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Josephson junctions loaded by transmission lines: A revisited problem

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The problem of evaluating dissipative effects in Josephson junctions loaded by transmission lines is reexamined, for either the symmetric or the asymmetric case, with particular consideration of the time domain in which the interaction between junction and load system occurs.

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Several works devoted to the problem of dissipation in macroscopic quantum tunneling have become available in the literature since the 1980s. In particular, the case of the Josephson junction has received special attention [1]. This argument has continued to attract interest even in recent years, either from the fundamental point of view or from the one of applications [2]. However, in spite of the numerous efforts made, certain aspects in the theory remained to be clarified, as has recently been evidenced in a paper dealing with a comparison of the theoretical predictions with some experimental results: i.e., those relative to the semiclassical traversal time of the barrier, as modified by dissipative effects [3].

Among the several theoretical approaches to the problem [4], a paper by Chakrawarty and Schmidt, dealing with Josephson junction loaded by transmission lines [5], deserved particular consideration. The peculiarity of this work consisted of the incorporation of a distributed circuit model—a transmission line that determines the dissipative effects—within the bounce formalism and, avoiding any *ad hoc* assumption, sometimes adopted in other methodics [4]. In a subsequent work dedicated to the same subject [6], an attempt was made searching for an agreement with the results of different approaches, that is, those relative to the phenomenological analyses reported in Ref. [4]. The purpose of the present paper is a reexamination of the problem in view of the possible obtainment of a better matching with the results relative to different methods. As will be demonstrated, a crucial role to this purpose is played by the temporal domain in which the processes are considered to occur. Already in Refs. [4,6], the halving of the time domain—there considered as an artifice—was assumed in order to obtain the above-mentioned agreement. Now, we intend to demonstrate that such an assumption is physically grounded.

The physical system. It is constituted by a Josephson junction coupled symmetrically to two open transmission lines (case of Ref. [5]), or asymmetrically to one open line (case of Ref. [6]); in both cases the total length of lines is L. By adopting the same notations of Refs. [5,6], let us denote by ρ and σ the capacitance and the inductance per unity of length, respectively. The characteristic impedance of the lines is $Z_0 = (\sigma/\rho)^{1/2}$ and the wave velocity is $c = (\sigma\rho)^{-1/2}$, the delay time is $\tau_0 = L/2c$ for each line in the symmetric case, and $\tau_0 = L/c$ in the asymmetric one. Without going into the details of the relative analysis based on the Green's function method [7], the action integral $S_{\rm int}$, due to the interaction between junction and the line system, is given by [5,8]

$$S_{\rm int} = \left(\frac{\Phi_0}{2\pi}\right)^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|\varphi(\omega)|^2}{2g(\omega)}, \tag{1}$$

where Φ_0 $(=2\pi\lambda$ in Ref. [5]) is the flux quantum, $\varphi(\omega)$ is the Fourier transform of the bounce trajectory $\varphi(\tau) = \varphi_B \mathrm{sech}^2(\Omega\tau/2)$, with φ_B the bounce amplitude and Ω the plasma frequency of the junction, and $g(\omega)$ is the Fourier transform of the Green's function for the transmission line. According to Ref. [5], $g(\omega)$, in the case of symmetric line, turns out to be

$$g(\omega) = \frac{Z_0}{2|\omega|}\coth(kL/2), \tag{2}$$

with $kL/2 = |\omega|\tau_0$. In the case of an asymmetric line, $g(\omega)$ should be given by

$$g(\omega) = \frac{Z_0}{|\omega|}\coth(kL). \tag{3}$$

However, following the reasoning in Ref. [6], based on a property of the δ function applied to the present case in which the line runs from $z = 0$ to $z = L$, we have that Eq. (3) must be halved; that is, apart from the argument of coth, $g(\omega)$ would be formally identical to Eq. (2) of the symmetric case [see Eq. (4) in Ref. [6]]. Still in Ref. [6], and following the criterion adopted in Ref. [4] where the time domain was necessarily halved [9], the agreement with other treatments was obtained by introducing a factor $1/2$ in Eq. (1), as due to the halving of the temporal domain, thus compensating for the halving of Eq. (3) above mentioned. This procedure could appear only as a formal artifice, but this is not the case, as will be explained in the following. In summary, we have that, with Eq. (1) rewritten in the form

$$S_{\rm int} = \left(\frac{\Phi_0}{2\pi}\right)^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\varphi(\omega)|^2 H(\omega), \tag{4}$$

$H(\omega)$ is given by

$$H(\omega) = \frac{|\omega|}{(2)Z_0}\tanh(|\omega|\tau_0) \xrightarrow[|\omega|\tau_0\gg 1]{} \begin{cases} |\omega|/Z_0, & \text{symm. line,} \\ |\omega|/2Z_0, & \text{asymm. line,} \end{cases} \tag{5}$$

where the factor (2) in the denominator is required for the case of an asymmetric line, while the limit $|\omega|\tau_0 \gg 1$ is considered

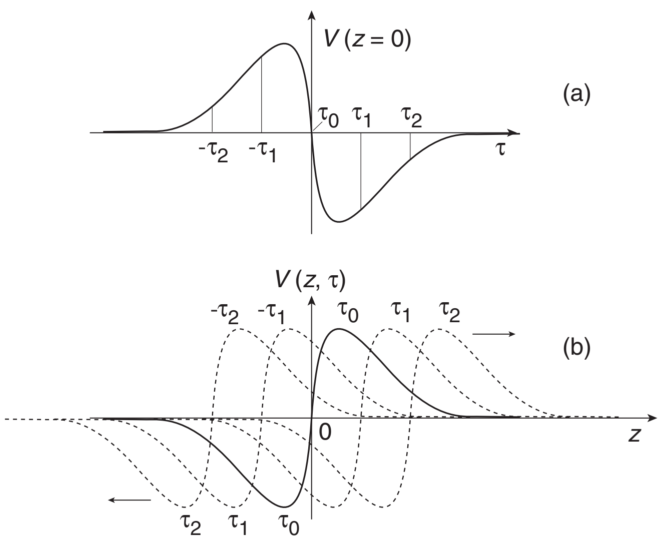

FIG. 1. Voltage pulse $V \propto \dot{\varphi}(\tau)$ generated by a Josephson junction loaded by transmission lines with characteristic impedance Z_0. In (a), the pulse $V(\tau)$ is represented as a function of time at $z = 0$. In (b), the pulse $V(z - c\tau)$ traveling for $z \geqslant 0$ and the pulse $V(-z - c\tau) = -V(z + c\tau)$ traveling for $z \leqslant 0$ are represented as a function of the spatial coordinate z of the line, at different instants $-\tau_2 < -\tau_1 < \tau_0 < \tau_1 < \tau_2$. However, the pulses represented in the negative region of z are not appropriate for negative times, whereas they are so for positive times.

in both cases, that is for lines sufficiently long. The results expressed by Eqs. (4) and (5) agree with Refs. [5,6].

Time domain and action variation. Turning now to the question of the temporal domain above mentioned, we have to consider the spatial-temporal evolution of the voltage pulse $V(z,\tau) \propto \dot{\varphi}(\tau)$ generated by the junction while performing the bounce trajectory. In the upper part of Fig. 1, the pulse $V(\tau)$ is represented as a function of the time, at $z = 0$ where the junction is situated. In the lower part, the pulse $V(z - c\tau)$, which travels in the direction of increasing z, is represented as a function of the spatial coordinate of the line at different instants $\tau_0 < \tau_1 < \tau_2$. The curves situated in the $z < 0$ region and relative to $-\tau_1$ and $-\tau_2$ instants would belong to the same progressive wave, provided that the $z < 0$ region is accessible, but they cannot be generated by the junction acting as a generator situated at $z = 0$ and centered around $\tau_0 = 0$. Rather, they belong to another progressive wave $-V(z + c\tau)$ running in the direction of $z < 0$, and corresponding to the same instants $\tau_0 < \tau_1 < \tau_2$ of the other progressive wave running in the direction of $z > 0$.

From these considerations, it follows that not only in the case of an asymmetric line, but also in that of a symmetric line, the active time domain is only the half-positive one. In this case, the result for S_{int} is obviously twice the one of the asymmetric line, since Z_0 is now halved by the parallel of the two lines, in agreement with Eq. (5).

By substituting from Eq. (5), $S_{\text{int}} \equiv \Delta S$ can be rewritten as

$$\Delta S = \eta \int_{-\infty}^{\infty} |\chi(\omega)|^2 \omega \tanh(|\omega|\tau_0) d\omega, \qquad (6)$$

where $\eta = (\Phi_0/2\pi)^2/Z_0$ and $\chi(\omega)$ is the Fourier transform of the bounce trajectory defined [that is, eliminating a factor 2π in the denominator of Eq. (4)] as

$$\chi(\omega) = \varphi_B \frac{2\sqrt{2}}{\sqrt{\pi}\Omega} \left(\frac{\pi\omega}{\Omega}\right) \operatorname{csch}\left(\frac{\pi\omega}{\Omega}\right). \qquad (7)$$

In the limit of $|\omega|\tau_0 \gg 1$, Eq. (6) gives as a result (in the case of a symmetric line) [4]

$$\Delta S_S \simeq 0.93\eta\varphi_B^2, \qquad (8)$$

while, in the case of an asymmetric line, the resulting ΔS is obviously halved:

$$\Delta S_A \simeq 0.465\eta\varphi_B^2. \qquad (9)$$

An identical result was also obtained in Ref. [10]. Therefore, the results obtained for ΔS are formally identical to the ones reported in all the previously given treatments [3–6]. However, according to the procedure here adopted, there is an important difference, especially for the case of a symmetric line, which is due to the use of halving the temporal domain in which the processes—namely the junction-lines interaction—occur.

In fact, it is true that the load impedance seen by the junction is equal to $Z_0/2$ for $|\omega|\tau_0 \gg 1$, thus reobtaining $H(\omega) = |\omega|/Z_0$ analogously to what happens if the Leggett's prescription, requiring the halving of the load admittance [11], is adopted [see Eq. (26) in Ref. [5]]. However, we remark that there is a noteworthy difference if our criterion is followed. Because of the use of transmission lines coupled in parallel to the junction, the load impedance is indeed halved to $Z_0/2$, but this is "seen" by the junction only in a half-temporal domain, that is for $\tau \geqslant 0$, and it is for this reason that a factor $1/2$ enters Eq. (1), thus giving the identical result with an "effective" admittance equal to $1/Z_0$.

As explained before, the pulse images represented in Fig. 1 at negative instants $(-\tau_1, -\tau_2, \ldots)$ are improper; rather, they have to be attributed to the same positive instants $(\tau_1, \tau_2, \ldots)$ for the opposite going wave, since they cannot be propagated before being generated by the junction around $\tau = 0$. This situation is evidently relative to the cases of sufficiently long line ($|\omega|\tau_0 \gg 1$ and short pulses), such that the traveling progressive waves are the only to be considered, while the waves reflected from the mismatched ends of the lines are less important, especially when we are in the presence of appreciable inherent losses in the line [12], or quite absent if the lines are terminated with Z_0 [13].

Different also is the case in the opposite (capacitive) limit ($|\omega|\tau_0 \ll 1$), in which the result turns out to be the same, independently of the position of the junction, that is proportional to the total capacitance $C = \rho L$ of the line in both cases [6]. However, see also Fig. 4 in Ref. [4] where, in the small Ω limit, the results [14] tend to be coincident with those of Ref. [5].

Still more different is the case of a purely resistive load Z_0 directly connected to the junction. In this case, we have to consider the complete temporal domain and the result obtained should be exactly the same of Eq. (8) [15]. This would confirm the nearly equal result of the phenomenological analysis [9], which supplied $\Delta S \simeq 0.87\eta\varphi_B^2$ (see point e) of Table I in Ref. [4]). This issue was, in a sense, anticipated in Ref. [4] where Fig. 3, relative to the case of an artificial asymmetric line terminated with the characteristic impedance Z_0, demonstrated the obtainment of results [14] tending, in the small Ω limit (long pulse duration or short line), to be twice the expected ones obtained dividing by two the results of Ref. [5] relative to the case of a symmetric line. In other terms, the ΔS values in Fig. 3 of Ref. [4] should be doubled if the complete time domain was adopted, thus obtaining results that can be considered as roughly representative for the case of a purely resistive load Z_0.

From the above arguments, it seems we can safely conclude that the consideration of the true temporal domain, in which the junction-load interaction occurs when the load consists of transmission lines, allows for a very good agreement of the results obtained by the different approaches, including the phenomenological ones. These conclusions also find confirmation by the results of Ref. [3], where the best agreement with experimental results, concerning the semiclassical traversal-time relative variation $\Delta\tau/\tau$, was obtained by considering the ratio $\Delta S/S_0$ [see Eq. (13) in Ref. [3]], with ΔS as given by Eq. (9) and S_0, the half-bounce action in absence of dissipation, being $S_J/2$ in the present notations.

[1] For a survey on the matter, see, e.g., *Proceedings of the International Workshop on Macroscopic Quantum Tunneling and Coherence* [J. Supercond. **12** (6) (1999)].

[2] See, e.g., A. L. Pankratov, A. S. Sobolev, V. P. Koshelets, and J. Mygind, Phys. Rev. B **75**, 184516 (2007) and references therein.

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[7] See, e.g., P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 7.

[8] The effective action determining the decay rate of the metastable superconductive state of the junction is $S_{\rm eff} = S_J + S_{\rm int}$, S_J being the action for the unloaded junction. According to the instanton-bounce method adopted here, the time scale relative to the bounce processes (which can be considered as "tunneling attempts") as in Fig. 1 does not give direct information on the mean escape time from the metastable state, which is determined by the inverse of the decay rate.

[9] This was required by the geometry of the considered junction load; otherwise, a double result should be obtained, similar to the one reported by A. Ranfagni, D. Mugnai, and R. Englman, Nuovo Cimento D **9**, 1009 (1987), for the case of a purely resistive load.

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[11] A. J. Leggett, Phys. Rev. B **30**, 1208 (1984).

[12] See F. E. Terman, *Electronic and Radio Engineering* (McGraw-Hill, New York, 1955), p. 95.

[13] Different is the case of Ref. [2], in which the reflected waves from the junction ends completely change the dynamics and the current-voltage characteristics.

[14] Results obtained ever adopting the halving of the time domain.

[15] In fact, if in the case of symmetric line the "effective" load admittance is given by $(1/2)(2/Z_0)$, this is exactly the same value of a purely resistive load equal to $1/Z_0$.