# Modeling asset price processes based on mean-field framework

Masashi Ieda* and Masatoshi Shiino
*Department of Physics, Faculty of Science, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan*
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We propose a model of the dynamics of financial assets based on the mean-field framework. This framework allows us to construct a model which includes the interaction among the financial assets reflecting the market structure. Our study is on the cutting edge in the sense of a microscopic approach to modeling the financial market. To demonstrate the effectiveness of our model concretely, we provide a case study, which is the pricing problem of the European call option with short-time memory noise.

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## I. INTRODUCTION

Numerous attempts have been made to describe the dynamics of prices of financial assets (stocks, bonds, foreign exchanges, and so on). Even if we discuss studies based only on stochastic processes, the history of these studies is not short: the first model based on stochastic processes to describe financial assets was introduced by Bachelier in 1900 [1]. In 1973 Black-Scholes model [2], one of the most famous model which employs geometric Brownian motions for describing the dynamics, was proposed. The Black-Scholes model has had an enormous impact on both academic studies and practical businesses, for example, implied volatilities are still calculated from Black-Scholes model in some practical business. The reasons for still employing the Black-Scholes model in a large number of scientific as well as business worlds are its simplicity and elegant mathematical properties.

After the 1980s, models taking into account more realistic features of markets have been studied extensively and intensively. One of the most notable market features is a long-time correlation in absolute values of price fluctuations and associated power-law tails in distributions of either price fluctuations [3] or assets and debts [4]. There exist some models proposed to describe the former feature from the viewpoint of econometrics [5,6]. Another famous feature of real markets is the volatility smile, a significant issue in mathematical finance. This is a phenomenon where an a implied volatility calculated from real market data is well fitted to a U-shape function, while the Black-Scholes model implies a flat-shape function. To explain the smile a lot of models have been designed [7,8]. These models are categorized as stochastic volatility models.

As mentioned above, econometrics and mathematical finance have designed models which are well fitted to real financial data; hence they have paid a little attention to including a structure of financial markets into models. Their most focused purposes are applications of models to issues such as the risk management or the pricing of derivative securities. Hence from their viewpoint, the better models are what describe real market features which they want more precisely. Whether the models include the structure of the markets or not is not a significant issue for them. However, for the purpose of understanding the financial markets, we are not able to avoid considering the model including the structure. Thus research whose primary purpose is considering the structure is required, and we argue that an approach from physics meets such a requirement. From a physics perspective, the structure should be of more concern than the applications. The market is regarded as a many-body complex interacting system. Theoretical statistical physics provides powerful tools to approach complex systems which are difficult to deal with numerically. For example, the mean-field framework is considered to describe the essence of phase transitions in spin systems under an assumption of reducing degrees of freedom in many-body systems. We expect that the mean-field framework grasps the essence of many-body interaction effects from financial markets. We are assured that the approach from physics is suitable for this purpose since it satisfies both motivation and method.

Our study provides a model of the dynamics of financial assets within the mean-field framework. One of the advantages of our model includes the market structure (also called interactions among financial assets), and this suits the purpose which we explained above. Almost all previous studies about describing the dynamics of financial assets from a physics perspective [9–14] take macroscopic approaches without considering the interactions, except for some studies (e.g., the studies are based on random matrix theory [15,16]). Thus our study is on the cutting edge in the sense that we take a microscopic approach by considering interactions among assets to emphasize the structure of financial markets.

The organization of the present paper is as follows. We introduce our model in Sec. II, where keywords are log return, the market structure (interactions among financial assets), and mean-field concept. We begin to build our model from the concept of log return. The structure we consider is trading volume changes due to asset price deviation from expected asset return rates. While the many-body model including the structure is difficult to treat, the mean-field concept allows us to reduce the degrees of freedom to obtain a solvable model. Section III provides a case study, a pricing problem of the European call option under the influence of short-time memory noise to discuss the effectiveness of our model. A European call option is a financial instrument which is a right to buy some underlying asset from the drawer of the option for the strike price at maturity [17,18]. We first mention that the time correlation of the noise does not affect the price

*mieda@mikan.ap.titech.ac.jp

 

of the option and we are not able to understand this issue intuitively. We extend the short-time memory noise model to the mean-field model discussed in Sec. II. Although there are a lot of methods to calculate the price of a European call option, we employ the risk neutral method [19,20]. In this method the price is calculated from computing the expectation under the corresponding risk neutral process of underlying asset. Due to the mean-field framework we are able to provide semianalytical solutions of this pricing problem. Time evolution equations of the order parameters and the Fokker-Planck equation play a central role for the pricing.

The main results of this case study are the following: (i) in the one-body problem, the time correlation effect vanishes, (ii) within the context of the mean-field model derived from the $N$-body model, the time correlation effect begins to appear, and (iii) the effect so obtained can be understood intuitively.

## II. MODEL BASED ON MEAN-FIELD FRAMEWORK

### A. Review of Black-Scholes model

Since our model is built from the log return and is then regarded as an extended Black-Scholes model, we briefly discuss the Black-Scholes model. The Black-Scholes model is constructed by a risky asset $S$ and a risk-free asset $B$ and is described by the following stochastic processes:

$$S(t) = S(0)e^{R(t)}, \tag{1}$$

$$dR(t) = \mu dt + \sigma dW(t), \quad R(0) = 0, \tag{2}$$

$$dB(t) = rB(t)dt, \tag{3}$$

where $\mu$ and $\sigma$ are constants, $W$ is a Wiener process, and $R$ is called a log return of the risky asset.

The main reason we deal with the Black-Scholes model is the simplicity of the model. Since our main purpose is to clarify the effects of interactions among many assets, each asset price model should be as simple as possible. A stochastic process characterized in terms of only mean value and variance is one of the simplest methods; hence choosing the Black-Scholes model as the base model satisfies this requirement.

Another reason is the clear concept regarding where we should include the market structure in the model, as is clarified below. Let us consider the meaning of the parameters $\mu$ and $\sigma$: $\mu$ represents the expected return rate, which means the average of predictions about the return rate of risky assets by market players, and $\sigma$ is the variance of $R$, which represents how the predictions of market players deviate from the expected return. If we assume that every market player has the same information, predictions about the return rate of market players does not differ as much. Hence we should not set the market structure in the expected return rate. Therefore we assume that the market structure should be included in the second term of Eq. (2).

### B. $N$-body interaction model

From the discussion in Sec. II A, we propose a general $N$-body interaction model in the following form:

$$S_i(t) = S_i(0)e^{R_i(t)} \quad i = 1,\cdots,N, \tag{4}$$

$$dR_i(t) = \mu_i dt + \sigma_i(t,\{S_i\}_{1<i<N})d\eta_i(t), \quad R_i(0) = 0, \tag{5}$$

$$dB(t) = rB(t)dt, \tag{6}$$

where $N$ is a number of assets in the market, $\eta_i(t)$ are certain stochastic processes, and they no longer need to be Wiener processes. The functions $\sigma_i$ are the terms that include the interaction among the assets to reflect the market structure. The variable $\eta_i(t)$ allows the noise-oriented effects (e.g., time correlation effect) to be included in the model.

We specify the functions $\sigma_i$ and suppose the following equations:

$$\mu_i = \mu, \tag{7}$$

$$\sigma_i(t,\{S_i\}_{1<i<N}) = \sqrt{\sigma_M + \sum_{k=1}^{N} J_{ik}V[z_k(t)]}, \tag{8}$$

$$z_k(t) = R_k(t) - \mu t, \tag{9}$$

where $\mu$,$\sigma_M$ and $J_{ik}$ are parameters that are assumed to take non-negative constant values, and $\mu$ is regarded as the common expected return rate among the all assets in the market. In this case $\sigma_i$ is characterized by two terms, a common term of the all assets $\sigma_M$ and an inherent term of asset $i$ representing the interaction. $J_{ik}$ represents the intensity of interaction between asset $i$ and $k$. The function $z_k(t)$, the key factor of the inherent term, is a deviation of the return rate of asset $k$ from the common expected return at time $t$. The non-negative function $V$ is a trading volume function which represents the volatility change effect due to the trading volume change.

We need to show why we assume that $z_k(t)$ becomes the argument of $V$. As mentioned in Sec. II A, $\mu t$ is regarded as the average expected return of the predictions of market players at time $t$. Hence if $z_k(t)$ is a positive (or negative, respectively) large value, the asset $k$ is regarded as an overestimated (or underestimated, respectively) asset by a lot of market players. For the overestimated (or underestimated) assets, market players will take the short (or long) positions and then the trading volumes of the assets are expected to increase. Therefore $V$ is a function of $z_k(t)$.

Another issue we need to explain is why the trading volumes change effect is responsible for occurrence of interaction. The costs for making a deal with asset $k$ will be funded by buying or selling some other assets. Thus the trading volumes of the associated assets are expected to increase. Hence this effect brings about a change in $\sigma_i$ through the interaction $J_{ik}$.

We do not give the specific form of the function $V$ in this section; $V$ should be determined from investigations regarding the trading volume changes (e.g., [21,22]).

### C. Mean-field interaction model

We have supposed several issues in the previous section; however, the evaluation is still difficult. We tackle this problem within the mean-field framework and we reduce the $N$-body problem to a one-body problem. To apply the mean-field framework we assume the following:

$$J_{ik} = \frac{\epsilon}{N}, \tag{10}$$

$$V(x) = x^2, \tag{11}$$

where $\epsilon$ is a constant which represents the intensity of the interaction, i.e., all interactions among the assets are the same and the order of $J_{ik}$ arises from the proper scaling with a number $N$ of the assets in the large $N$ limit. The dynamics of

$z_i(t)$ is written by

$$dz_i(t) = \sqrt{\sigma_M + \sum_{k=1}^{N} \frac{\epsilon}{N} z_k^2(t)}\, d\eta_i(t). \quad (12)$$

We also assume that $\eta_i(t)$ are independent and identically distributed random variables, and then $z_i(t)$ follows the same dynamics. Hence the value of $\sum_{k=1}^{N} J_{ik} z_k^2(t)$ in the limit $N \to \infty$ is represented by the following form:[1]

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{\epsilon}{N} z_k^2(t) = \epsilon \mathbb{E}[z^2(t)], \quad (13)$$

where $\mathbb{E}$ denotes the expectation of a random variable and $z(t)$ is called the representative process of $z_i(t)$ and is defined by

$$dz(t) = \sqrt{\sigma_M + \epsilon \mathbb{E}[z^2(t)]}\, d\eta(t). \quad (14)$$

Here $z(t)$ is defined self-consistently, and $z(t)$ includes many body interaction effects.

From this discussion we define the mean-field interaction model as follows:

$$S(t) = S(0)e^{R_i(t)}, \quad (15)$$

$$dR(t) = \mu_i dt + \sigma(t) d\eta(t), \quad R(0) = 0, \quad (16)$$

$$dB(t) = rB(t)dt, \quad (17)$$

where

$$\sigma(t) = \sqrt{\sigma_M + \epsilon \mathbb{E}[z^2(t)]}, \quad (18)$$

$$z(t) = R(t) - \mu t, \quad (19)$$

and $\eta(t)$ is a certain stochastic process. The key points of the mean-field interaction model are as follows:

(1) Assuming the interactions are same with the order $\frac{1}{N}$, and

(2) A reduced one-body problem including many-body interactions as mentioned above.

[1] The validity of this method is given by Dawson and Gärtner [23].

## III. CASE STUDY: PRICING PROBLEM OF EUROPEAN CALL OPTION

To show the effectiveness of our model, we provide a case study—a pricing problem of the European call option which starts from a simple extension of the Black-Scholes model with a short-time memory noise.

### A. A paradox in short-time memory noise Black-Scholes model

We consider a simple extension of the Black-Scholes model using a short-time memory noise. We define a short-time memory noise as given by the following stochastic differential equation:

$$d\eta(t) = -a\eta(t)dt + dW(t), \quad (20)$$

where $a$ is a positive constant. This is the well-known Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process has the transient states and a stationary state attained in the limit $t \to \infty$. We write probability density of $\eta(t)$ as $p(t,\eta_t)$, and then the time evolution equation of $p(t,\eta_t)$ is well known to be given by the corresponding Fokker-Planck equation. To avoid the transient states, we set $p(0,\eta_0)$ to the stationary probability density:

$$p(0,\eta_0) = \sqrt{\frac{a}{\pi}} e^{-a\eta_0^2}. \quad (21)$$

If such an initial condition is given, we call $d\eta(t)$ the stationary noise. We call $d\eta(t)$ the nonstationary noise if the initial condition is given as

$$p(0,\eta_0) = \delta(\eta_0 - \eta_{\rm init}), \quad (22)$$

where $\eta_{\rm init}$ is a constant value. We consider both the stationary case and the nonstationary case.

We now discuss the time correlation of our noise. For a precise discussion, we treat $d\eta(t)$ more carefully. Since $d\eta(t)$ is a limit of $\eta(t+\Delta t) - \eta(t)$ ($\Delta t \searrow 0$), let us consider the time correlation of the noise up to $(\Delta t)^2$ order (see Appendix):

$$\mathbb{E}[(\eta(t+\Delta t) - \eta(t))(\eta(u+\Delta t) - \eta(u))] \approx \begin{cases} -\frac{1}{2}ae^{-a(t-u)}(\Delta t)^2 + a^2\left\{\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right\}e^{-a(t+u)}(\Delta t)^2 & (t \neq u) \\ \Delta t - \frac{1}{2}a(\Delta t)^2 + a^2\left\{\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right\}e^{-2at}(\Delta t)^2 & (t = u). \end{cases} \quad (23)$$

Notice the coefficients of $(\Delta t)^2$, and we understand the noise exhibits the short-time memory and its time correlation length depends on the parameter $a$. We mention the time correlation intensity also depends on $a$. If $a \ll 1$, then $e^{-a(t-u)} \simeq 1$ and $e^{-a(t+u)} \simeq 1$; thus the intensity increases with $a$. If $a$ has a large value, then $e^{-a(t-u)} \simeq 0$ and $e^{-a(t+u)} \simeq 0$, and hence the time correlation vanishes. At first the time correlation intensity increases with increasing $a$, and after $a$ exceeds a certain level the time correlation intensity starts decreasing.

Let us consider the case of replacing $dW(t)$ in the Black-Scholes risky asset process Eq. (2) with $d\eta(t)$:

$$dR(t) = \mu dt + \sigma d\eta(t) = (\mu - a\sigma\eta(t))dt + \sigma dW(t). \quad (24)$$

Since the drift part of the risky asset does not affect the pricing of derivative securities under the arbitrage theory, the time correlation parameter $a$ does not affect the price of the European call option. However, prices of derivative securities are regarded as the risk premium, and the time correlation gives information which reduces a risk due to uncertainties. Hence we are not able to understand this result intuitively.

### B. Mean-field interaction model with short-time memory noise and the corresponding risk neutral process

To resolve the paradox in Sec. III A we extend the risky asset (24) to a many-body system and apply the mean-field framework discussed in Sec. II. The corresponding mean-field interaction model is defined by the following form:

$$S(t) = S(0)e^{R(t)}, \tag{25}$$

$$dR(t) = \mu dt + \sigma(t)d\eta(t) \tag{26}$$

$$= (\mu - a\sigma(t)\eta(t))dt + \sigma(t)dW(t), \tag{27}$$

$$dB(t) = rB(t)dt, \tag{28}$$

where

$$d\eta(t) = -a\eta(t)dt + dW(t), \tag{29}$$

$$\sigma(t) = \sqrt{\sigma_M + \epsilon\mathbb{E}[z^2(t)]}, \tag{30}$$

$$z(t) = R(t) - \mu t. \tag{31}$$

As previously mentioned in the Introduction section, our option pricing method is an evaluation of certain expectations under the corresponding risk neutral process. We donate by $\tilde{S}(t)$ the corresponding risk neutral process of Eqs. (25) and (28) and then the equation for $\tilde{S}(t)$ is given by[2]

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = rdt + \sigma(t)d\tilde{W}(t). \tag{32}$$

### C. Time evolution equations of the order parameters

Before we tackle the calculation of expectations, we have to solve the problem about $\mathbb{E}[z^2(t)]$ in $\sigma(t)$. Since $\mathbb{E}[z^2(t)]$ is an order parameter of $z(t)$, a method using the time evolution equations of the order parameters is convenient for the purpose of dealing with $\mathbb{E}[z^2(t)]$. From Eqs. (26) and (31), the stochastic process of $z(t)$ reads

$$dz(t) = \sigma(t)d\eta(t)$$

$$= \sqrt{\sigma_M + \epsilon\mathbb{E}[z^2(t)]}[-a\eta(t)dt + dW(t)]. \tag{33}$$

Hence the time evolution of $\mathbb{E}[z^2(t)]$ is calculated from the following ordinary differential equations:

$$\frac{d}{dt}\{\mathbb{E}[z^2(t)]\} = -2a\sqrt{\sigma_M + \epsilon\mathbb{E}[z^2(t)]}\mathbb{E}[\eta(t)z(t)] + \{\sigma_M + \epsilon\mathbb{E}[z^2(t)]\}, \tag{34}$$

$$\frac{d}{dt}\{\mathbb{E}[\eta(t)z(t)]\} = -a\mathbb{E}[\eta(t)z(t)] - a\sqrt{\sigma_M + \epsilon\mathbb{E}[z^2(t)]} \times \left\{\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right\}e^{-2at}. \tag{35}$$

From Eqs. (26) and (31), initial conditions for the above equations read

$$\mathbb{E}[z^2(0)] = 0, \quad \mathbb{E}[\eta(0)z(0)] = 0. \tag{36}$$

Basically, we have to resort to a numerical method to calculate $\mathbb{E}[z^2(t)]$; however, in the case of $a = 0$ or the case where the noise is stationary we obtain the analytical solution of $\mathbb{E}[z^2(t)]$:

$$\mathbb{E}[z^2(t)] = \frac{\sigma_M}{\epsilon}(e^{\epsilon t} - 1). \tag{37}$$

[2] $\tilde{W}(t)$ means an another Wiener process which differs from $W(t)$.

### D. Fokker-Planck equation

To calculate the expectations under the risk neutral process $\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t)$, the conditional probability density of $\tilde{S}$ is a powerful tool. $\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t)$ is a solution of the corresponding Fokker-Planck equation of Eq. (32), which reads

$$\frac{\partial}{\partial t'}\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t) = -\frac{\partial}{\partial \tilde{s}_{t'}}[r\tilde{s}_{t'}\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t)] + \frac{1}{2}\frac{\partial^2}{\partial \tilde{s}_{t'}^2}\left[\tilde{s}_{t'}^2\sigma^2(t)\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t)\right]. \tag{38}$$

Fortunately, we are able to obtain a semianalytical solution of Eq. (38) in the following form:

$$\tilde{p}(t',\tilde{s}_{t'}|t,\tilde{s}_t) = \frac{1}{\tilde{s}_{t'}\sqrt{2\pi J(t,t')}}e^{-\frac{1}{2J(t,t')}(\ln\frac{\tilde{s}_{t'}}{\tilde{s}_t} - r(t'-t) + \frac{1}{2}J(t,t'))^2}, \tag{39}$$

where

$$J(t,t') = \int_t^{t'} \sigma^2(u)du. \tag{40}$$

### E. Pricing of the European call option

We now tackle the pricing problem of the European call option. Let us consider a European call option whose underlying asset is $S$, maturity is $T$, and the strike price is $K$. We write its price at time $t$ as $C(t,S(t))$ and then $C(t,S(t))$ has to satisfy the termination condition

$$C(T,S(T)) = (S(T) - K)^+, \tag{41}$$

where the function $(x)^+$ represents $x$ when $x \geqslant 0$ and 0 when $x < 0$. According to the risk neutral method, $C(t,s)$ is represented by the following expectation:

$$C(t,s) = e^{-r(T-t)}\tilde{\mathbb{E}}[(\tilde{S}(T) - K)^+|t,s]. \tag{42}$$

We already obtained the conditional probability density $\tilde{p}(T,\tilde{s}_{t'}|t,\tilde{s}_{t'})$, so then $C(t,s)$ reads

$$C(t,s) = e^{-r(T-t)}\int_{-\infty}^{\infty}(\tilde{s}_T - K)^+\tilde{p}(T,\tilde{s}_T|t,s)d\tilde{s}_T. \tag{43}$$

From Eq. (39) we obtain $C(t,s)$ in the following semianalytical form:

$$C(t,s) = sN(d_+) - Ke^{-r(T-t)}N(d_-), \tag{44}$$

where

$$d_+ = \frac{1}{\sqrt{J(t,T)}}\left[\ln\frac{s}{K} + r(T-t) + \frac{1}{2}J(t,T)\right], \tag{45}$$

$$d_- = \frac{1}{\sqrt{J(t,T)}}\left[\ln\frac{s}{K} + r(T-t) - \frac{1}{2}J(t,T)\right], \tag{46}$$

and

$$N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy. \tag{47}$$

To show the validity of the present model, let us consider a case of $\epsilon = 0$. Since $\sigma(t) = \sigma$ at this time, $J(t,T) = \sigma_M(T-t)$ and hence $C(t,s)$ coincides with the Black-Scholes formula.

To discuss the effects of the interactions, we illustrate $C(0,s)$ with focusing on the parameter $\epsilon$ as in Fig. 1.

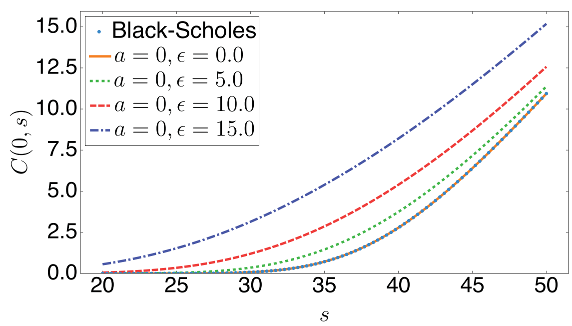


FIG. 1. (Color online) $C(0,s)$, the price of European call option at time $t = 0$ as a function of underlying asset price $s$ for $T = 1/4$ year, $r = 8\%$/year, $\sigma_M = 0.09$, and $K = 40$. The illustration demonstrates the effect of $\epsilon$. Circle mark represents the Black-Scholes case. The orange line, green dotted line, red dashed line, and blue chain line represent $C(0,s)$ with $\epsilon = 0.0$, $\epsilon = 5.0$, $\epsilon = 10.0$, and $\epsilon = 15.0$, respectively.

We can see that the option price behaves in the same manner as $\epsilon$ qualitatively. Since prices of financial derivatives are regarded as risk premiums and $\epsilon$ is the coefficient of an active deal effect, this result shows that the price changes reflect propagations of risk from other assets.

We next discuss the effect of time correlation. We illustrate $C(0,s)$ with focusing on the parameter $a$ as is shown in Fig. 2.

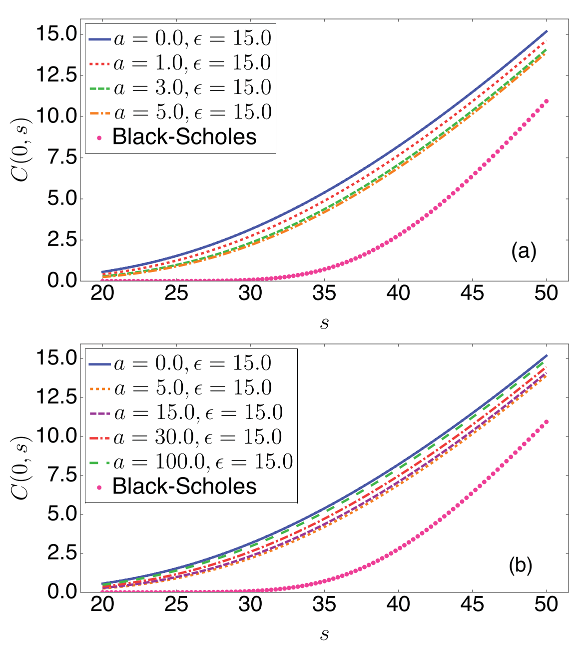


FIG. 2. (Color online) $C(0,s)$, the price of the European call option at time $t = 0$ as a function of underlying asset price $s$ for $T = 1/4$ year, $r = 8\%$/year, $\sigma_M = 0.09$, and $K = 40$. In this case we employ the nonstationary noise with $\eta_{\rm init} = 0$. The illustration demonstrates the effect of $a$. In (a), The circle mark represents Black-Scholes case. The blue line, red dotted line, green dashed line, and orange chain line represent $C(0,s)$ with $a = 0.0$, $a = 1.0$, $a = 3.0$, and $a = 5.0$, respectively. In (b), the circle mark represents the Black-Scholes case. The blue line, orange dotted line, purple dashed line, red chain line, and green sparse line represent $C(0,s)$ with $a = 0.0$, $a = 5.0$, $a = 15.0$, $a = 30.0$, and $a = 100.0$, respectively.

We observe that once the price $C(0,s)$ decreases with increasing $a$ and after $a$ exceeds a certain level, $C(0,s)$ goes back to the value when $a = 0$ with increasing $a$. As previously mentioned, prices of options are regarded as risk premiums; hence this result shows that the time correlation effect reduces the risk. This is quite natural and we comprehend the mechanism which underlies the obtained result by the following steps: (i) $a$ is the parameter which represents the time correlation intensity, (ii) hence $a$ is a part of information that we know about the market, and (iii) since increasing $a$ corresponds to increasing amount of information, it implies reducing the risk. We are also able to understand the increasing $C(0,s)$ after $a$ exceeds a certain level in terms of the behavior of a. As we discussed in Sec. III A, at first the time correlation intensity increases with increasing $a$ and after $a$ exceeds a certain level, the time correlation intensity start decreasing. This corresponds to the behavior of $C(0,s)$ with changing $a$, and $a$ represents amount of our information; hence the behavior of $C(0,s)$ with changing $a$ is explained by the amount of information.

The most important result in this section is observation of the risk reduction effect through the parameter $a$ when we include the market structure in the model. Since we are able to understand this intuitively, the paradox in Sec. III A is resolved.

## IV. SUMMARY

We have studied a model of the dynamics of financial assets based on the mean-field framework and the effectiveness of the model. On the basis of the concept of log return, we have constructed the $N$-body model for the purpose of including the market structure. Thus our model is regarded as an extended Black-Scholes model. We extended it to the $N$-body model for the purpose of building a model which includes the market structure. The market structure is represented by the interactions among the financial assets, and we call the outcome of the interaction the active deal effect. Although the $N$-body model is difficult to treat, we are able to make it solvable by employing the mean-field framework. We emphasize two significant features of our model:

(1) The model includes the mean-field-type interactions among the assets as the active deal effect.

(2) The model takes the form of a reduced one-body model including many-body interactions.

The effectiveness of our model has been demonstrated by the case study, which is the pricing problem of the European call option. We first have considered a simple extension of the Black-Scholes model which takes the short-time memory noise into account. It leads a paradox where the short-time memory effect does not bring about a change in the price of the European call option. To resolve the paradox we have extended the model, using the short-time memory noise, to the corresponding mean-field model. We have obtained a semianalytical expression of the price of the European call option by taking advantage of use of the time evolution equations of the order parameters and the Fokker-Planck equation. The obtained results show that the price evaluated from the mean-field model undergoes the influence of the short-time memory effect, and this means that the paradox is resolved. This result is significant in the sense that the model successfully grasps the effect which disappears in the system without interactions.

We compare our study with previous ones. Our study is on the cutting edge in the sense of a microscopic approach to modeling the financial market; hence this study has few correlations with previous studies. However, the Fokker-Planck equation approach developed in physics [24–27] allows us to obtain the semianalytical solution of the European call option. Since our $N$-body model belongs to a type of stochastic volatility model, the mean-field model is considered as an approximation model of the stochastic volatility model. This aspect implies that the models developed further from our model allow us to understand the stochastic volatility model more completely.

## APPENDIX: EVALUATION OF THE TIME CORRELATION OF THE NOISE Eq. (20)

Let us evaluate the time correlation of the noise Eq. (20). Suppose $t \geqslant u$, then we have

$$\begin{aligned}
\mathbb{E}[\eta(t)\eta(u)] &= \mathbb{E}\left[\left(\eta(0)e^{-at} + \int_0^t e^{-a(t-v)}dW(v)\right)\left(\eta_0 e^{-au} + \int_0^u e^{-a(u-v)}dW(v)\right)\right] \\
&= \mathbb{E}[\eta_0\eta_0]e^{-a(t+u)} - \mathbb{E}\left[\eta_0\right]e^{-at}\mathbb{E}\left[\int_0^u e^{-a(u-v)}dW(v)\right] - \mathbb{E}\left[\int_0^t e^{-a(t-v)}dW(v)\right]\mathbb{E}[\eta_0]e^{-au} \\
&\quad + \mathbb{E}\left[\int_0^t e^{-a(t-v)}dW(v)\int_0^t e^{-a(t-v)}dW(v)\right] \\
&= \mathbb{E}[\eta^2(0)]e^{-a(t+u)} + \mathbb{E}\left[e^{-at}\left(\int_0^u e^{av}dW(v) + \int_u^t e^{au}dW(v)\right) + e^{-av}\int_0^u e^{av}dW(v)\right] \\
&= \mathbb{E}[\eta^2(0)]e^{-a(t+u)} + e^{-a(t+u)}\left\{\mathbb{E}\left[\left(\int_0^u e^{av}dW(v)\right)^2\right] + \mathbb{E}\left[\int_0^u e^{av}dW(v)\right]\mathbb{E}\left[\int_u^t e^{av}dW(v)\right]\right\}.
\end{aligned}$$

Applying the Ito isometry, we obtain

$$\begin{aligned}
\mathbb{E}[\eta(t)\eta(u)] &= \mathbb{E}[\eta^2(0)]e^{-a(t+u)} + e^{-a(t+u)}\mathbb{E}\left[\int_0^u \left(e^{av}\right)^2 dv\right] \\
&= \mathbb{E}[\eta^2(0)]e^{-a(t+u)} + \frac{1}{2a}e^{-a(t+u)}(e^{2au} - 1) \\
&= \frac{1}{2a}e^{-a(t-u)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u)}.
\end{aligned}$$

We require the different treatments whether $t > u$ or $t = u$. We first evaluate in the case of $t > u$:

$$\begin{aligned}
\mathbb{E}\left[(\eta(t+\Delta t) - \eta(t))(\eta(u+\Delta t) - \eta(u))\right] &= \mathbb{E}\left[\eta(t+\Delta t)\eta(u+\Delta t)\right] - \mathbb{E}\left[\eta(t+\Delta t)\eta(u)\right] - \mathbb{E}\left[\eta(t)\eta(u+\Delta t)\right] + \mathbb{E}\left[\eta(t)\eta(u)\right] \\
&= \frac{1}{2a}e^{-a(t+\Delta t-u-\Delta t)} + \left(\mathbb{E}\left[\eta^2(0)\right] - \frac{1}{2a}\right)e^{-a(t+\Delta t+u+\Delta t)} + \frac{1}{2a}e^{-a(t+\Delta t-u)} \\
&\quad + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+\Delta t+u)} + \frac{1}{2a}e^{-a(t-u-\Delta t)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u+\Delta t)} \\
&\quad + \frac{1}{2a}e^{-a(t-u)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u)}.
\end{aligned}$$

Thus we have

$$\mathbb{E}\left[(\eta(t+\Delta t) - \eta(t))(\eta(u+\Delta t) - \eta(u))\right] = \frac{1}{2a}e^{-a(t-u)}[2 - e^{-a\Delta t} - e^{a\Delta t}] + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u)}[e^{-(2a\Delta t)} - 2e^{-a\Delta t} + 1].$$

Expanding this up to $(\Delta t)^2$ order, we obtain

$$\begin{aligned}
\mathbb{E}[(\eta(t+\Delta t) - \eta(t))(\eta(u+\Delta t) - \eta(u))] &= \frac{1}{2a}e^{-a(t-u)}\left[2 - \left(1 - a\Delta t + \frac{1}{2}a^2(\Delta t)^2\right) - \left(1 + a\Delta t + \frac{1}{2}a^2(\Delta t)^2\right)\right] \\
&\quad + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u)}\Bigg[(1 - 2a\Delta t + 2a^2(\Delta t)^2) \\
&\quad - 2\left(1 - a\Delta t + \frac{1}{2}a^2(\Delta t)^2\right) + 1\Bigg] \\
&= -\frac{1}{2}ae^{-a(t-u)}(\Delta t)^2 + a^2\left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+u)}(\Delta t)^2.
\end{aligned}$$

We next consider the case of $t = u$:

$$\begin{aligned}\mathbb{E}[(\eta(t+\Delta t)-\eta(t))(\eta(t+\Delta t)-\eta(t))] &= \mathbb{E}[\eta(t+\Delta t)\eta(t+\Delta t)] - 2\mathbb{E}[\eta(t+\Delta t)\eta(t)] + \mathbb{E}[\eta(t)\eta(t)]\\ &= \frac{1}{2a}e^{-a(t+\Delta t-t-\Delta t)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+\Delta t+t+\Delta t)}\\ &\quad -2\left\{\frac{1}{2a}e^{-a(t+\Delta t-t)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+\Delta t+t)}\right\}\\ &\quad +\frac{1}{2a}e^{-a(t1t)} + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-a(t+t)}\\ &= \frac{1}{a}[1-e^{-a\Delta t}] + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-2at}[e^{-2a\Delta t} - 2e^{-a\Delta t} + 1].\end{aligned}$$

Expanding this up to $(\Delta t)^2$ order, we obtain

$$\begin{aligned}\mathbb{E}[(\eta(t+\Delta t)-\eta(t))(\eta(t+\Delta t)-\eta(t))] &= \frac{1}{a}\left[1-\left(1-a\Delta t+\frac{1}{2}a^2(\Delta t)^2\right)\right] + \left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-2at}\\ &\quad\times\left[(1-2a\Delta t+2a^2(\Delta t)^2) - 2\left(1-a\Delta t+\frac{1}{2}a^2(\Delta t)^2\right)+1\right]\\ &= \Delta t - \frac{1}{2}a(\Delta t)^2 + a^2\left(\mathbb{E}[\eta^2(0)] - \frac{1}{2a}\right)e^{-2at}(\Delta t)^2.\end{aligned}$$


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