PHYSICAL REVIEW E **84**, 066107 (2011)

Effect of the nature of randomness on quenching dynamics of the Ising model on complex networks

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(Received 3 October 2011; published 13 December 2011)

Randomness is known to affect the dynamical behavior of many systems to a large extent. In this paper we investigate how the nature of randomness affects the dynamics in a zero-temperature quench of the Ising model on two types of random networks. In both networks, which are embedded in a one-dimensional space, the first-neighbor connections exist and the average degree is 4 per node. In random model A the second-neighbor connections are rewired with a probability p, while in random model B additional connections between neighbors at a Euclidean distance l ($l > 1$) are introduced with a probability $P(l) \propto l^{-\alpha}$. We find that for both models, the dynamics leads to freezing such that the system gets locked in a disordered state. The point at which the disorder of the nonequilibrium steady state is maximum is located. The behavior of dynamical quantities such as residual energy, order parameter, and persistence are discussed and compared. Overall, the behavior of physical quantities are similar, although subtle differences are observed due to the difference in the nature of randomness.

DOI: 10.1103/PhysRevE.84.066107 PACS number(s): 89.75.Hc, 75.78.Fg, 81.40.Gh, 64.60.aq

I. INTRODUCTION

The dynamics of Ising models is a much studied phenomenon and has emerged as a rich field of present-day research. An important dynamical feature commonly studied is the quenching phenomenon below the critical temperature. In a quenching process, the system has a disordered initial configuration corresponding to a high temperature and its temperature is suddenly dropped. This results in quite a few interesting phenomena such as domain growth [1,2] and persistence [3–6].

In one dimension, a zero-temperature quench of the Ising model starting with a completely random configuration (which corresponds to a very high temperature) and evolving according to the usual Glauber dynamics always leads the system to the equilibrium configuration (all spins up or all spins down). The average domain size D increases in time t as $D(t) \sim t^{1/z}$, where z is the dynamical exponent associated with growth. As the system coarsens, the magnetization also grows in time as $m(t) \sim t^{1/2z}$. In two or higher dimensions, however, the system does not always reach equilibrium [6], although these scaling relations still hold. In a zero-temperature quench, another important dynamical behavior commonly studied is persistence, which is the probability that a spin does not flip until time t. In regular lattices, in one or higher dimensions, the persistence probability $P(t)$ at time t is usually seen to follow a power-law decay given by $P(t) \propto t^{-\theta}$, where θ is called the persistence exponent and is unrelated to any other known static or dynamic exponents.

The dynamical behavior of Ising models may change drastically when randomness is introduced in the system. Randomness can occur in many ways and its effect on dynamics can depend on its precise nature. For example, randomness in the Ising model can be incorporated by introducing dilution in the site or bond occupancy in regular lattices and consequently the percolation transition plays an important role [7,8]. Scaling behavior is completely different from power laws here. One can also consider the interactions to be randomly distributed, either all ferromagnetic type or mixed type (e.g., as in a spin glass) [9]; the system goes to a frozen state following a zero-temperature quenching in both cases. Another way to introduce randomness is to consider a random field in which case the scaling behavior is also completely different from power laws [10].

Here we consider Ising models on random graphs or networks where the nearest-neighbor connections exist. In addition, the spins have random long-range interactions that are quenched in nature. In general, here the dynamics, instead of leading the system toward its equilibrium state, makes it freeze into a metastable state such that the dynamical quantities attain saturation values different from their equilibrium values. Moreover, rather than showing a conventional power-law decay or growth, the dynamical quantities exhibit completely different behavior in time.

A point to be noted here is that when long-range links are introduced, the domains are no longer well defined as interacting neighbors could be well separated in space. This results in freezing of Ising spins on random graphs as well as on small-world networks [11,12]. The phase-ordering dynamics of the Ising model on a Watts-Strogatz network [13], after a quench to zero temperature, produces dynamically frozen configurations, disordered at large length scales [12,14]. Even on small-world networks, the dynamics can depend on the nature of the randomness; it was observed that while in a sparse network there is freezing, in a densely connected network freezing disappears in the thermodynamic limit [15].

In this paper we investigate the dynamical behavior of an Ising system on two different networks following a zero-temperature quench. In these two networks, both of which are sparsely connected, the nature of randomness is subtly different and we study whether this difference has any effect on the dynamics. Both these networks are embedded in a one-dimensional lattice and the nearest-neighbor connections always exist and the nodes have degree 4 on average. They differ as in one of the networks the random long-range interactions have a spatial dependence.

It is also quite well known that many dynamical social phenomena can be appropriately mapped to the dynamics of

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spin systems. At the same time, social systems have been shown to behave like complex networks (having small-world and/or scale-free features, etc.). So the present study may be particularly interesting in the context of studying social phenomena described by Ising-type models.

In Sec. II we describe the two different networks, which we call random model A (RMA) and random model B (RMB). In Sec. III we give a list of the quantities calculated. In Secs. IV and V we discuss the detailed dynamical behavior of Ising spin systems on random model A and random model B, respectively. A comparison of the results of the quenching dynamics between the two models is discussed in Sec. VI. A qualitative analysis of the quenching dynamics is also presented. We summarize in Sec. VII.

II. DESCRIPTION OF THE NETWORK MODELS

The two network models under consideration were introduced in Ref. [16]. Random model A is in fact very similar to the Watts-Strogatz (WS) network [13]. Here a spin is initially connected to its four-nearest neighbors and then only the second-nearest-neighbor links are rewired with probability p (Fig. 1). In the RMB each spin is connected to its two-nearest neighbors and then two extra bonds (on average) are attached randomly to each spin. The extra bonds are attached to spins located at a distance $l > 1$ with probability $P(l) \propto l^{-\alpha}$ (Fig. 1). We keep the first neighbors intact in both cases to ensure that the networks are connected. The average degree per node is 4 in both networks. The dynamical evolution is considered on the static networks after the process of rewiring or the addition of links is completed.

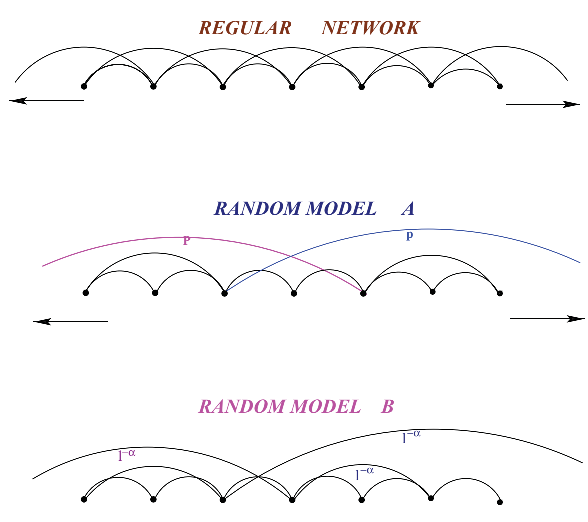

FIG. 1. (Color online) Schematic diagram for different network models. The average degree is $2K = 4$ in each network. In the regular network both the first- and second-nearest neighbors are present. In random model A only second-nearest neighbors are rewired with probability p. In random model B first-nearest neighbors are always linked, while other nodes are linked with probability $l^{-\alpha}$ with $l \geqslant 2$.

The general form of the Hamiltonian in a one-dimensional Ising spin system for RMA and RMB can be written as

$$H = -\sum_{i<j} J_{ij} S_i S_j, \tag{1}$$

where $S_i = \pm 1$ and $J_{ij} = J$ when sites i and j are connected and zero otherwise. (We set $J = 1$ in this paper.) The ground state (minimum-energy state at zero temperature) of the Ising spin system in both RMA and RMB is a state with all spins up or all spins down.

Random model A is a variant of the WS model with identical static properties. It is regular for $p = 0$, random for $p = 1$, and for any $p > 0$, the nature of RMA is small-world-like [13,16]. Euclidean models of RMB type have been studied in a few earlier works [16–18]. While it is more or less agreed that for $\alpha \leqslant 1$ the network is random and for $\alpha > 2$ it behaves as a regular network, the nature of the network for intermediate values of α is not very well understood. According to the earlier studies [16–18], it may either have a small-world characteristic or behave like a finite-dimensional lattice. In the present work we assume that RMB has random nature for $\alpha < 1$ and for $1 < \alpha < 2$ it is small-world-like (at least for the system sizes considered here) following the results of Ref. [16], which are based on an exact numerical evaluation of the shortest distance and clustering coefficients. This is also because the Euclidean model considered in Ref. [16] is exactly identical to RMB with average degree 4, while the average degree of the Euclidean models considered in the earlier studies is not necessarily equal to 4.

In the case of RMA, the network is regular and random for only two extreme values $p = 0$ and 1, respectively, whereas for RMB the random and the regular behavior of the network are observed over an extended region. The regular network corresponding to these two models is the one-dimensional Ising spin system with nearest-neighbor and next-nearest-neighbor interactions. We have studied the zero-temperature quenching dynamics for this model also and the results for the dynamics are identical to that of the nearest-neighbor Ising spin model. So it will be interesting to note how the dynamics is affected by the introduction of randomness in the Ising spin system and also how the difference in the nature of randomness of RMA and RMB shows up in the dynamics.

In the simulations, single-spin-flip Glauber dynamics is used in both cases and the spins are oriented randomly in the initial state. We have taken one-dimensional lattices of size L with $100 \leqslant L \leqslant 1500$ to study the dynamics. The results are averaged over (a) different initial configurations and (b) different network configurations. For each system size the number of networks considered is 50 and for each network the number of initial configuration is also 50. Periodic boundary conditions have been used.

III. QUANTITIES CALCULATED

We have estimated the following quantities in the present work.

(i) Magnetization $m(t)$. For a Ising spin system with regular connections and having only the ferromagnetic interaction, the order parameter is usually the magnetization $m = \frac{|\sum_i S_i|}{L}$, where L is the size of the system. Magnetization can be

considered as the order parameter, even when the connections are random. We have calculated the growth of magnetization with time and also the variation of the saturation value of the magnetization m_{sat} with p and α for RMA and RMB, respectively.

(ii) Persistence probability $P(t)$. As already mentioned, this is the probability that a spin does not flip until time t.

(iii) Energy $E(t)$. In these networks, the domain-wall measurement is not very significant as domains are ill defined. The presence of domain walls in regular lattices causes an energy cost [14]. So instead of the number of domain walls, the appropriate measure for disorder is the residual energy per spin $\varepsilon = E - E_0 = E + 4$, where $E_0 = -4$ is the known ground-state energy per spin and E is the energy of the dynamically evolving state. In fact, the magnetization is not a good measure of the disorder either since even when the energy is close to the ground state, magnetization may be very close to zero (this is also true for the models without randomness). So a residual energy measurement is the best way to find out whether the system has reached the equilibrium ground state or it is stuck in a higher-energy nonequilibrium steady state. We measure the decay of residual energy ε with time and the variation of its saturation value $\varepsilon_{\mathrm{sat}}$ with p and α for RMA and RMB, respectively.

(iv) Freezing probability. The probability with which any configuration freezes, i.e., does not reach the ground state (the state with magnetization $m = 1$ or the state with zero residual energy), is defined as the freezing probability.

(v) Saturation time. It is the time taken by the system to reach the steady state. It has been observed in some earlier studies [19] that it also shows a scaling behavior with the system size with the dynamical exponent z. This in fact provides an alternative method to estimate z when straightforward methods fail.
Both magntization and energy are regarded as dimensionless quantities (ϵ and E scaled by J) in this paper.

IV. DETAILED RESULTS OF QUENCHING DYNAMICS ON RMA

The results of a zero-temperature quench for the Ising model on RMA are presented in this section. Starting from an initial random configuration following a quench to zero temperature, the system cannot reach the ground state for *all* initial configurations for any $p \neq 0$. The magnetization, energy, persistence all attain a saturation value in time. The saturation values of all the quantities show nonmonotonic behavior as a function of p.

Figure 2 shows the decay of residual energy per spin and the growth of magnetization with time for different values of the rewiring probability. It should be noted that the dynamic quantities do not show any obvious power-law behavior beyond a few time steps. For small p, there is apparently a power-law behavior for a larger range of time, which we believe is the effect of the $p = 0$ point where such a scaling definitely exists.

The saturation value of the residual energy per spin $\varepsilon_{\mathrm{sat}}$ increases with the rewiring probability p (for small p), reaches a maximum for an intermediate value of p ($p < 1$), and then decreases again. This implies that the disorder of the spin system is maximum for a nontrivial value of $p = p_{\mathrm{maxdis}}$, which can be termed the point of maximum disorder. The saturation value of magnetization, in contrast, decreases for small p, takes its minimum value for another intermediate value of p ($p < 1$), and then increases again (Fig. 3).

The value of p_{maxdis} increases with the system size L for small L and then appears to saturate for larger system sizes. The value of the residual energy at p_{maxdis} also increases with the system size (Fig. 4). This establishes the existence of the point of maximum disorder at an intermediate value of p ($p \simeq 0.62$) even in the thermodynamic limit.

Magnetization reaches a minimum at a value of p that is *less* than p_{maxdis}. This implies that there exists a region where both magnetization and energy increase as p increases. This is also apparent from Fig. 3. The physical phenomenon responsible for this intriguing feature is conjectured and discussed in detail in Sec. VI B.

The saturation time decreases very fast with the rewiring probability p for small p and remains almost constant as p increases (Fig. 5). It is known that for $p = 0$ the saturation time varies as L^2; here it appears that for any $p > 0$, there is no noticeable size dependence.

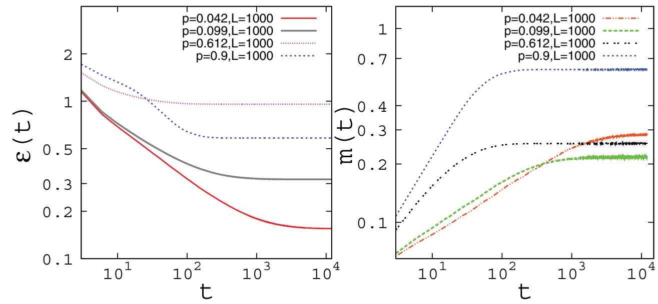

FIG. 2. (Color online) Decay of the residual energy per spin and the growth of magnetization with time for RMA for different probabilities.

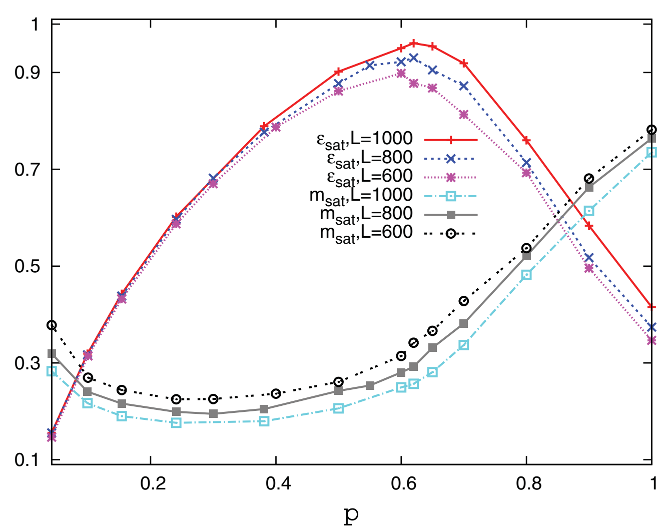

FIG. 3. (Color online) Saturation values of the residual energy per spin $\varepsilon_{\mathrm{sat}}$ and the magnetization m_{sat} plotted with the probability of rewiring p for RMA.

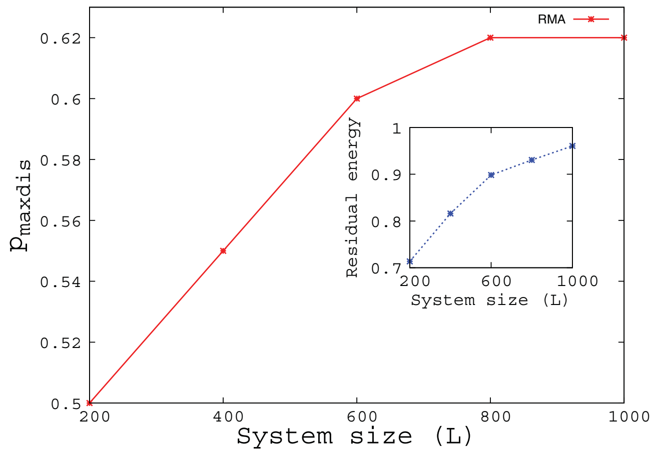

FIG. 4. (Color online) Rewiring probability at the point of maximum disorder plotted with the system size. The inset shows the increase of the residual energy at the point of maximum disorder with the increment of the system size.

For RMA the freezing probability is almost unity for small p. However, when the disorder is increased beyond $p \simeq 0.5$, the freezing probability shows a rapid decrease (Fig. 5, inset). In one dimension, we checked that the freezing probability is *zero* for the regular network ($p = 0$), but here we find that even for very small values of p, the freezing probability is unity. So there is a discontinuity in the freezing probability at $p = 0$. This also supports the fact that any finite p can make the dynamics different from a conventional coarsening process.

An interesting observation may be made about the behavior of the saturation value of the residual energy in the region $p < 0.5$. If one allows p to decrease from 0.5 to 0, the saturation value of the residual energy also decreases, although the freezing probability is unity in the entire region. This implies that in this range of the parameter, although the system does not reach the real ground state in any realization of the network (or initial configuration), such that $\epsilon \neq 0$ in each case,

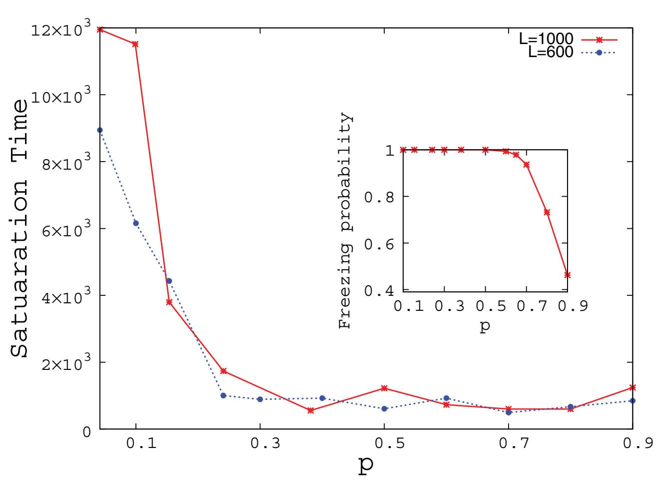

FIG. 5. (Color online) Time of saturation with the probability of rewiring plotted for two different sizes for RMA. The inset shows the variation of the freezing probability with the probability of rewiring for RMA.

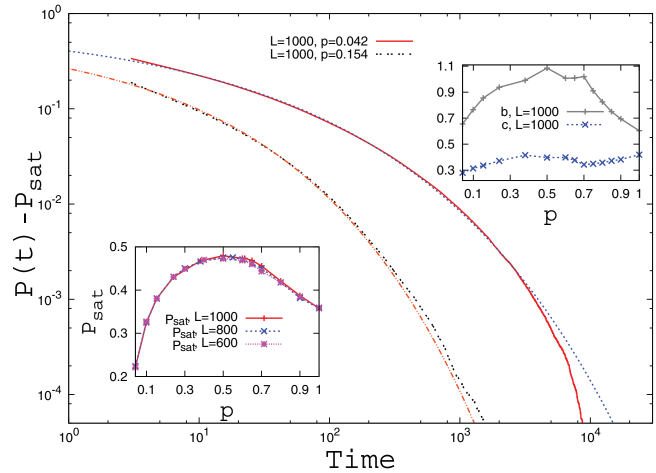

FIG. 6. (Color online) Decay of $P(t) - P_{\text{sat}}$ with time t along with the stretched exponential function found to fit its form. The bottom left inset shows the variation of the saturation value of persistence P_{sat} with p. The inset on the top right shows the variation of b and c with p.

the system has a tendency to approach the actual ground state monotonically with p for $p < 0.5$ (Fig. 3).

The persistence probability follows a stretched exponential behavior with time for any nonzero p, fitting quite well the form

$$P(t) - P_{\text{sat}} \simeq a \exp(-bt^c), \tag{2}$$

where P_{sat} is the saturation value of the persistence. It does not depend on the system size, but changes with the rewiring probability p; also there exists an intermediate value of p where the value of P_{sat} is maximum. The values of b and c vary nonmonotonically with p (Fig. 6).

V. DETAILED RESULTS OF QUENCHING DYNAMICS ON RMB

In this section we present the results of the zero-temperature quenching dynamics of the Ising model on RMB. Here also the system does not reach the ground state for *all* initial configurations for any finite value of α. The magnetization, energy, and persistence all attain a saturation value in time as in RMA. Figure 7 shows the decay of residual energy per spin and the growth of magnetization with time for different

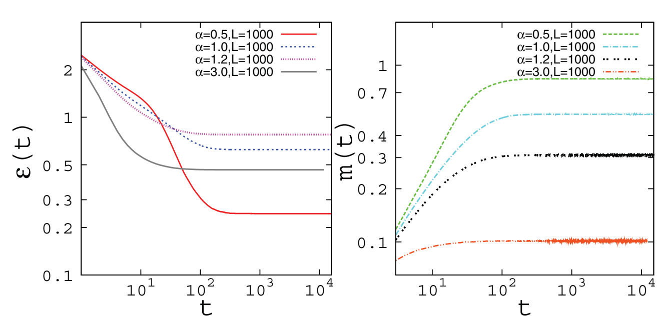

FIG. 7. (Color online) Decay of the residual energy per spin and the growth of magnetization with time for RMB for different probabilities.

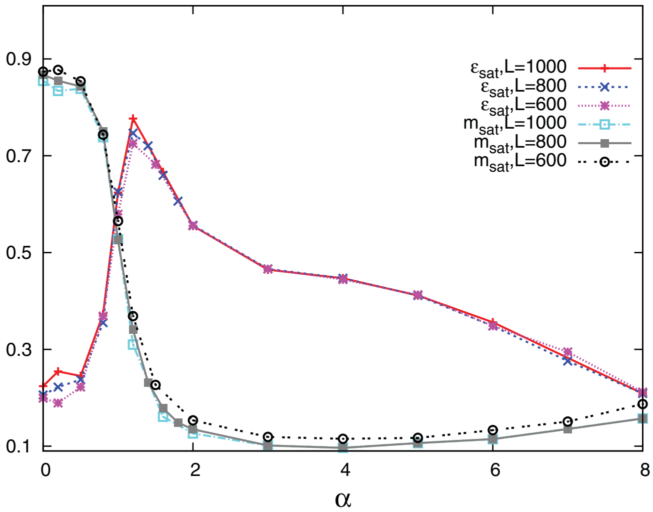

FIG. 8. (Color online) Saturation values of the residual energy per spin $\varepsilon_{\mathrm{sat}}$ and the magnetization m_{sat} plotted with α for RMB.

values of α. It should be noted that the dynamical quantities do not show any obvious power-law behavior for RMB either.

The saturation values of all the quantities show nonmonotonic behavior as a function of α. The saturation value of residual energy per spin $\varepsilon_{\mathrm{sat}}$ increases with α for small α, reaches a maximum for a finite value of α, and then decreases again. This implies that for RMB also, the disorder of the spin system is maximum for a finite value of α, which is the point of maximum disorder here. In contrast, the saturation value of the magnetization decreases for small α, takes its minimum value for another finite value of α, and then slowly increases (Fig. 8).

The value of $\alpha = \alpha_{\mathrm{maxdis}}$, at which the maximum disorder occurs, decreases with system size L for small L and then saturates for larger system sizes. The value of the residual energy at α_{maxdis} also increases with the system size (Fig. 9). This establishes the existence of the point of maximum

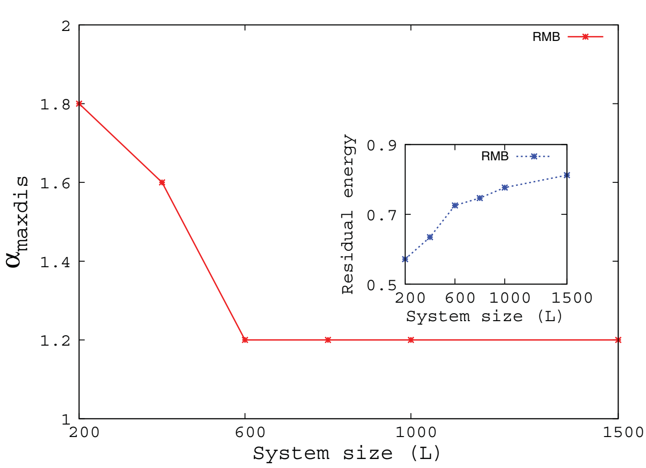

FIG. 9. (Color online) Value of α at the point of maximum disorder plotted with the system size. The inset shows the increase of residual energy at the point of maximum disorder with the increment of the system size.

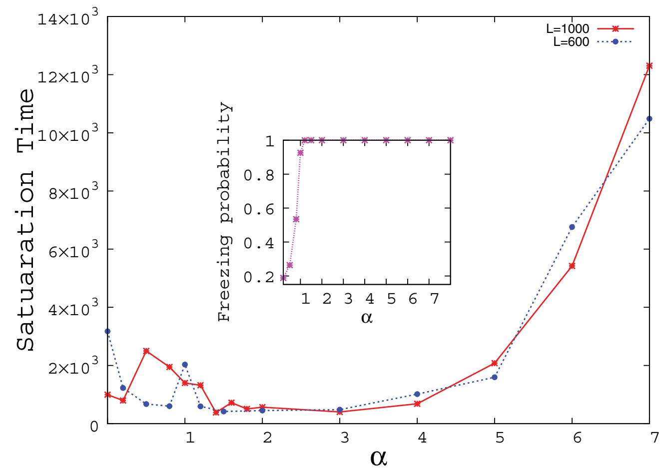

FIG. 10. (Color online) Time of saturation with the value of α plotted for two different sizes for RMB. The inset shows the variation of the freezing probability with α for RMB.

disorder, for RMB, at a finite value of α ($\alpha \simeq 1.2$) even in the thermodynamic limit. Similar to RMA, there is a region beyond $\alpha = 1.2$ where the energy and the magnetization both decrease, until the magnetization starts growing again. As already mentioned, this issue is addressed in Sec. IV B.

The saturation time for RMB in the random network in the region $0 \leqslant \alpha < 1$ shows fluctuations that are too large to let one conclude whether it is a constant in this region or varies with α. Beyond $\alpha = 1$ and up to $\alpha = 3.0$, it is almost independent of α. For $\alpha > 3$ the saturation time increases with α. There is no remarkable finite-size effect in the saturation time for RMB for any finite value of α. The saturation time varies as L^2 for a regular lattice corresponding to $\alpha \to \infty$, where it appears that for any finite α, however large, there is no remarkable size dependence.

The freezing probability is small for $\alpha = 0$ ($\simeq 0.2$) and increases rapidly with α for small α. The freezing probability becomes almost unity beyond $\alpha \simeq 1.2$ and remains the same for large α. It seems that for any finite $\alpha > 1.2$ the freezing probability remains unity and it will be zero only at $\alpha \to \infty$ (Fig. 10); as in one dimension, the freezing probability is *zero* for the regular network. So for RMB there is a discontinuity of the freezing probability at $\alpha = \infty$ that corresponds to the $p = 0$ point of RMA.

Beyond $\alpha \simeq 1.2$, the energy decreases with α though the freezing probability remains unity. This implies that although the system definitely reaches a frozen state, it approaches the real ground state monotonically as $\alpha \to \infty$ (Fig. 8). The preceding results indicate that even though for $\alpha > 2$ the network behaves as a regular one, dynamically the network is regular only at its extreme value $\alpha \to \infty$.

We find that the persistence probability follows roughly a stretched exponential form with time [given by Eq. (2)] for any finite α. The saturation value of the persistence P_{sat} does not depend on the system size, but changes with α; also there exists an intermediate value of α where the value of P_{sat} is maximum. For RMB also b and c vary nonmonotonically with α (Fig. 11).

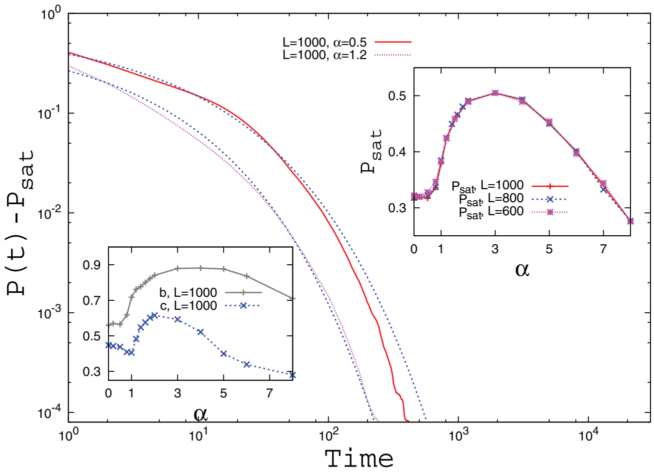

FIG. 11. (Color online) Decay of $P(t) - P_{\rm sat}$ with time t along with the stretched exponential function found to fit it approximately. The top right inset shows the variation of the saturation value of persistence $P_{\rm sat}$ with α. The inset on the bottom left shows the variation of b and c with α.

VI. DISCUSSION OF THE RESULTS

A. Comparison of the results for RMA and RMB

In Secs. IV and V the results of a quench at zero temperature for the Ising model on RMA and RMB have been presented separately. In this section we compare the results to understand how the difference in the nature of randomness affects the dynamics of the Ising spin system.

The gross features of the results are similar: In both models we have a freezing effect that makes the system get stuck in a higher-energy state compared to the static equilibrium state in which all spins are parallel. No power-law scaling behavior with time is observed in the dynamic quantities in either model. There exists a point in the parameter space where the deviation from the static ground state is maximum. The behavior of the saturation times and freezing probability as functions of the disorder parameters are also quite similar qualitatively.

The saturation values of magnetization and persistence attain minimum and maximum values, respectively, at an intermediate value of the relevant parameters in both models. The decay of the persistence probability also follows the same functional form in the entire parameter space. The saturation values of the persistence has no size dependence for either model. This indicates that as a whole, the dynamics is not affected much due to the change in the nature of randomness of the Ising spin system.

Let us consider the parameter values at which RMA and RMB are equivalent as a network: RMA and RMB behave as random networks at $p = 1$ and $\alpha = 0$, respectively. So one can expect that the saturation values of the residual energy per spin, magnetization, and the numerical value of the saturation time would be same at these values. However, the numerical values of these quantities are quite different. For RMA, at $p = 1$ the saturation value of the residual energy per spin $\varepsilon_{\rm sat} \simeq 0.415$, whereas for RMB at $\alpha = 0$, $\varepsilon_{\rm sat} \simeq 0.224$ for $L = 1000$. Similarly, we found numerically that for RMA the value of saturation magnetization $m_{\rm sat} \simeq 0.735$ for RMA and $m_{\rm sat} \simeq 0.855$ for RMB for the same system size. This is because even though the networks are both random here, the connections have a subtle difference. For RMA the number of second-nearest neighbor is exactly zero at $p = 1$ and all the other long-range-neighbor connections are equally probable. In contrast, for RMB second-nearest neighbors can still be present in the network and the probability is the same for this and any other longer-range connection. This difference in the nature of randomness affects the dynamics of the Ising spin system sufficiently to make the saturation values different. This means that the systems are locked at *different* nonequilibrium steady states. For RMB it is closer to the actual ground state as it is more short ranged in comparison.

The other values at which the two networks are equivalent are $p = 0$ and $\alpha > 2$, where regular network behavior is found as far as the network properties are concerned. Interestingly, the behavior of RMB even when α is finite and greater than 2 is not quite like the dynamics of a regular one-dimensional lattice with nearest- and next-nearest-neighbor links only. In fact, the point at which the magnetization becomes minimum is well inside the region $\alpha > 2$ and not within the small-world region as in RMA. Actually, there is an extended region of regular and random network behavior for RMB and as a result a few more interesting points are possible to observe here. Only at the extreme point $\alpha = \infty$ can the one-dimensional Ising exponents $z = 2.0$ and $\theta = 0.375$ be recovered as the frozen states continue to exist even for finite values of $\alpha > 2$ for RMB. For the model of the regular network with nearest and next-nearest neighbors, we have checked that there is no freezing at all. So discontinuities of the freezing probabilities occur at $p = 0$ and $\alpha = \infty$ on RMA and RMA, respectively.

Though the nature of randomness is different for RMA and RMB, for both models there exists a point of maximum disorder where the saturation value of the residual energy per spin attains a maximum value. For RMB maximum disorder of the Ising spin system occurs near the static phase transition point (small world to random phase), whereas for RMA the point of maximum disorder is well within the small-world region.

We try to explain this by considering the deviation from the point $p = 1$ (for RMA) and $\alpha = 0$ (for RMB). Two processes occur simultaneously here: (a) The number of connections with farther neighbors decreases and (b) clustering becomes more probable. As a result of these two processes, freezing occurs. For RMA the effect is *smaller* as there is less clustering [16]. For RMB, however, the effect is *greater* and spans the entire parameter space $\alpha > 1$ and therefore the point of maximum disorder of the Ising spin system is very close to the random–small-world phase transition point $\alpha = 1$.

The question may arise whether this difference prevails when the models are made even more similar. In RMB the probability $p_3(\alpha)$ that $l \geqslant 3$ can be expressed as a function of α,

$$p_3(\alpha) = \frac{\sum_{l=3}^{l=L/2} l^{-\alpha}}{\sum_{l=2}^{l=L/2} l^{-\alpha}}. \quad (3)$$

A further correspondence between the two networks can be established by imposing $p = p_3(\alpha)$, which makes the number of second-neighbor links in RMA and RMB also the same (but the rest of the extra links are connected differently).

Using Eq. (3), we can obtain the value of p corresponding to a given value of α and vice versa. However, it is immediately seen that the two networks are not equivalent even after making them similar up to the second-neighbor connections. For example, for $\alpha = 2.0$ the corresponding value of $p = 0.612$ in this scheme. However, we have already seen that while the point of maximum disorder occurs close to this value of p in RMA, the point of maximum disorder for RMB is considerably far from $\alpha = 2.0$. So the nature of randomness continues to affect the dynamics at least quantitatively.

B. Analysis of some general features of the quenching phenomena on networks

We find several interesting features in the quenching phenomena of Ising spin systems on both networks and in this section we attempt to provide an understanding of the same. It is intriguing that the results indicate that the minimum amount of randomness can make the system freeze. For a small amount of randomness the interactions are still dominantly nearest-neighbor type and domains in the conventional sense should grow, which will be of both plus and minus signs. The system will freeze as there will be some stable domain walls due to the few long-range interactions present. As the system attains saturation the domains will be small in number and large in size, irrespective of their signs. As a result, the magnetization attains a small value while the residual energy is still small.

This effect continues for some time until something more interesting happens. Take, for example, the case of quenching on RMA. There is a distinct region $0.4 < p < 0.6$ where the energy and magnetization grow simultaneously, an apparently counterintuitive result. Similar behavior can be noted for the quenching on RMB in a certain region in its parameter space. A problem in analyzing the situation for different p (or α) values is that the final frozen states are not, in principle, related in any way. This is because the energy landscapes change as p is changed and the initial configurations that undergo evolution are completely uncorrelated. In fact, in such a situation, even if the energy landscape is the same, with a number of local minima, different initial configurations will end up in different final nonequilibrium steady states. Nevertheless, one can attempt to explain this counterintuitive result by assuming that the final states are not largely different when p is changed slightly in the following way. This assumption and explanation are supported by the actual final states obtained for small system sizes.

Let us consider as an example RMA and take two values of p, $p_2 > p_1$, for which the magnetization and residual energy of the final state corresponding to p_2 are both larger than those for p_1. Now this can be possible due to the fragmentation of a larger domain into several domains such that the magnetization increases. This can be demonstrated with a simple example: Let us imagine a situation where one has only two domains of size N^+ (of up spins) and N^- (of down spins) for p_1 with magnetization equal to $m_1 = |(N^+ - N^-)|/L$ and assume that for p_2 the domain with N^+ up spins remains the same while the domain with N^- down spins gets fragmented into three domains of size ${N^-}_1, {N^+}_1$ and ${N^-}_2$ in the final state. For p_2, therefore, the magnetization is

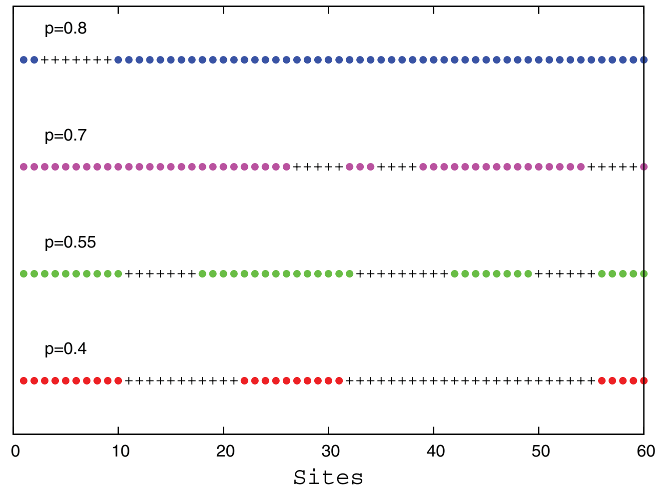

FIG. 12. (Color online) Snapshots of the final spin configurations for different values of the disorder parameter p for quenching on RMA. The + and • indicate up and down spins, respectively. The domains in the conventional sense are clearly visible.

$m_2 = |(N^+ + 2{N^+}_1 - N^-)|/L$, which is larger than m_1. Here in this hypothetical case we have assumed that $N^+ > N^-$ and p_2 is very close to p_1. One can also assume that the energy increases for p_2 as the system is still sufficiently short ranged and the new domain walls cause an extra energy compared to the state obtained for p_1. Of course this is an oversimplified picture where we have assumed that the final states for p_1 and p_2 are identical except for the fragmentation of one domain. However, we find that the final configurations obtained for small systems for different values of p as shown in Fig. 12 are consistent with our conjecture. These snapshots are representative of the real situation in the sense that they give a typical picture and are not just rare cases; we have obtained a similar picture from almost all such configurations generated for small systems.

As p increases further the domains do not fragment into even smaller pieces as the increasing number of long-range interactions again helps in the growth of so-called domains of one particular sign only such that the magnetization grows and the energy decreases. However, domains of both signs still survive, although the sizes are no longer comparable. It can therefore be expected that the region for which both magnetization and energy increase as a function of p or α would continue until the short-range interactions are dominating; our results are consistent with this expectation.

VII. CONCLUSION

In this paper we addressed the question of how the quenching dynamics of Ising spins depend on the nature of randomness of the underlying network by considering two networks in which the randomness is realized differently. The networks are the same up to the first-neighbor links and have the same average degree per node. While the qualitative features are the same, there are intricate differences occurring in the behavior of the saturation values of the dynamical quantities.

Overall, we found some interesting features. The saturation values of the dynamical quantities do not have monotonic behavior as a function of the disorder parameters. In particular, we found that increasing randomness does not necessarily make the system get locked in a higher-energy state. The dynamics takes the system to a steady state very quickly and the saturation times are not dependent on the system size. No scaling behavior is obtained from the studies either with time or with system size for any of the dynamic quantities. The most surprising result is perhaps the existence of a region in the parameter space where both the residual energy and the magnetization increase, which can be explained phenomenologically.

The zero temperature quench of Ising spins on the Euclidean model shows some surprising behavior in both the random and regular regions. We find that decreasing randomness makes the system end up in a higher-energy state in the random region, while in the regular region the familiar behavior of the Ising dynamics with short-range interactions is not obtained; in fact, the probability of freezing is unity here, indicating that in none of the realizations could the system end up in the static ground state. The saturation time also does not show scaling with time.

As already mentioned, the present study is relevant for dynamical social phenomena on complex networks. For example, the evolution of binary opinions on a complex network (where the initial states are randomly $+1$ and -1) is analogous to the dynamical study reported in the present paper. Of course, in the case of the opinion dynamics, the interactions could be more complex compared to the the simple Ising model. Our result indicates that the qualitative features of the results will not be much different for different complex networks.

Dynamic frustration [20] is responsible for freezing in many Ising systems where there is no frustration in the conventional sense. One interesting observation is that the nature of dynamic frustration in regular lattices of dimension greater than one and that in systems with random interaction (but no frustration) are, in general, quite different as in the latter one does not encounter the familiar scaling laws.

ACKNOWLEDGMENTS

Financial support from DST Project No. SR/S2/CMP-56/2007 and partial computational support from the UPE project are acknowledged (S.B. and P.S.). S.B. acknowledges financial support from CSIR (Grant No. 09/028(0818)/2010-EMR-I).

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