Parametric correlations versus fidelity decay: The symmetry breaking case

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We provide formulas for fidelity decay and parametric energy correlations for random matrix ensembles where time-reversal invariance of the original Hamiltonian is broken by the perturbation. Like in the case of a symmetry conserving perturbation a simple relation between both quantities exists. Fidelity freeze is observed for systems with even and odd spin.

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I. INTRODUCTION

Fidelity presently attracts considerable attention in diverse fields like quantum information, quantum chaotic systems, and others [1,2]. It measures the change of quantum dynamics of a state under a modification of the Hamiltonian. In quantum information, fidelity measures the deviation between a mathematical algorithm and its physical implementation.

Since fidelity requires knowledge of the entire wave function for the original and for the modified system the measurement of fidelity is a notoriously difficult task. However a number of experimental results have been obtained in microwave billiards, where the perturbation was achieved by varying some geometric parameter. There are two qualitatively different ways to do this, either by a global perturbation, e.g., by moving one wall [3], or a local perturbation, e.g., by varying the position of an impurity [4]. For the first case random matrix theory is applicable, and indeed a perfect agreement between experiment and theory has been found [3].

On the other hand statistical properties of energy correlations between spectra of complex quantum systems which differ by a parameter-dependent variation have been studied experimentally and theoretically [5]. This quantity can be obtained with great accuracy from scattering experiments by analyzing the fluctuations of the resonances in the scattering cross section [6,7].

From an experimental point of view it is interesting to relate fidelity with spectral quantities. This allows an indirect measurement of fidelity via an analysis of the (parametric) scattering data and the problem of measuring the entire wave function is circumvented.

A simple differential relation between fidelity decay and parametric energy correlations has been established in the case where the parameter-dependent perturbation falls into the same symmetry class as the unperturbed system [8,9]. This differential relation was derived earlier in energy space by Taniguchi *et al.* [10] and Simons and Altshuler [11], and it was identified with a continuity equation of the Calogero–Moser–Sutherland model [12]. In Ref. [13] similar expressions are derived for parametric energy correlations in the case where the perturbation breaks the global symmetry of the orginінal unperturbed system.

Recently billiard experiments have been performed in microwave resonators, where time reversal symmetry (TRS) was broken by a piece of ferrite [14] which plays the role of the perturbation. From the experimental results S-matrix elements could be determined and an estimate of the strength of the TRS breaking could be made. The experimental setup seems adequate for the measurement of parametric energy correlations and of fidelity decay by a TRS breaking perturbation.

In this paper we therefore analyze the expressions found in Refs. [11,13] for TRS breaking perturbations under the aspect of fidelity and provide formulas for fidelity and parametric form factor as well as differential relations between them and discuss their consequences.

II. DEFINITIONS AND RESULTS

Fidelity amplitude is defined as a functional of the initial wave function. In an ergodic situation it seems reasonable to replace a specific initial state by a random one. In Ref. [15] the corresponding random matrix model for the fidelity amplitude is defined by ($\hbar = 1$)

$$f(\lambda_\parallel,\lambda_\perp,t) = \frac{1}{N}\langle \mathrm{tr}\exp(itH)\exp(-itH_0)\rangle. \qquad (1)$$

The Fourier transform of parametric energy correlations is defined by

$$\widetilde{K}(\lambda_\parallel,\lambda_\perp,t) = \frac{1}{N}\langle \mathrm{tr}\exp(itH)\mathrm{tr}\exp(-itH_0)\rangle. \qquad (2)$$

It was named cross form-factor in Ref. [8]. The brackets denote an ensemble average. The perturbed Hamiltonian H is given as

$$H = H_0 + \lambda_\parallel V_\parallel + \lambda_\perp V_\perp. \qquad (3)$$

Let us first discuss the unperturbed Hamiltonian. We assume that for the unperturbed system H_0 TRS is conserved. The time reversal operator $\mathcal{T}$ acts differently on systems with integer spin and on systems with half-integer spin [16]. For even spin, $\mathcal{T}_1 = \hat{C}$, where $\hat{C}$ is the complex conjugation operator. In this case (called case I in the following) H_0 is chosen from the ensemble of real symmetric matrices, called the Gaussian orthogonal ensemble (GOE, $\beta = 1$). For odd spin

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systems, $\mathcal{T}_2 = \hat{C}\hat{J}$, where $\hat{J}$ acts via conjugation with the symplectic metric. In this case (case II) H_0 is chosen from the Gaussian symplectic ensemble (GSE, $\beta = 4$), consisting of all Hermitian matrices which are invariant under $\mathcal{T}_2$. The ensembles are defined by the averages

$$\langle (H_0)_{ij}(H_0)_{kl}\rangle = \begin{cases} \frac{N}{\pi^2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) & \text{(GOE)},\\ \frac{N}{\pi^2}(\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ik}\delta_{jl}) & \text{(GSE)}. \end{cases} \quad (4)$$

For the GSE the matrix entries are quaternions. Zirnbauer and Altland classified random matrix ensembles along Cartan's classification of symmetric spaces with curvature zero [17,18]. They called the GOE of AI type and the GSE of AII type.

In contrast to Ref. [19] we now assume that the perturbation contains two parts. One part, named $V_\parallel$, shares the symmetry of H_0 and is taken either from a GOE or a GSE

$$\langle (V_\parallel)_{ij}(V_\parallel)_{kl}\rangle = \begin{cases} \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} & \text{(GOE)},\\ \delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ik}\delta_{jl} & \text{(GSE)}. \end{cases} \quad (5)$$

The second part, $V_\perp$, breaks the symmetry of H_0 and is taken for case I from the Gaussian ensemble of matrices which change sign under time reversal $\mathcal{T}_1$, being antisymmetric matrices with purely imaginary entries. The ensemble is of type B using the classification of Ref. [18]. For case II it is taken from the ensemble of matrices which change sign under $\mathcal{T}_2$. They are of the block form

$$V_\perp = \left[\begin{array}{c|c} A & B \\ \hline B^\dagger & -A^* \end{array}\right], \quad A = A^\dagger, \quad B = B^T, \quad (6)$$

where A and B are $N/2 \times N/2$ matrices (N even). The corresponding ensemble is termed C type in Ref. [18]. The B type and C type ensembles are defined by the averages

$$\langle (V_\perp)_{ij}(V_\perp)_{kl}\rangle = \begin{cases} \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl} & \text{(B type)},\\ \delta_{il}\delta_{jk} + \frac{1}{2}\delta_{ik}\delta_{jl} & \text{(C type)}. \end{cases} \quad (7)$$

The variance of the matrix elements has been chosen to have a mean level spacing, $D = 1$, for H_0, and to be of the order $1/\sqrt{N}$ for $V_\parallel$ and $V_\perp$.

One parameter $\lambda_\parallel$ characterizes the strength of the perturbation, which conserves TRS. The second one, $\lambda_\perp$, is the strength of the TRS breaking perturbation. Thereby we consider a much wider class of TRS breaking Hamiltonians as before. Observe that for $\lambda_\parallel = \lambda_\perp$ this corresponds to a perturbation by a Hermitian matrix, i.e., to a perturbation which is taken from the Gaussian unitary ensemble (GUE). This ensemble is called type A in Ref. [18]. Thus time reversal symmetry breaking can occur in different ways. Symbolically we may write the left-hand side of Eq. (3) as $\mathrm{AI} + \lambda_\parallel\mathrm{AI} + \lambda_\perp\mathrm{B}$ (case I) or as $\mathrm{AII} + \lambda_\parallel\mathrm{AI} + \lambda_\perp\mathrm{C}$ (case II). Usually only the transition $\mathrm{AI} + \lambda\mathrm{A}$ is considered, when time reversal invariance is discussed [20].

Analyzing Eq. (4) and the following ones of Ref. [13], we find expressions for fidelity amplitude and for cross form-factor. To present them concisely we define for case I the function

$$\mathcal{Z}^{(\mathrm{I})}(\lambda_\parallel,\lambda_\perp,\tau) = \int_{\max(0,\tau-1)}^{\tau} du \int_0^u v\,dv \times \frac{1 + 4\pi^2\lambda_\perp^2(\tau^2 - v^2)}{\sqrt{[u^2 - v^2][(u+1)^2 - v^2]}} \frac{(\tau - u)(1 - \tau + u)}{(v^2 - \tau^2)^2} \times e^{-2\pi^2(\lambda_\parallel^2+\lambda_\perp^2)\tau(2u+1-\tau) - 2\pi^2(\lambda_\parallel^2 - \lambda_\perp^2)v^2}, \quad (8)$$

and we define for case II the function

$$\mathcal{Z}^{(\mathrm{II})}(\lambda_\parallel,\lambda_\perp,\tau) = \int_{-1}^{+1} du \int_0^{1-|u|} \frac{(u+t)^2 - 1}{(t^2 - v^2)^2} \times \theta(u - 1 + t)\frac{v\,dv[1 + \pi^2\lambda_\perp^2(\tau^2 - v^2)]}{\sqrt{[(u-1)^2 - v^2][(u+1)^2 - v^2]}} \times e^{-\pi^2(\lambda_\parallel^2+\lambda_\perp^2)\tau(2u+\tau) - \pi^2(\lambda_\parallel^2 - \lambda_\perp^2)v^2}, \quad (9)$$

where τ is time measured in units of Heisenberg time $t_H = 2\pi/D$. Then in the large N-limit the fidelity as defined in Eq. (1) is given in both cases by

$$f(\lambda_\parallel,\lambda_\perp,\tau) = -\frac{1}{\pi^2}\frac{\partial}{\partial(\lambda_\parallel^2)}\mathcal{Z}(\lambda_\parallel,\lambda_\perp,\tau). \quad (10)$$

The cross form-factor is given by

$$\widetilde{K}(\lambda_\parallel,\lambda_\perp,\tau) = \frac{4}{\beta}\tau^2\mathcal{Z}(\lambda_\parallel,\lambda_\perp,\tau). \quad (11)$$

From this follows the relation between fidelity and cross form-factor [11]:

$$f(\lambda_\parallel,\lambda_\perp,\tau) = -\frac{\beta}{4\pi^2\tau^2}\frac{\partial}{\partial(\lambda_\parallel^2)}\widetilde{K}(\lambda_\parallel,\lambda_\perp,\tau). \quad (12)$$

This relation can be derived through a universality argument without going through a lengthy supersymmetric calculation and comparing results. In Appendix A we present this derivation extending the method of Refs. [8,9] to the case of TRS breaking.

Some details on the the derivation of Eqs. (8) to (12) from the pertinent formulas of Ref. [13] are given in Appendix B.

III. DISCUSSION

The double integrals (8) and (9) can be evaluated numerically (see Appendix B of Ref. [21] for a convenient parametrization). Figure 1 shows the fidelity decay in case I for different perturbation strengths and for a perturbation taken from a GUE ($\mathrm{AI} + \lambda\mathrm{A}$), from a GOE ($\mathrm{AI} + \lambda_\parallel\mathrm{AI}$), and from the B-type ensemble of purely antisymmetric matrices ($\mathrm{AI} + \lambda_\perp\mathrm{B}$). For weak perturbations there is nearly no difference between the fidelity decay with a GOE and a GUE perturbation, respectively. This feature is closely related with the extremely weak fidelity decay for an imaginary antisymmetric perturbation. The latter is called fidelity freeze [22] and is discussed in the context with random matrix theory in Ref. [19]. The diagonal elements of the perturbation in the eigenbasis of the original Hamiltonian cause a Gaussian decay, which dominates for times larger than Heisenberg time. It was therefore predicted [22] that fidelity decay is much slower for

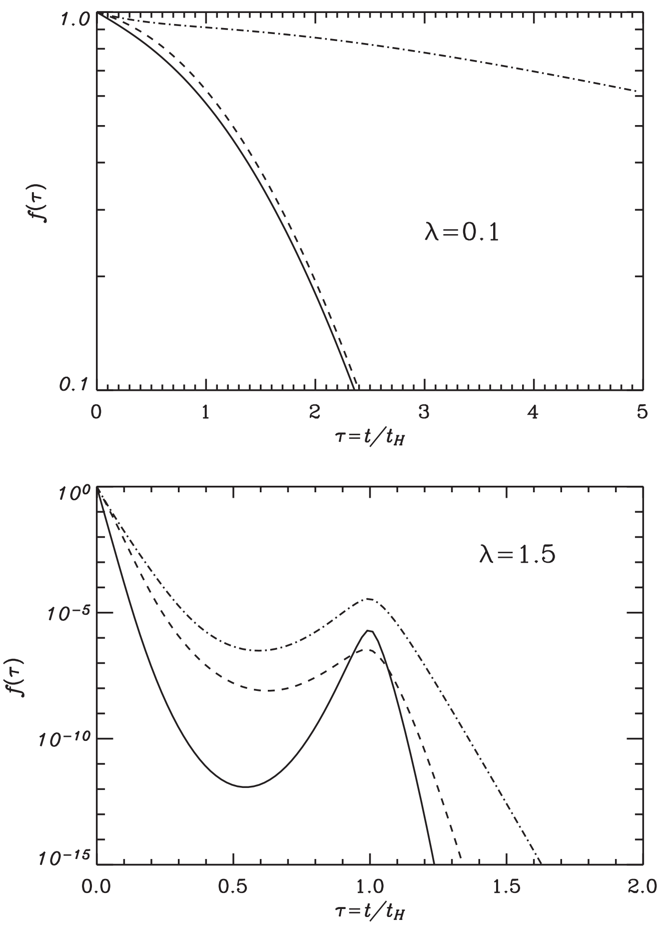

FIG. 1. Ensemble average of the fidelity amplitude $f(\lambda,\lambda,\tau)$ (solid line) with H_0 taken from the GOE (case I) and the perturbation taken from the GUE for different perturbation strengths λ. The results $f(\lambda,0,\tau)$ (a pure GOE perturbation, dashed lines) and $f(0,\lambda,\tau)$ (a purely perpendicular perturbation, dashed-dotted lines) are shown as well for the same parameter.

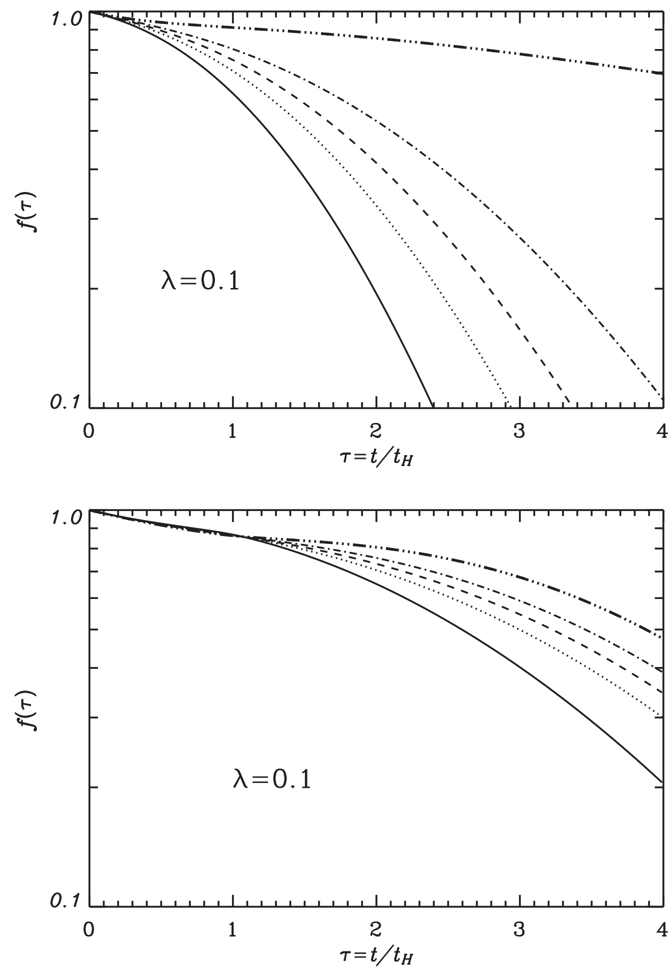

FIG. 2. Fidelity amplitude $f(\lambda_\parallel,\lambda_\perp,\tau)$ for case I (upper picture) and for case II (bottom picture) for different values of $\lambda_\parallel$ and $\lambda_\perp$ and for fixed overall perturbation $\lambda \equiv \sqrt{\lambda_\parallel^2+\lambda_\perp^2} = 0.1$. The values of $\lambda_\parallel$ and $\lambda_\perp$ are given by the five ratios $\lambda_\parallel^2/\lambda_\perp^2 = \infty$ (solid line), 2 (dotted line), 1 (dashed line), $1/2$ (dashed-dotted line), and 0 (dashed-dotted-dotted-dotted line).

perturbations which are purely off-diagonal in the eigenbasis of the original Hamiltonian.

With increasing perturbation the decays for the GOE and the GUE perturbation separate, and the freeze behavior gets lost. For strong perturbations a recovery of fidelity at Heisenberg time is seen. This is already known from Ref. [23] where the cases A + λA and AI + λAI are discussed.

For small perturbations and for times much smaller than Heisenberg time, fidelity decay is governed by Fermi's golden rule. In this regime the crucial parameter is $\lambda^2 = \lambda_\parallel^2 + \lambda_\perp^2$, which is related to the spreading width $\Gamma = 2\pi\lambda^2 D$ of an unperturbed state. This result holds independently of the universality class of the background. It is therefore interesting to look at the fidelity amplitude for fixed λ but different ratios between orthogonal and parallel perturbation.

In Fig. 2 fidelity amplitude is plotted for small perturbation strength $\lambda = 0.1$ and for different ratios between $\lambda_\parallel$ and $\lambda_\perp$ for case I and case II.

In case I we see that fidelity amplitude is a monotonous function of this ratio for all times. The slowest decay happens for $\lambda_\parallel = 0$, i.e., when the perturbation in the direction of H_0 is zero (freeze). In case II the fidelity shows qualitatively the same behavior, i.e., a slower decay for perpendicular perturbations for times beyond Heisenberg time. This suggests that one should define, for a general perturbation V,

$$\mathrm{tr} H_0 V = 0 \tag{13}$$

as a condition for a fidelity freeze, which is slightly more general than the one proposed in Ref. [22]. However in case II the freeze is much less pronounced than in case I, indicating that the diagonal elements of $V_\perp$, albeit $\mathrm{tr} V_\perp = \mathrm{tr} V_\perp H_0 = 0$, have some impact on the decay.

A careful look reveals that for times beyond Fermi's golden rule but smaller than Heisenberg time in case II fidelity decay is slower for a parallel perturbation than for a perpendicular perturbation.

This becomes evident for strong perturbations. In Fig. 3 the fidelity amplitude is plotted for the same ratios of $\lambda_\parallel$ and $\lambda_\perp$ as before but for strong overall perturbation $\lambda = 1.5$. Case I fidelity decay shows monotonous behavior as a function $\lambda_\parallel/\lambda_\perp$ and fidelity decay is for all times slowest for a perpendicular perturbation. However case II fidelity decay is more

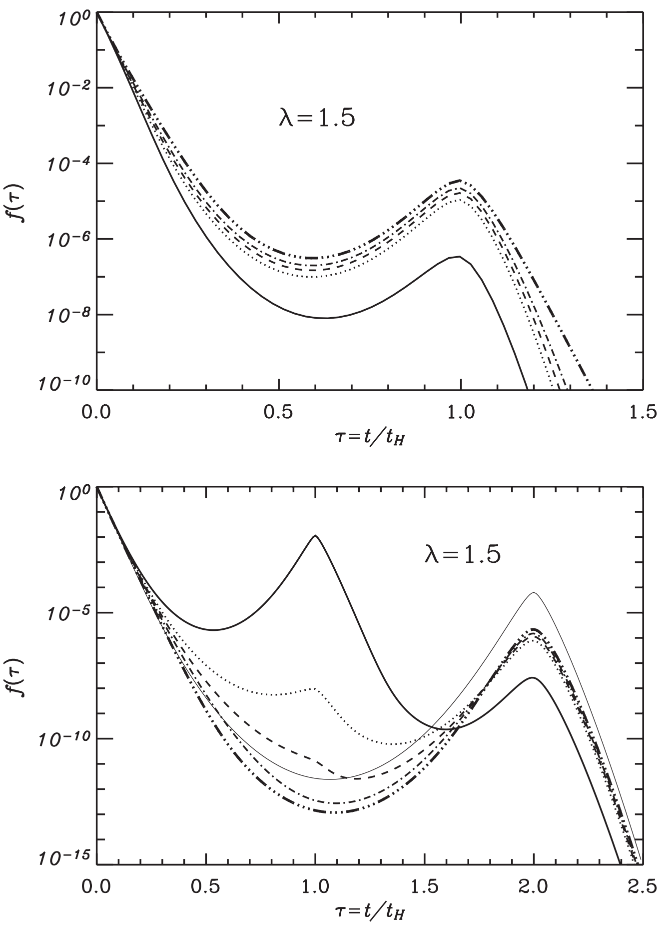

FIG. 3. Same as Fig. 2 but for a strong overall perturbation $\lambda = 1.5$. Fidelity amplitude $f(\lambda_\parallel,\lambda_\perp,\tau)$ is shown for case I (upper panel) and case II (lower panel) and for the ratios $\lambda_\parallel^2/\lambda_\perp^2 = \infty$ (solid line), 2 (dotted line), 1 (dashed line), 1/2 (dashed-dotted line), and 0 (dashed-dotted-dotted-dotted line). In the lower panel for comparison fidelity amplitude $f(\sqrt{2}\lambda,\tau/2)$ for a GUE with a GUE perturbation is shown as well (thinner solid line).

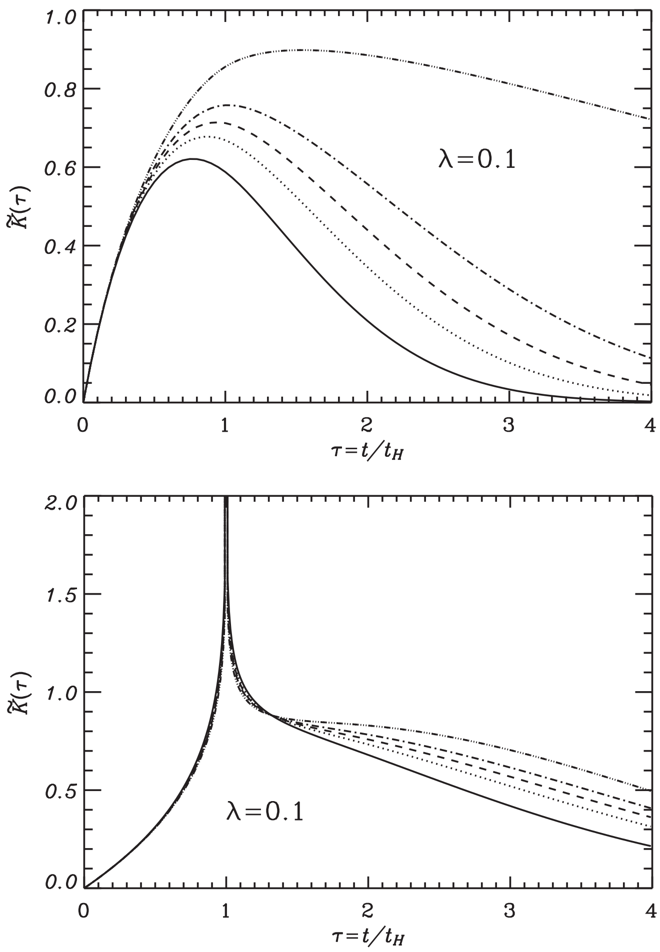

FIG. 4. Cross form-factor $\tilde{K}(\lambda_\parallel,\lambda_\perp,\tau)$ for case I (upper panel) and case II (lower panel) for different values of $\lambda_\parallel$ and $\lambda_\perp$ and for small overall perturbation $\lambda = 0.1$. The values of $\lambda_\parallel$ and $\lambda_\perp$ are given by the the ratios $\lambda_\parallel^2/\lambda_\perp^2 = \infty$ (solid line), 2 (dotted line), 1 (dashed line), 1/2 (dashed-dotted line), and 0 (dashed-dotted-dotted-dotted line).

complicated. For times smaller than Heisenberg time decay is slowest for a purely parallel perturbation and fastest for a purely perpendicular one. At Heisenberg time a pronounced revival is seen for a purely parallel perturbation. The peak decreases as the share of the perpendicular perturbation increases. Finally for a purely perpendicular perturbation there is a minimum at Heisenberg time and no revival at all.

After Heisenberg time things change. Now decay becomes fastest for a purely parallel perturbation with only a tiny second revival at twice the Heisenberg time. For a purely perpendicular perturbation the freeze behavior comes in and at twice the Heisenberg time a sizable revival occurs, such that just as in case I for long times decay is slowest for a purely perpendicular perturbation. Somewhere between Heisenberg time and twice the Heisenberg time the two curves cross.

To understand this behavior qualitatively, we recall two peculiarities of the GSE: first the spectral rigidity is much higher than that for the GUE or the GOE. It has been argued [23] that the revival at Heisenberg time is a signature of the high spectral rigidity. More generally high spectral rigidity favors a slow decay. Second the eigenvalues of the GSE are twofold degenerate (Kramers degeneracy).

Thus a perpendicular perturbation has two effects: first it breaks time reversal invariance and drives the GSE into a GUE. Since the latter has lower spectral rigidity, this has as a consequence that the peak at Heisenberg time becomes less and less pronounced and for times smaller than Heisenberg time decay is enhanced by the perpendicular perturbation. Second it breaks Kramers degeneracy, thus the number of independent levels and therefore level density and Heisenberg time double. This leads to the pronounced peak at twice the (original) Heisenberg time. A comparison with the plot of fidelity amplitude $f(\sqrt{2}\lambda,\tau/2)$ of a GUE with a GUE perturbation$(A+\lambda A)$ shows indeed good agreement.

In Fig. 4 the cross form-factor is plotted in both cases for the same five ratios between $\lambda_\parallel$ and $\lambda_\perp$ as before. Qualitatively the behavior is similar to the fidelity amplitude. In case I the form factor is smallest for a purely parallel perturbation for all times. In case II before Heisenberg time the form factor is smallest for a purely perpendicular perturbation and largest for a purely parallel one. After Heisenberg time the order is inverted. At Heisenberg time a logarithmic singularity occurs, which is typical for the GSE. For strong perturbations the cross form-factor develops peaks at Heisenberg time and for case II at twice the Heisenberg time (not shown here). It has its cause in the algebraic decay of the cross form-factor at these specific times [8]. At all other times it decays exponentially.

IV. CONCLUSION

In conclusion we presented the analytic formulas for the fidelity amplitude and cross-form factor for parametric RMT ensembles, where the time reversal invariance of the unperturbed system is broken by the perturbation. The general perturbation is split into a parallel component, sharing the symmetries of the original Hamiltonian and a perpendicular component which maximally breaks this symmetry.

Both possibilities of TRS breaking, even spin GOE → GUE and odd spin GSE → GUE, were discussed on equal footing. In the first case a strong freeze effect occurs for a purely perpendicular perturbation. It can be explained by the absence of diagonal elements of the perturbation in the eigenbasis of the unperturbed Hamiltonian. In case II long time decay is slowest for a purely perpendicular perturbation as well. This leads us to propose $\mathrm{tr}H_0V = 0$ as a more general condition for a reduced fidelity decay. However in case II the perturbation has diagonal entries in the eigenbasis of H_0 and the attenuation of decay is much less pronounced than in case I. We are reluctant to call this behavior "freeze." We propose to call it "weak fidelity freeze."

The full Hilbert space is involved in the condition $\mathrm{tr}H_0V = 0$. Therefore it is only applicable to fidelity decay with respect to a random initial state as considered here, to which all states of the Hilbert space contribute. For an arbitrary initial state this condition will in general not suffice to attenuate fidelity decay.

In the differential relation between fidelity and cross form-factor only the parallel perturbation strength enters. The relation might be verified experimentally for instance in a billiard experiment as described in Ref. [14]. It might be used to measure fidelity indirectly via spectral correlations.

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APPENDIX A: DERIVATION OF EQUATION (12) BASED ON A UNIVERSALITY ARGUMENT

In this section we demonstrate on the example of the transition GOE → GUE (case I) how the method of Refs. [8,9] can be extended to the case of symmetry breaking. We introduce new variables:

$$\tilde{\lambda}_{\parallel} = \frac{\lambda_{\parallel}}{2}, \quad \tilde{H}_0 = H_0 + \frac{\lambda_{\parallel}}{2}V_{\parallel}. \tag{A1}$$

For $\tilde{H}_0$ we allow for a general probability measure in the GOE universality class and denote it by $d\nu(\tilde{H}_0)$, while those of $V_\parallel$ and $V_\perp$ are Gaussian measures as before ($dV_\parallel$ and $dV_\perp$ include the normalization constants). Since the probability measure of $\tilde{H}_0$ is assumed to be general, it should be typical and free from any special constraint besides the matrix symmetry.

Now we define

$$\begin{aligned}\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} &= \left(\frac{1}{z_1 - H_0}\right)_{\alpha\beta}\left(\frac{1}{z_2 - H}\right)_{\gamma\delta} \\ &= \left(\frac{1}{z_1 - \tilde{H}_0 + \tilde{\lambda}_\parallel V_\parallel}\right)_{\alpha\beta} \\ &\quad\times\left(\frac{1}{z_2 - \tilde{H}_0 - \tilde{\lambda}_\parallel V_\parallel - i\lambda_\perp V_\perp}\right)_{\gamma\delta}.\end{aligned} \tag{A2}$$

Introducing δ distributions of matrix arguments we can express $\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta}$ as

$$\begin{aligned}\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} &= \int dH_1 dH_2\, \delta(H_1 - \tilde{H}_0 + \tilde{\lambda}_\parallel V_\parallel)\mathcal{F}_{\alpha\beta,\gamma\delta} \\ &\quad\times \delta\big(H_2^{(R)} - \tilde{H}_0 - \tilde{\lambda}_\parallel V_\parallel\big)\,\delta\big(H_2^{(I)} - \lambda_\perp V_\perp\big),\end{aligned} \tag{A3}$$

where H_1 is an $N \times N$ real symmetric matrix, H_2 is an $N \times N$ Hermitian matrix, $H_2^{(R)} = \mathrm{Re}H_2$, and $H_2^{(I)} = \mathrm{Im}H_2$. Moreover

$$\mathcal{F}_{\alpha\beta,\gamma\delta} = \left(\frac{1}{z_1 - H_1}\right)_{\alpha\beta}\left(\frac{1}{z_2 - H_2}\right)_{\gamma\delta}. \tag{A4}$$

All three δ distributions can be Fourier transformed. We find

$$\begin{aligned}\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} &= \int d\Lambda_1 d\Lambda_2 d\Lambda_3 dH_1 dH_2\, e^{2\pi i \mathrm{tr}\Lambda_1(H_1 - \tilde{H}_0 + \tilde{\lambda}_\parallel V_\parallel)} \\ &\quad\times e^{2\pi i \mathrm{tr}\Lambda_2(H_2^{(R)} - \tilde{H}_0 - \tilde{\lambda}_\parallel V_\parallel)} \\ &\quad\times e^{2\pi i \mathrm{tr}\Lambda_3(H_2^{(I)} - \lambda_\perp V_\perp)}\,\mathcal{F}_{\alpha\beta,\gamma\delta}.\end{aligned} \tag{A5}$$

Here $\Lambda_{1,2,3}$ are matrices which have the same symmetry as their real space counterparts, namely, H_1, $H_2^{(R)}$, and $H_2^{(I)}$. This means Λ_1 and Λ_2 are $N \times N$ real symmetric matrices and Λ_3 is an $N \times N$ real antisymmetric matrix. The integration domain is the real axis for all independent entries of Λ_n, $n = 1,2,3$. The expectation value of $\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta}$ can be written as

$$\begin{aligned}\langle\mathcal{F}_{\alpha\beta,\gamma\delta}\rangle &= \int d\nu(\tilde{H}_0)dV_\parallel dV_\perp \tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta} e^{-(1/4)\mathrm{tr}V_\parallel^2 + (1/4)\mathrm{tr}V_\perp^2} \\ &= \int d\nu(\tilde{H}_0)d\Lambda_1 d\Lambda_2 d\Lambda_3 dH_1 dH_2 \mathcal{F}_{\alpha\beta,\gamma\delta} \\ &\quad\times e^{-(2\pi\tilde{\lambda}_\parallel)^2\mathrm{tr}(\Lambda_1 - \Lambda_2)^2 + (2\pi\lambda_\perp)^2\mathrm{tr}(\Lambda_3)^2} \\ &\quad\times e^{2\pi i\,\mathrm{tr}\{\Lambda_1(H_1 - \tilde{H}_0) + \Lambda_2(H_2^{(R)} - \tilde{H}_0) + \Lambda_3 H_2^{(I)}\}}.\end{aligned} \tag{A6}$$

Here the brackets $\langle\cdots\rangle$ do not simply mean the expectation value. Rather $\langle\mathcal{F}_{\alpha\beta,\gamma\delta}\rangle$ is defined to be the expectation value of $\tilde{\mathcal{F}}_{\alpha\beta,\gamma\delta}$.

Now we introduce the notation

$$\mathrm{tr}\frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} = \sum_{j=1}^{N} \frac{\partial^2}{\partial (H_1)_{jj}\, \partial \big(H_2^{(R)}\big)_{jj}} + \frac{1}{2}\sum_{j<l}^{N} \frac{\partial^2}{\partial (H_1)_{jl}\, \partial \big(H_2^{(R)}\big)_{jl}}. \tag{A7}$$

Then it follows from partial integrations that

$$\left\langle \mathrm{tr}\frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} \mathcal{F}_{\alpha\alpha,\beta\beta} \right\rangle = -(2\pi)^2 \langle \mathrm{tr}(\Lambda_1\Lambda_2)\mathcal{F}_{\alpha\alpha,\beta\beta}\rangle \tag{A8}$$

and

$$\frac{\partial}{\partial(\tilde{\lambda}_\parallel^2)}\langle \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle = -(2\pi)^2 \langle \mathrm{tr}(\Lambda_1 - \Lambda_2)^2 \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle. \tag{A9}$$

Here repeated indices stand for summations from 1 to N.

Let us note that a simultaneous shift of H_1 and $H_2^{(R)}$ in Eq. (A6) induces a shift of $\tilde{H}_0$. Although such a shift modifies the measure $d\nu(\tilde{H}_0)$, the universality of the spectral correlation function implies that $\langle \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle$ is asymptotically invariant in the limit $N \to \infty$. Therefore we obtain the following estimate

$$\left\langle \mathrm{tr}\left(\frac{\partial}{\partial H_1} + \frac{\partial}{\partial H_2^{(R)}}\right)^2 \mathcal{F}_{\alpha\alpha,\beta\beta} \right\rangle = -(2\pi)^2 \langle \mathrm{tr}(\Lambda_1 + \Lambda_2)^2 \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle \approx 0. \tag{A10}$$

From this it follows that

$$\frac{\partial}{\partial(\lambda_\parallel^2)}\langle \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle + \left\langle \mathrm{tr}\frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} \mathcal{F}_{\alpha\alpha,\beta\beta} \right\rangle = -\pi^2 \langle \mathrm{tr}(\Lambda_1 + \Lambda_2)^2 \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle \approx 0. \tag{A11}$$

In order to show that the estimate (A10) is indeed correct, let us pay attention to Eq. (A6). Proper unfolding of the energy level correlations requires an $\mathcal{O}(1)$ scaling of the eigenvalue density of $\tilde{H}_0$. Each element of the perturbation $V_\parallel$ is set to be $\mathcal{O}(1)$, because it should equally scale as the mean level spacing. When the eigenvalue density is scaled as $\mathcal{O}(1)$, since there are N eigenvalues, each eigenvalue $\tilde{E}_{0j}$ of $\tilde{H}_0$ should typically be $\mathcal{O}(N)$. Then the right-hand side of the identity

$$\mathrm{tr}(\tilde{H}_0)^2 = \sum_{j=1}^{N} (\tilde{E}_{0j})^2 \tag{A12}$$

becomes $\mathcal{O}(N^3)$. In the left-hand side, on the other hand, we have $\mathcal{O}(N^2)$ terms, each of which is the square of an element of $\tilde{H}_0$. Therefore each element of $\tilde{H}_0$ is estimated as $\mathcal{O}(N^{1/2})$. Then the main contribution to the integral over the matrix $\tilde{H}_0$ with respect to the measure $d\nu(\tilde{H}_0)$ in Eq. (A6) comes from a region where the elements of $\Lambda_1 + \Lambda_2$ are of order $\mathcal{O}(N^{-1/2})$. Only in that region a rapid oscillation of the exponential factor is avoided.

It can be seen from the Gaussian factor in Eq. (A6) that the elements of $\Lambda_1 - \Lambda_2$ are scaled as $\mathcal{O}(1)$. Because of the identity

$$(\Lambda_1 - \Lambda_2)^2 = -2(\Lambda_1\Lambda_2 + \Lambda_2\Lambda_1) + (\Lambda_1 + \Lambda_2)^2, \tag{A13}$$

the elements of $(\Lambda_1 - \Lambda_2)^2$ are approximated by the elements of $-2(\Lambda_1\Lambda_2 + \Lambda_2\Lambda_1)$. Hence we find an estimate

$$\mathrm{tr}(\Lambda_1 - \Lambda_2)^2 \approx -4\mathrm{tr}(\Lambda_1\Lambda_2), \tag{A14}$$

which implies Eq. (A11). We notice that this estimate can only be fulfilled when $\mathrm{tr}(\Lambda_1\Lambda_2)$ is negative.

On the other hand, we can readily find

$$\mathrm{tr}\frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} \mathcal{F}_{\alpha\alpha,\beta\beta} = \mathrm{tr}\left(\frac{1}{z_1 - H_1}\right)^2 \left(\frac{1}{z_2 - H_2}\right)^2 \tag{A15}$$

and

$$\frac{\partial^2}{\partial z_1 \partial z_2}\mathcal{F}_{\alpha\beta,\beta\alpha} = \frac{\partial^2}{\partial z_1 \partial z_2}\left(\frac{1}{z_1 - H_1}\right)_{\alpha\beta}\left(\frac{1}{z_2 - H_2}\right)_{\beta\alpha} = \mathrm{tr}\left(\frac{1}{z_1 - H_1}\right)^2 \left(\frac{1}{z_2 - H_2}\right)^2, \tag{A16}$$

so that

$$\left\langle \mathrm{tr}\frac{\partial^2}{\partial H_1 \partial H_2^{(R)}} \mathcal{F}_{\alpha\alpha,\beta\beta} \right\rangle = \frac{\partial^2}{\partial z_1 \partial z_2}\langle \mathcal{F}_{\alpha\beta,\beta\alpha}\rangle. \tag{A17}$$

Comparing Eqs. (A11) and (A17), we arrive at

$$\frac{\partial}{\partial(\lambda_\parallel^2)}\langle \mathcal{F}_{\alpha\alpha,\beta\beta}\rangle \approx -\frac{\partial^2}{\partial z_1 \partial z_2}\langle \mathcal{F}_{\alpha\beta,\beta\alpha}\rangle, \tag{A18}$$

which gives the required relation (12) between the fidelity and parametric spectral correlation, namely, cross form-factor.

APPENDIX B: DERIVATION OF EQUATIONS (8) AND (12) FROM REFERENCE [13]

In Ref. [13], called THSA in the following, the Fourier transform of the cross form-factor was derived as a threefold integral

$$K(\bar{x}, x_o, x_u, \omega) = \mathrm{Re} \int d\lambda d\lambda_1 d\lambda_2 W e^{F_\pm}, \tag{B1}$$

where the integration domains are in case I defined by $\lambda \in [-1,1]$, $\lambda_1 \in [1,\infty]$, and $\lambda_2 \in [1,\infty]$ and in case II by $\lambda \in [1,\infty]$, $\lambda_1 \in [-1,1]$, and $\lambda_2 \in [0,1]$. Setting the parameter $\bar{x} = x_u/2$ the expressions for F and W [Eqs. (5) and (6) of THSA] are given by

$$F_\pm = \pm\kappa i\pi\omega(\lambda_1\lambda_2 - \lambda) \pm \frac{x_u^2\pi^2}{2}\left(\lambda_1^2 + \lambda_2^2 - \lambda^2 - 1\right) \pm \frac{x_o^2\pi^2}{4}\left(2\lambda_1^2\lambda_2^2 - \lambda^2 - \lambda_1^2 - \lambda_2^2 + 1\right) \tag{B2}$$

$$W = \frac{(\lambda_1\lambda_2 - \lambda)^2(1 - \lambda^2)}{\left(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1\lambda_2 - 1\right)^2} \times \left[1 + \frac{\pi^2 x_u^2}{\kappa}\left(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda\lambda_1\lambda_2 - 1\right)\right]. \tag{B3}$$

Here the plus sign applies to case I and the minus sign to case II. The parameter κ has the value $\kappa = 1$ (case I) and $\kappa = 2$ (case II). This factor does not appear in THSA; however it does appear in Ref. [24]. We introduced it, such that $\tilde{K}(t)$ is related to $K(\omega)$ in both cases via

$$\tilde{K}(\tau) = \int d\omega e^{-2\pi i\omega\tau} K(\omega). \quad \text{(B4)}$$

In THSA the function W differs in case I and case II by a relative minus sign between two summands in the last line of Eq. (B3). This seems to be wrong. Moreover in the same line the factor $1/\kappa$ in the second summand is missing in THSA.

Fourier transformation yields $\delta(\tau - \lambda_1\lambda_2/2 + \lambda)$ in case I and $\delta(\tau - \lambda + \lambda_1\lambda_2)$ in case II, which allows one to integrate over λ. Equations (8) to (12) are obtained through the following transformations:

$$\left.\begin{aligned} u &= \tfrac{1}{2}(\lambda_1\lambda_2 - 1) \\ v &= \tfrac{1}{2}\sqrt{\lambda_1^2\lambda_2^2 - \lambda_1 - \lambda_2 + 1} \end{aligned}\right\} \text{case I,} \quad \text{(B5)}$$

$$\left.\begin{aligned} u &= \lambda_1\lambda_2 \\ v &= \sqrt{\lambda_1^2\lambda_2^2 - \lambda_1 - \lambda_2 + 1} \end{aligned}\right\} \text{case II.} \quad \text{(B6)}$$

The parameters are identified as $\lambda_\parallel = x_o/2$ and $\lambda_\perp = x_u/\sqrt{2}$.

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