ASM, we consider the conservative fixed energy sandpile (FES). The extensive Monte Carlo simulations show that the threshold density $\rho_{\text{th}}(h_0)$ of the FES converges rapidly to ρ_s for $h_0 < 1$.

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I. INTRODUCTION

A long-standing discussion of the comparative critical properties of the dissipative Abelian sandpile model (ASM) [1] and the conservative fixed-energy sandpile (FES) [2] has gained recently renewed impetus due to the works by Fey *et al.* [3]. The point of discussion in a laconic form can be reduced to the single question of, given the same lattice with open and closed boundary conditions, whether the stationary density of the dissipative ASM ρ_s coincides with the threshold density of the FES ρ_{th}. Using large-scale simulations on the square lattice, the authors of works [3,4] gave a negative answer to this question and supported their numerical findings by exact solutions for some graphs of higher symmetry. A more detailed answer lies in a description of the average density of grains $\rho(\tau)$ for the given density of added particles τ of the dissipative sandpile:

$$\rho(\tau) = \frac{(\text{no. grains at time } t)}{N^2}, \quad \tau = \frac{t}{N^2}, \qquad (1)$$

where N is the linear size of the lattice and t is the number of added particles, which plays the role of the discrete time. It was shown in [3] that $\rho(\tau)$ for the ASM exhibits a transition at the point ρ_c, which coincides with the threshold density of the FES ρ_{th}. For $\tau > \rho_c$, the system continues evolution and its density tends to the stationary value ρ_s of the ASM when $\tau \to \infty$.

In this paper, we continue the analysis of the correspondence between the ASM and FES. The focus of the work will be three characteristics of sandpiles:

(i) the stationary density in the dissipative ASM ρ_s;
(ii) the critical density in the dissipative ASM ρ_c;
(iii) the threshold density of FES ρ_{th}.

A peculiarity of this study is detailed consideration of dependence $\rho_c(h_0)$ and $\rho_{\text{th}}(h_0)$ on the initial height h_0. Confirming the main result of [3] on the existence of critical density ρ_c, we, however, present arguments that the strict inequality $\rho_{\text{th}} \neq \rho_s$ in [3] should be changed into the more sophisticated statement that ρ_{th} actually *converges* to ρ_s for appropriate initial conditions.

Instead of usual ASM with open boundaries, we find it useful to consider a situation when the ASM is as close as possible to the FES, namely, the ASM on the lattice with periodic boundary conditions with a single dissipative site (sink). The dynamics of the nearly closed ASM is very close to that of the FES as dissipation through the single site is strongly restricted. To fix our notations, we will specify the two-dimensional ASM as in [5], where all stable sites have heights $1 \leqslant h_i \leqslant 4$ while sites with $h_i \geqslant 5$ are unstable and topple.

Traditionally, empty sites ($h_i = 0$) and negative heights are not considered either analytically or numerically because they exist in transient states only and vanish when the system approaches criticality. Nevertheless, these sites may affect the evolution of the ASM from an initial state to the critical point $\rho_c(h_0)$. Nonergodicity of the FES causing a dependence of the threshold density upon the initial conditions has been discussed recently in [4] and [15] for the positive initial heights. In this paper, we consider functions $\rho_c(h_0)$ and $\rho_{\text{th}}(h_0)$ for the ASM and FES in the whole interval, including zero and negative values of h_0: $-1 \leqslant h_0 \leqslant 4$. We confirm the equality $\rho_c(h_0) = \rho_{\text{th}}(h_0)$ obtained in [3] for $h_0 = 1$ and check that this result holds for all considered initial heights.

Our main observation is the behavior of $\rho_c(h_0)$ and $\rho_{\text{th}}(h_0)$ in the interval $h_0 = 1$ to $h_0 = -1$. For the ASM with initial conditions $h_0 = 1$, the value $\rho_c(1) = 3.125\,28$ is close to the threshold density of the FES obtained by Fey *et al.* [3] (3.125 288). For $h_0 = 0$, the density ρ increases linearly with τ, reaches the value $\rho_c(0) \simeq \rho_{\text{th}}(0) \simeq 3.125\,0224$, very close to ρ_s, and then keeps fluctuating around it. For $h_0 = -1$ the difference $\rho_c(-1) - \rho_s$ vanishes up to the standard deviations. The extensive Monte Carlo simulations of the FES allow us to compute the difference $\rho_{\text{th}}(h_0) - \rho_s$ with greater accuracy. We see that values $\rho_{\text{th}}(h_0) - \rho_s$ are 0.000 288,0.000 022,0.000 005 for $h_0 = 1, 0, -1$, respectively, i.e., demonstrate the rapid convergence to zero.

In Sec. II, we give the definition of the model and remind of the origins of the conjectured (now proved) exact value $\rho_s = 25/8$. In Sec. III we present details of the numerical calculations of the dissipative ASM and describe five different scenarios of evolution for various initial conditions. Section IV contains the results of simulations of the FES. Section V is a short conclusion.

II. MODEL AND CRITICAL DENSITY

We consider the standard Abelian sandpile model [1] on the $N \times N$ square lattice. To simplify the comparison between the ASM and FES, we impose the nearly closed boundary conditions: the lattice is a torus with a single open site i_0. The height h_i at any site i beside i_0 takes values $h_i \leqslant 4$ in stable configurations. Particles are added one by one at a random site increasing the height at that site by 1. If the height exceeds 4, then the site becomes unstable and topples, transferring one particle to each of its four neighboring sites. The site i_0 serves as a sink where particles disappear. During a long evolution, the system gets eventually into a set of recurrent configurations. The theory of recurrent states of the ASM was developed by Dhar [5]. An important consequence of this theory is the bijection between the set of recurrent configurations and the set of spanning trees on the same lattice. The bijection allows one to find the single site height probabilities in the recurrent state and therefore to find the stationary density ρ_s. Majumdar and Dhar [6] calculated the probability of height $h = 1$ in the thermodynamic limit,

$$P_1 = \frac{2}{\pi^2} - \frac{4}{\pi^3}. \tag{2}$$

The probabilities P_2, P_3, and $P_4 = 1 - P_1 - P_2 - P_3$ were found in [7]:

$$P_2 = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_1}{4}, \tag{3}$$

$$P_3 = \frac{1}{4} + \frac{3}{2\pi} + \frac{1}{\pi^2} - \frac{12}{\pi^3} - \frac{I_1}{2} - \frac{3I_2}{32}, \tag{4}$$

$$P_4 = \frac{1}{4} - \frac{1}{\pi^2} + \frac{4}{\pi^3} + \frac{I_1}{4} + \frac{3I_2}{32}, \tag{5}$$

where I_ν, $\nu = 1,2$ are fourfold integrals.

Later on, an exact relation between P_2 and P_3 was proved in [8], thereby eliminating one of the two integrals. A conjecture on the value of the remaining one, based on its numerical evaluation to 12 decimal digits, then led to the following values for the P_i [8]:

$$P_2 = \frac{1}{4} - \frac{1}{2\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3}, \tag{6}$$

$$P_3 = \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3}, \tag{7}$$

and for the stationary density,

$$\rho_s = P_1 + 2P_2 + 3P_3 + 4P_4 = \frac{25}{8}. \tag{8}$$

Recently, this conjecture has been proved in [10] (see also [11]) by the mapping of spanning trees onto monomer-dimer tiling and independently in [12] by the method of vector bundle Laplacian [13].

Another conjecture, coined in [3] as the "density conjecture," identifies ρ_s with the threshold density of the FES, ρ_{th} [2]. The threshold density is defined as the average maximal number of randomly dropped particles per site, which allows the relaxation of the closed sandpile to stop. The value ρ_{th} marks the border of stability: for any $\rho > \rho_{\text{th}}$, an avalanche process started by adding a particle never stops with probability tending to 1 as $N \to \infty$. The conjecture $\rho_{\text{th}} = \rho_s$ was supported by numerical calculations [2]. However, recently, starting from the initial condition $h_0 = 1$, Fey *et al.* [3] have obtained $\rho_{\text{th}} = \rho_s + \Delta$, where $\Delta = 0.000\,288$ to six decimal digits. Together with the discussion in [4] and [15] mentioned in the Introduction, this makes the correspondence between critical points of the ASM and FES rather problematic.

We consider here a nearly closed sandpile with a single open site i_0. The idea is to retrace carefully the process of adding particles from different initial heights h_0 to the critical state of the ASM. At the initial stage of the process, the probability of dissipation via the sink at i_0 is negligibly small for the large lattices and the evolution of the system almost coincides with the dynamics of the FES. When the density approaches a critical value, the large avalanches appear and the probability of dissipation increases. If the transition to the dissipative regime is sufficiently sharp, we can identify the transition point with $\rho_c(h_0)$. According to the process described in [3], the system evolves with further adding of particles and reaches eventually the stationary density of the ASM ρ_s. The main difference from previous studies is in the initial conditions. Besides the usual initial conditions $1 \leqslant h_0 \leqslant 4$, we consider zero and negative h_0 and show that the difference $\rho_c(h_0) - \rho_s$ decreases rapidly when h_0 goes down.

III. SIMULATIONS FOR ASM WITH ONE SINK

Consider the Abelian sandpile model with one sink on a $N \times N$ square lattice $\mathcal{L}$ with periodic boundary condition in both directions. The initial configuration is uniform, i.e., the heights of all vertices have the same initial value h_0. We add one particle per unit time at a uniformly random site, and let the system stabilize before a new particle is added. The resulting density of particles is measured as a function of time.

At the first stage of the evolution, the system loses a negligibly small fraction of particles to the sink, and the density of particles increases with time linearly. Eventually, the system reaches some critical density $\rho_c(h_0)$ where it starts losing a macroscopic amount of particles. At this point, the behavior of density changes sharply. During further evolution the system relaxes and the density tends to the stationary density of the ASM $\rho_s = 25/8$. At the critical point, the density $\rho(\tau)$ has a kink, which enables one to determine $\rho_c(h_0)$ with reasonable accuracy. We monitor the evolution of the system for various h_0.

A. $h_0 = 4$

The simulations show that after dropping a finite number of particles to the configuration of heights $h_0 = 4$, the system loses a large amount of particles and its density decreases down to $\rho_c(4) \approx 3.44$ (see Fig. 1). After this big avalanche,

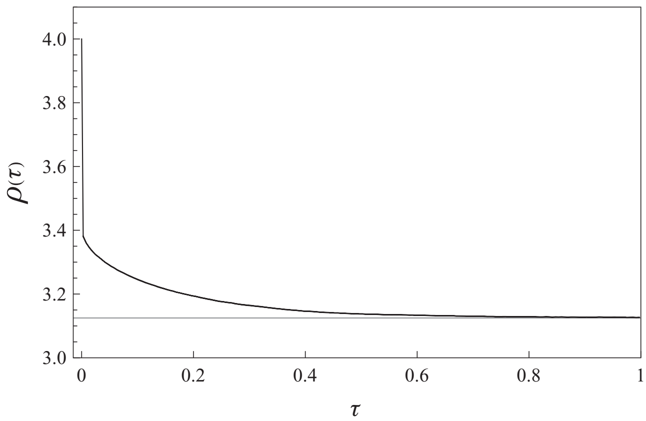

FIG. 1. The evolution of the dissipative ASM with one sink for $h_0 = 4$, number of samples is 1000, $N = 500$.

the system relaxes to its asymptotic value ρ_s. For large τ the convergence is exponential. The rate of the exponential convergence is universal, i.e., independent on initial conditions. It defines the relaxation time, and is nothing but the spectral gap of the transition matrix of the process.

B. $h_0 = 3$

During the initial stage of evolution, the number of particles in the finite system grows almost linearly with τ. At the critical density $\rho_c(3)$, a kind of percolation occurs which can be termed a "weak percolation.' It implies that sites with $h = 4$ do not percolate yet, but additional sites increasing their height during an avalanche produce together with them a percolation set, which causes big avalanches spreading through the system. The weak percolation point for $N = 500$ is visible in Fig. 2 as a kink following the linear part of the function $\rho(\tau)$.

The initial value $h_0 = 3$ shows some similarity with bootstrap percolation [14]. The differences between the two models, however, prevent us from having a clear view of the exact relationship between them.

The position of the kink $\rho_c(3)$ [which coincides with $\rho_{\rm th}(3)$ of FES] in the limit $N \to \infty$ can be found by using the following observation proved in [9] (see Proposition 1.4). Consider an infinite square lattice of height 3 on all sites. To each site add one grain with probability $\varepsilon > 0$. Then, the addition of a finite number of particles to the origin will produce an infinite avalanche, which covers all lattices with probability 1.

To apply this statement to our case, let us add randomly εN^2 particles to the $N \times N$ lattice with periodic boundary conditions and initial sandpile configuration $h_0 = 3$, where $\varepsilon > 0$ is an arbitrary small constant (see Fig. 2). Then, the probability to find a site with height $h(\varepsilon)$ higher than a fixed value tends to 1 when N tends to infinity. Considering this site as the origin in the referred proposition, we conclude that $\rho_{\rm th}(3)$ [and therefore $\rho_c(3)$ too] tends to 3, when $N \to \infty$.

C. $h_0 = 2$

Due to lowering of the initial density, the percolation picture of the sites with $h = 4$ becomes more pronounced, getting close to the usual site percolation. The percolation dynamics versus the avalanche dynamics for different background densities in the FES has been discussed recently by Park [15]. It was demonstrated that the critical density of transition into the unstable state strongly depends on the background density and may differ considerably from the accepted values of ρ_s and $\rho_{\rm th}(1)$. Our results for $h_0 = 2$ confirm this conclusion (see Fig. 3). In the limit of large lattices the density of particles $\rho(\tau)$ grows strictly linearly in time up to the transition point $\rho_c(2) \approx 3.134$.

At the critical point $\rho_c(2)$, the avalanche mechanism is activated and the dissipation reduces density toward ρ_s. It may seem strange that lowering of the initial density from $h_0 = 3$ to $h_0 = 2$ leads to the increase of the critical density from $\rho_c(3) = 3$ to $\rho_c(2) \approx 3.134$. However, this is the result of the competition between a larger number of percolation sites with $h = 4$ for $h_0 = 2$ and the reduced density of particles outside the percolation cluster. Of course, the value of the critical point $\rho_c(2)$ cannot be derived directly from the site-percolation critical probability $P_s = 0.592\ldots$, because the distribution of heights outside the percolation cluster remains unknown.

D. $h_0 = 1$

This case has been investigated with high precision in [3]. We repeat the simulations for $h_0 = 1$ to test the basic

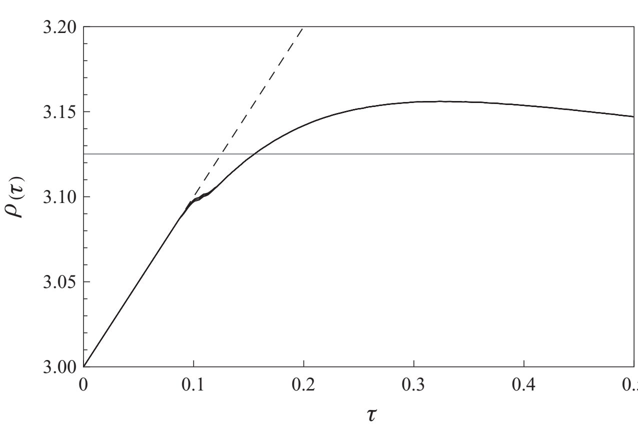

FIG. 2. The evolution of the dissipative ASM with one sink for $h_0 = 3$, number of samples is 1000, and lattice size $N = 500$.

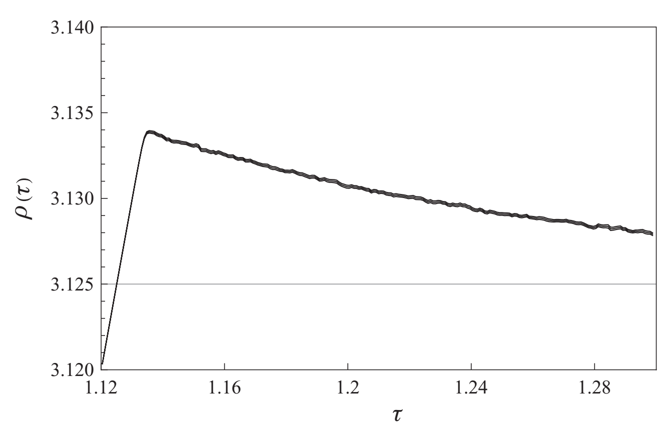

FIG. 3. The evolution of the dissipative ASM with one sink for $h_0 = 2$, number of samples is 1000, and lattice size $N = 500$.

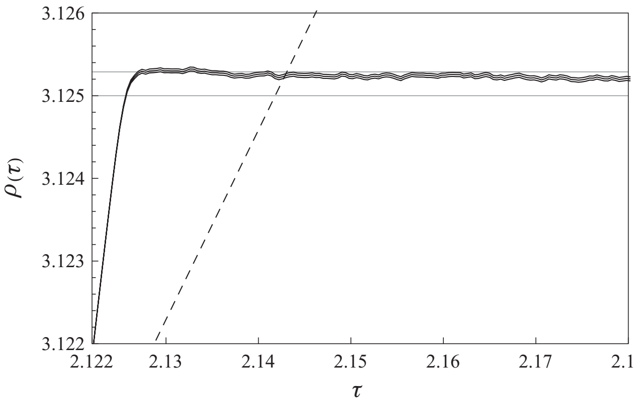

FIG. 4. The evolution of the dissipative ASM with one sink for $h_0 = 1$, number of samples is 1000, and lattice size $N = 500$. Two horizontal lines correspond to values $\rho_{\text{th}}(0) = 3.125$ and $\rho_{\text{th}}(1) = 3.125\,288\ldots$.

observations of [3] in our case of the lattice with a single open site (see Fig. 4). We found that the value $\rho_c \approx 3.125\,28$ for the lattice with $N = 500$ is very close to the threshold value of the FES $\rho_{\text{th}} = 2.125\,278 \pm 0.000\,0004$ reported in [3] for the lattice $N = 512$. This confirms that the nearly closed sandpile behaves almost identically to the FES during the nondissipative part of evolution. The results of our simulations of the FES in the case $h_0 = 1$ for different lattice sizes are shown in the next section.

E. $h_0 = 0$ and $h_0 = -1$

The initial conditions $h_0 = 0$ and $h_0 = -1$ are crucial and turn out to be different from the previous cases. Adding particles to the empty lattice, we observe as above the linear growth of density $\rho(\tau)$ which indicates an almost nondissipative character of evolution. The linear part changes at the critical point $\rho_c \approx 3.125$ into a function slightly fluctuating around $\rho(\tau) = \text{const}$ up to error bars (not shown in Fig. 5, as they are of order of thickness of the curves). It means that the excess density growth observed for initial conditions $h_0 = 3,2,1$, and which leads to the considerable difference

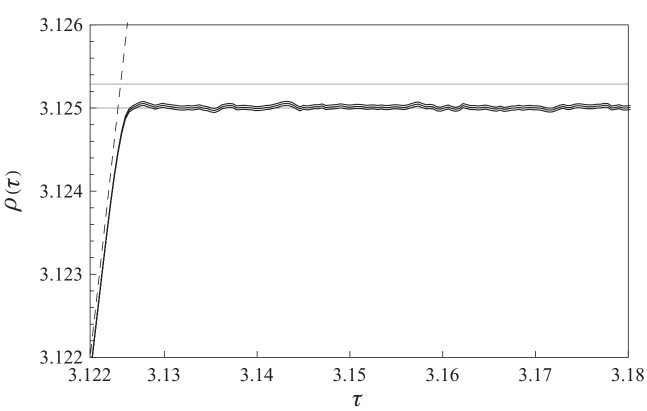

FIG. 5. The evolution of the dissipative ASM with one sink for $h_0 = 0$, number of samples is 1000, and lattice size $N = 500$. Two horizontal lines correspond to values $\rho_{\text{th}}(0) = 3.125$ and $\rho_{\text{th}}(1) = 3.125\,288\ldots$.

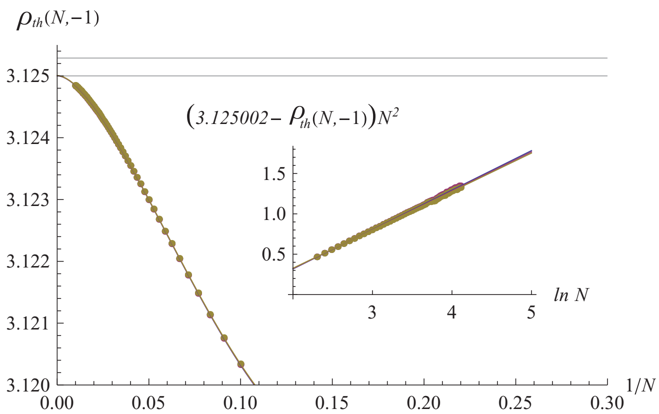

FIG. 6. (Color online) The threshold densities of the fixed energy sandpile for $h_0 = -1$, number of samples is 10^7, and $N = 10,11,\ldots,100$. Two horizontal lines correspond to values $\rho_{\text{st}} = 3.125$ and $\rho_{\text{th}}(1) = 3.125\,288\ldots$.

between ρ_c and ρ_s, is strongly suppressed as h_0 decreases. The closeness of ρ_c for $h_0 = 0$ to that for $h_0 = -1$ makes us continue investigations of these initial conditions with higher accuracy. In the next section, we do this for the conservative FES.

IV. THRESHOLD DENSITY OF FES

Along with the nearly closed sandpile, we considered the FES for initial conditions $h_0 = -1,0,1,2,3$ to confirm the relation $\rho_c = \rho_{\text{th}}$ established in [3] for $h_0 = 1$ and to determine the convergence laws of the threshold density to its asymptotical value. To do that, we consider $N \times N$ lattices with periodic boundary conditions in both directions for various N. For each N, we run 10^7 samples of the FES and measure the average threshold density $\rho_{\text{th}}(N,h_0)$ together with its standard deviation $D\rho_{\text{th}}(N,h_0)$. Extrapolating the so-obtained data for $\rho_{\text{th}}(N,h_0)$ for $1/N \to 0$, we find numerically the asymptotic values of the threshold density and its finite-size corrections for large N.

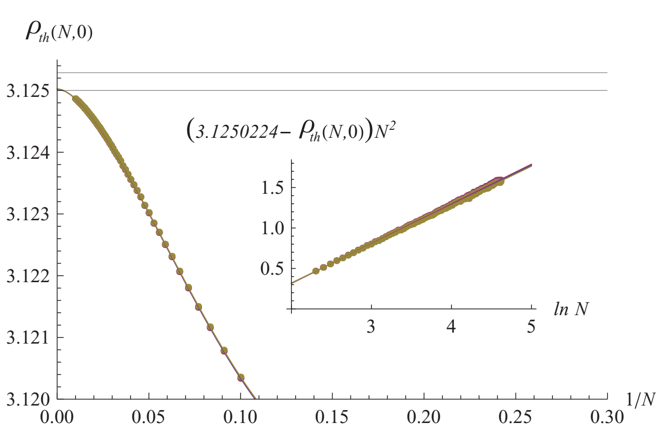

FIG. 7. (Color online) The threshold densities of the fixed energy sandpile for $h_0 = 0$, number of samples is 10^7, and $N = 10,11,\ldots,100$. Two horizontal lines correspond to values $\rho_{\text{st}} = 3.125$ and $\rho_{\text{th}}(1) = 3.125\,288\ldots$.

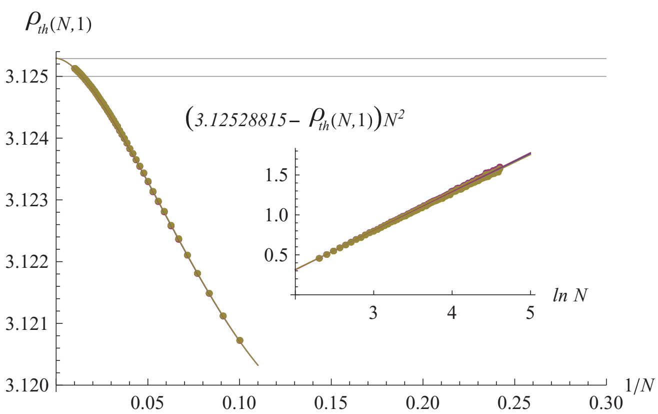

FIG. 8. (Color online) The threshold densities of the fixed energy sandpile for $h_0 = 1$, number of samples is 10^7, and $N = 10,11,\ldots,100$. Two horizontal lines correspond to values $\rho_{\mathrm{st}} = 3.125$ and $\rho_{\mathrm{th}}(1) = 3.125\,288\ldots$.

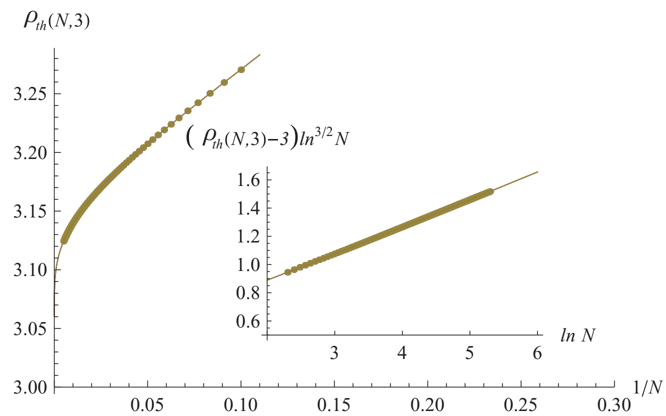

FIG. 10. (Color online) The threshold densities of the fixed energy sandpile for $h_0 = 3$, number of samples is 10^7, for $N = 10,11,\ldots,200$.

The results of simulations for the FES with initial conditions $h_0 = -1,0,1,2,3$, are shown in Figs. 6–10. The sizes of dots in the figures are proportional to their statistical errors. After extrapolation of the numerical data, we found the following asymptotical values and finite-size corrections for $\rho_{\mathrm{th}}(N,h_0)$:

$$\begin{aligned}
\rho_{\mathrm{th}}(N,-1) &\simeq 3.125\,00(5) - \frac{0.48(4)\ln N}{N^2} + \frac{0.64(7)}{N^2},\\
\rho_{\mathrm{th}}(N,0) &\simeq 3.125\,02(2) - \frac{0.48(4)\ln N}{N^2} + \frac{0.64(8)}{N^2},\\
\rho_{\mathrm{th}}(N,1) &\simeq 3.125\,2881(5) - \frac{0.49(1)\ln N}{N^2} + \frac{0.67(1)}{N^2}, \qquad (9)\\
\rho_{\mathrm{th}}(N,2) &\simeq 3.1339(6) - \frac{0.06(5)}{N^{\alpha_2}} + \frac{18.3(5)}{N^{\beta_2}},\\
\rho_{\mathrm{th}}(N,3) &\simeq 3.00(3) + \frac{0.18(1)}{(\ln N)^{\alpha_3}} + \frac{0.51(6)}{(\ln N)^{\beta_3}}.
\end{aligned}$$

Remarkably, the first three asymptotical expansions have a universal form where coefficients at terms $\frac{\ln N}{N^2}$ and $\frac{1}{N^2}$ are very similar. Therefore they can be used for determination of the leading terms in $\rho_{\mathrm{th}}(N,-1)$, $\rho_{\mathrm{th}}(N,0)$, $\rho_{\mathrm{th}}(N,1)$ with high accuracy. The asymptotical expansions for $\rho_{\mathrm{th}}(N,2)$, $\rho_{\mathrm{th}}(N,3)$ do not display any universal behavior. We estimated exponents in these expansions as $\alpha_2 \simeq 5/4$, $\alpha_3 \simeq 1/2$, $\beta_2 \simeq 15/4$, and $\beta_3 \simeq 3/2$.

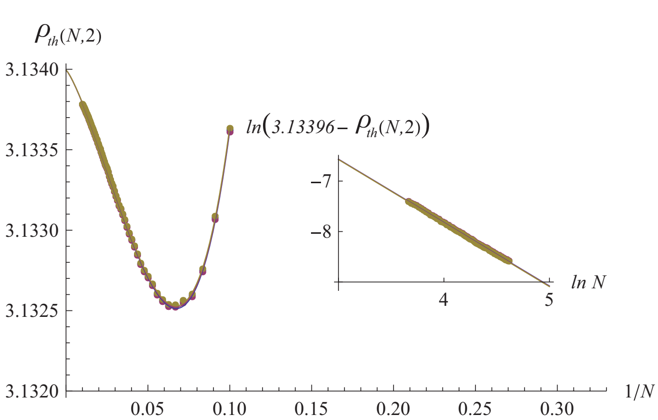

FIG. 9. (Color online) The threshold densities of the fixed energy sandpile for $h_0 = 2$, number of samples is 10^7, and $N = 10,11,\ldots,100$.

The obtained asymptotical values of the threshold density of the FES $\rho_{\mathrm{th}}(h_0)$ show that the difference between $\rho_{\mathrm{th}}(0)$ and $\rho_{\mathrm{th}}(-1)$ is approximately ten times smaller than that between $\rho_{\mathrm{th}}(1)$ and $\rho_{\mathrm{th}}(0)$. However, it is not zero and we may expect that it continues to decrease for larger negative values of h_0. If so, the strict inequality $\rho_{\mathrm{th}} \neq \rho_s$ in [3] has the more realistic alternative that ρ_{th} actually converges to ρ_s for sufficiently large negative initial heights.

V. CONCLUSION

The results of our numerical simulations suggest the following three statements: (1) the density of sand held by the lattice with a single dissipative site (or with dissipation rate tending to zero), as a function of sand deposed, shows a kink at a value ρ_c, which can be identified with the threshold density of the corresponding FES with the same initial condition; (2) the critical value ρ_c depends on the initial condition, and consequently, there is presumably also a range of values for the FES threshold density, also depending on the initial condition and potentially on the way sand is dropped; (3) when the height h_0 of the uniform initial configuration goes down, the difference between threshold ρ_{th} and the stationary density, $\rho_s = 25/8$, becomes smaller and most probably tends to 0.

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