# Modification of the adiabatic invariants method in the studies of resonant dissipative systems

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We study the system of equations for the canonically conjugate variables $p$ and $q$ specified by the one-dimensional Hamiltonian $H = H(p,q,\Lambda_1,\ldots,\Lambda_N)$ dependent on $N$ self-consistent slightly changing parameters obeying the equations: $\dot{\Lambda}_n = \varepsilon f_n(\Lambda_1,\ldots,\Lambda_N,p,q)$. A broad range of oscillatory and wave processes with weak dissipation is described by analogous systems. The general method of adiabatic invariant construction for this system is proposed. Self-consistent averaged equations for the evolution of the action integral and the parameters $\Lambda_n$ are obtained. The constructed theory is applied to a generalized model of the nonlinear resonance. The autoresonance (phase locking) regime of decay parametric instability in a dissipative medium is revealed.

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## I. INTRODUCTION

The use of adiabatic invariants is an efficient way to analyze different physical systems [1–3]. A standard object for such an approach is the Hamiltonian system $H = H(p,q,\Lambda_1,\ldots,\Lambda_N)$, in which the parameters $\Lambda_n$ can be considered as given functions of the independent variable $\tau$. The applicability of the adiabatic invariants method is justified by a certain slowness of variation of the parameters $\Lambda_n$ on the scale of characteristic times of the conservative system. However, a lot of real physical objects are modeled by more complicated systems, where the dynamics of the parameters is self-consistent with the dynamics of the canonic variables. In such a case, the division of a set of quantities $p,q,\Lambda_1,\ldots,\Lambda_N$ into "dynamic variables" and "parameters" is conditional and makes sense, generally speaking, only if the dynamic equations for the quantities $\Lambda_n$ depend on a small parameter.

A broad range of oscillatory and wave processes in dissipative systems can be described by the canonic equations, specified by the Hamiltonian $H = H(p,q,\Lambda_1,\ldots,\Lambda_N)$, together with the first-order equations for the quantities $\Lambda_n$ (hereafter we call them parameters): $\dot{\Lambda}_n = \varepsilon f_n(\Lambda_1,\ldots,\Lambda_N,p,q)$. In this symbolic notation, the small parameter $\varepsilon$ (responsible for the dissipation) guarantees that the relative variation of $\Lambda_n$ at characteristic times of "stationary" dynamics is small.

An important step in this direction was made in [4], where a modified adiabatic invariant (different from the standard action integral $I_0 = \oint p dq$) was constructed for the definite type of these complicated systems. The procedure for constructing an adiabatic invariant in a general case is proposed in the present paper. However, such modified invariants cannot always be found in an explicit form, so the derivation of "truncated" equations describing averaged trends of the action and the parameters of the system is a more efficient method. Note that such an approach can be especially effective for studying dynamic regimes similar to the autoresonance (phase locking) effect [5–14]. This effect consists in "trapping" the phase trajectory in the vicinity of the elliptic equilibrium states with the adiabatic trend of the system parameters. In a standard case, the realization of such a regime is guaranteed by the conservation of the quantity $I_0$, which has the meaning of an area enclosed by an orbit in phase space. Correspondingly, in the general case, it is exactly the evolution of the action integral that determines the feasibility of such a trapping. Let us also mention such an important effect of the dynamics of nonlinear systems as a transition of the phase trajectory through a separatrix. It takes place when the quantity $I_0$ approaches the area encircled by the corresponding branch of the separatrix [5–7,15]. Obviously, the description of the self-consistent averaged dynamics of the action of the system and the parameters is an effective method for investigating processes of this type as well.

The paper is organized as follows. In Sec. II, first, a general approach to obtaining the modified adiabatic invariants is formulated and, second, self-consistent equations for evolution of the action integral and an averaged trend of parameters are obtained. In Sec. III, we present some examples of systems reducible to the one-dimensional (1D) Hamiltonian form with self-consistent parameters discussed here:

(i) A two-level atom controlled by a resonant quasi-monochromatic field within the framework of the Weisskopf-Wigner approximation

(ii) A system of nonlinear interactions of $M$ damping rotators that can be reduced to the universal model of nonlinear resonance generalized to the case of finite dissipation [16,17]. This model is widely used in celestial mechanics, the theory of particles motion in resonant high-frequency (HF) fields, and the theory of waves nonlinear interaction

(iii) A four-wave mixing of radiation in the regime of electromagnetically induced transparency in an ensemble of three-level atoms

In Sec. IV, the developed approach is applied to the generalized nonlinear resonance model. In particular, it is used to study the decay parametric instability of waves in a nonlinear dissipative medium. An autoresonance regime of such instability is detected. Appendixes A and B are devoted to an important example of a system reduced to the generalized model of nonlinear resonance, namely, the waveguide modes in a nonlinear medium.

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## II. MODIFICATION OF THE ADIABATIC INVARIANTS METHOD

Consider the system specified by the Hamiltonian $H = H(p,q,\Lambda_1,\ldots,\Lambda_N)$ dependent on $N$ parameters $\Lambda_n$.

$$\frac{dp}{d\tau} = \frac{\partial H}{\partial q}, \quad \frac{dq}{d\tau} = -\frac{\partial H}{\partial p}. \tag{1a}$$

The equations for $\Lambda_n$ have the form[1]

$$\frac{d\Lambda_n}{d\tau} = \varepsilon f_n(\Lambda_1,\ldots,\Lambda_N,p,q), \tag{1b}$$

where $\varepsilon$ is the small parameter. Let the closed trajectories $p = p(\Lambda_1,\ldots,\Lambda_N,E,q)$ correspond to this system for $\varepsilon \to 0$ ($\Lambda_n = \text{const}$). Here $E = H(\Lambda_1,\ldots,\Lambda_N,p,q) = \text{const}$ is the energy integral.[2] Following [1], we find the small (because of the smallness of the parameter $\varepsilon$) variation of some function $I(E,\Lambda_1,\ldots,\Lambda_N)$ over the period of motion $T$ along the unperturbed closed trajectory:

$$\Delta I = \sum_{n=1}^{N} \frac{\partial I}{\partial \Lambda_n} \int_{\tau}^{\tau+T} \frac{d\Lambda_n}{d\tau} d\tau + \frac{\partial I}{\partial E} \int_{\tau}^{\tau+T} \frac{dE}{d\tau} d\tau. \tag{2}$$

Substitute the exact relations

$$\frac{dE}{d\tau} = \sum_{n=1}^{N} \frac{\partial H}{\partial \Lambda_n} \frac{d\Lambda_n}{d\tau}, \quad d\tau = \frac{dq}{(\partial H/\partial p)},$$

together with the equations which are valid for the motion over an unperturbed trajectory (i.e., for small values of $\varepsilon$, see [1])

$$\frac{\partial H}{\partial p} = \left(\frac{\partial p}{\partial E}\right)^{-1}, \quad \frac{\partial H}{\partial p}\frac{\partial p}{\partial \Lambda_n} + \frac{\partial H}{\partial \Lambda_n} = 0$$

into Eq. (2). As a result, we obtain

$$\Delta I = \varepsilon \sum_{n=1}^{N} \left( A_E^n \frac{\partial I}{\partial \Lambda_n} - A_\Lambda^n \frac{\partial I}{\partial E} \right), \tag{3a}$$

where

$$A_E^n = \oint f_n(\Lambda_1,\ldots,\Lambda_N,p,q) \frac{\partial p}{\partial E} dq,$$
$$A_\Lambda^n = \oint f_n(\Lambda_1,\ldots,\Lambda_N,p,q) \frac{\partial p}{\partial \Lambda_n} dq. \tag{3b}$$

In Eq. (3b) the integration is performed over an unperturbed trajectory, i.e., for $p = p(\Lambda_1,\ldots,\Lambda_N,E,q)$, where $E = \text{const}$ and $\Lambda_n = \text{const}$.

If[3] $A_E^n, A_\Lambda^n \neq 0$, then the function $I(E,\Lambda_1,\ldots,\Lambda_N)$, which satisfies the condition $\Delta I = 0$ and, correspondingly, the equation

$$\sum_{n=1}^{N} A_E^n \frac{\partial I}{\partial \Lambda_n} - \sum_{n=1}^{N} A_\Lambda^n \frac{\partial I}{\partial E} = 0, \tag{4a}$$

can be found. A solution to Eq. (4a) is the integral of the system of equations for the characteristics (see, e.g., [37]):

$$\frac{d\Lambda_1}{dE} = -\frac{A_E^1}{\sum_{n=1}^N A_\Lambda^n}, \ldots, \frac{d\Lambda_N}{dE} = -\frac{A_E^N}{\sum_{n=1}^N A_\Lambda^n}. \tag{4b}$$

If the derivatives $d\Lambda_n/d\tau$ depend on the dynamic variables only via the Hamilton function,[4] i.e., $f_n = f_n(\Lambda_1,\ldots,\Lambda_N,H)$, then it follows from Eq. (3b) that $(A_E^n, A_\Lambda^n) = f_n(\Lambda_1,\ldots,\Lambda_N,E)(\frac{\partial I_0}{\partial E}, \frac{\partial I_0}{\partial \Lambda_n})$. In this case one of the integrals of the system of equations (4b) is the standard action $I = I_0 = \oint p dq$. For the factorized functions $f_n = u_n(\Lambda_1,\ldots,\Lambda_N,H)\eta(p,q)$, from Eq. (3b) we obtain $(A_E^n, A_\Lambda^n) = u_n(\Lambda_1,\ldots,\Lambda_N,E)(\frac{\partial I_S}{\partial E}, \frac{\partial I_S}{\partial \Lambda_n})$, where $I = I_S = \oint \nu dq$, $\frac{\partial \nu}{\partial p} = \eta$. In this case, the integral of the system of equations (4b) is the quantity $I_S$.

Earlier the expression for the adiabatic invariant $I_S$ was found in [4] for the particular case, when the Hamiltonian depends on one parameter $\Lambda$. In [4], it is shown that the modified adiabatic invariant is preserved with exponential accuracy. The proof presented in [4] can easily be generalized to the case of many parameters.

In the general case, the search for the integral of Eq. (4b) can appear as a fairly complicated problem. For actual physical systems (see the examples presented in Sec. III) in a typical situation some parameters are known functions of $\tau$ and the evolution of other parameters depends on the evolution of $p$ and $q$. Considering the more general case, let Eq. (1b) for the parameters $\Lambda_n$ with the numbers $n$ from 1 to $P-1$ comprise the functions $f_n = f_n(\Lambda_1,\ldots,\Lambda_N,H)$ independent of canonical variables. In such a situation, it seems natural to derive the differential equation for the action of the system. For this purpose, we pass from the finite variation of the function $I$ over the period to the temporal derivative $\frac{dI}{d\tau} \approx \frac{\Delta I}{T}$ and then substitute $I = I_0$ in Eq. (3a). As a result, with allowance for the expression $T = \frac{\partial I_0}{\partial E}$, the derivatives with respect to the parameters with numbers $n < P$ will be excluded in the final equation for the action:

$$\frac{dI_0}{d\tau} = \varepsilon \left(\frac{\partial I_0}{\partial E}\right)^{-1} \sum_{n=P}^{N} \left( A_E^n \frac{\partial I_0}{\partial \Lambda_n} - A_\Lambda^n \frac{\partial I_0}{\partial E} \right). \tag{5a}$$

Equation (5a) obviously comprises the values of all quantities averaged over the period $T$ of "fast" motions. Since in the general case the trend of the average values of the parameters $\Lambda_n$ is unknown, Eq. (5a) should be supplemented with the corresponding relations for these quantities. For this purpose,

[1] An admissible explicit dependence of the Hamiltonian on the "slow" time can formally be taken into account by introduction of the parameter, $\Lambda_{n=n^*} = \varepsilon\tau$, for which $f_{n^*} \equiv 1$.

[2] To avoid any misunderstanding, hereafter we use the designation for a constant $E$ which is equal to the Hamilton function on the given trajectory for $\Lambda_n = \text{const}$ and for the Hamiltonian $H(\Lambda_1,\ldots,\Lambda_N,p,q)$ which depends on the canonical variables.

[3] V. Khudik drew the author's attention to the latter condition.

[4] Obviously, all the results discussed in what follows are true if the functional dependence on the Hamiltonian is replaced by the functional dependence on the action $I_0$.

we apply the averaging operator[5] $\langle\cdots\rangle = \frac{1}{T}\oint(\cdots)\frac{dq}{(\partial H/\partial p)}$ to Eq. (1b); as a result, we obtain the following averaged equations for the parameters:

$$\frac{d\Lambda_n}{d\tau} = \varepsilon\left(\frac{\partial I_0}{\partial E}\right)^{-1} A_E^n. \tag{5b}$$

It can be shown that the first integral of the system of Eqs. (5a) and (5b) is the solution of Eq. (4a), written in corresponding variables $(I_0,\Lambda_n)$. The question of possible discrepancy between the solution of truncated Eqs. (5a) and (5b) and the solution of the initial system in Eqs. (1a) and (1b) is reduced in this way to the accuracy problem for the modified adiabatic invariant $I(I_0,\Lambda_n)$. This problem lies beyond the present analysis. Anyway, at worst the inaccuracy is defined by the small parameter $\varepsilon$ in correspondence with general asymptotic averaging methods (see, for example [18]).

## III. EXAMPLES OF HAMILTONIAN SYSTEMS WITH SELF-CONSISTENT DYNAMICS OF THE PARAMETERS

### A. Two-level quantum system in the Weisskopf-Wigner approximation

As a simple example, we consider the transition between two quantum states under the action of a quasimonochromatic external field with allowance for the relaxation processes within the framework of the Weisskopf-Winger model [19]. Let the states $|1\rangle$ and $|2\rangle$ correspond to the energy $W_1 < W_2$. The matrix element of the operator of the external-field interaction with the considered system can be represented in the form $V_{12} = d_{12}F(t)\cos\phi(t)$, where $d_{12}$ is a complex constant of coupling (e.g., a matrix element of the dipole moment), $F(t)$ is the real amplitude of the external field, $\phi = \int_0^t \omega(t)dt$ is the phase, and $\omega(t)$ is the "instantaneous" value of the frequency. The Weisskopf-Wigner approximation corresponds to the case in which the lower level is metastable, and the particles escape from the upper level to other levels of the system (i.e., the sum of populations at the considered pair of levels $|1\rangle$ and $|2\rangle$ is not preserved). Within the framework of such an approach, the Hamiltonian quantum operator is supplemented with a relaxation operator (see [19]):

$$\hat{H} = W_1|1\rangle\langle 1| + W_2|2\rangle\langle 2| + V_{12}|1\rangle\langle 2| + V_{21}|2\rangle\langle 1| - i\hbar\kappa|2\rangle\langle 2|, \tag{6}$$

where $V_{21} = V_{12}^*$, and $\kappa$ is the inverse lifetime of particles at the upper level. From the Schrödinger equation $i\hbar\dot{\Psi} = \hat{H}\Psi$ for the state vector $\Psi = c_1|1\rangle + c_2|2\rangle$ and the expression for the Hamiltonian in Eq. (6), using the so-called rotating-wave approximation [19], one can easily find the canonical system corresponding to the Hamiltonian

$$H(p,q) = \delta(t)p + 2\Omega_R(t)\sqrt{p(\Lambda - p)}\cos q, \tag{7a}$$

and the equation for the parameter

$$\dot{\Lambda} = 2\kappa(p - \Lambda). \tag{7b}$$

The canonical variables $p$ and $q$ are determined by the state vector, in which we put $c_{1,2} = |c_{1,2}|e^{i\varphi_{1,2}}$: $p = |c_1|^2$ and $q = \varphi_1 - \varphi_2 + \phi - \vartheta$; the angle $\vartheta$ is determined by the relationship $d_{12} = |d_{12}|e^{i\vartheta}$. The parameters of the Hamiltonian in Eq. (7a) are the effective Rabi frequency $\Omega_R = \frac{|d_{12}|F(t)}{2\hbar}$, the frequency-resonance detuning $\delta = \omega(t) - \frac{W_1 - W_2}{\hbar}$, and the sum of population $\Lambda = |c_1|^2 + |c_2|^2$, which is preserved at the limit $\kappa \to 0$.

Thus, we obtained a Hamiltonian system comprising both the "external" parameters $\delta(t)$ and $\Omega_R(t)$ and the parameter $\Lambda$, whose evolution depends self-consistently on the dynamics of the canonic variables. The role of the small parameter $\varepsilon$ in this case is played by the ratio $\kappa/\Omega_R$.

### B. Generalization of the universal model of nonlinear resonance to dissipative systems

The system of Eqs. (7a) and (7b) is a particular case of the more general model. For its demonstration we consider $M$ interacting rotators with eigenfrequencies $\omega_m$, where $m = 1,\ldots,M$. Let each rotator correspond to the complex equation of the first order:[6]

$$\frac{d\zeta_m}{d\tau} + i[\omega_m + \delta_m(\tau) - i\kappa_m]\zeta_m = iJ_m(\tau,\mathrm{Re}\zeta_1,\ldots,\mathrm{Re}\zeta_M). \tag{8}$$

Here, $\delta_m$ is the frequency shift (which in the general case depends on the independent variable $\tau$) and $\kappa_m$ is the dissipation coefficient. The time scale of variation of the oscillation amplitudes are assumed large compared to the oscillation periods $2\pi/\omega_m$. That is guaranteed by the smallness of the ratios $J_m/\zeta_m\omega_m$, $\kappa_m/\omega_m$, and $\delta_m/\omega_m$, as well as by a certain slowness of the explicit functions $\delta_m(\tau)$ and $J_m(\tau)$. The character of the interaction between separate rotators, and the nonlinear frequency shifts are determined by the nonlinear terms on the right-hand side of Eq. (8). The nonlinear terms $J_m$ are assumed to be real functions. In the case of sufficiently small amplitudes of oscillations these terms can be expanded into a series such as

$$J_m = \sum_j\left(\sum_{n=1}^{M} u_{mn}^{(j)}(\tau)\mathrm{Re}\zeta_n\right)^j. \tag{9}$$

Let the synchronism condition in an ensemble of rotators [20,22] be realized in the considered system:

$$\sum_{n=1}^{M}\sigma_n N_n\omega_n = 0, \tag{10}$$

where $N_n$ are the numbers of the resonant harmonics, the numbers $\sigma_n = \pm 1$ determine the particular synchronism condition (e.g., for the synchronism condition $\omega_1 + \omega_2 = 2\omega_3$ we

[5]To avoid cumbersome formulas, hereafter we omit the averaging symbol $\langle\cdots\rangle$. The use of averaged equations for the parameters together with the corresponding exact equations for the same parameters should not lead to a misunderstanding since the idea will always be clear from the context.

[6]Equations of this type correspond, in particular, to the complex amplitudes of the resonator or propagating wave modes, quantized-field operators, etc.

have $N_{1,2} = 1, \sigma_{1,2} = +1$ and $N_3 = 2, \sigma_3 = -1$). The resonant interaction between rotators under condition (10) is defined by the terms of series (9) with powers $j = q$, where[7]

$$q = \left(\sum_{n=1}^{M} N_n\right) - 1. \quad (11)$$

In this case, the resonant terms proportional to $(\zeta_m^*)^{N_m-1}\prod_{n\neq m}^{M}(\hat{\Theta}_{mn}\zeta_n)^{N_n} \propto e^{-i\omega_m\tau}$ appear on the right-hand side of Eq. (8). Here, the operator $\hat{\Theta}_{nm}$ performs complex conjugation if $\sigma_m\sigma_n = +1$ and is equal to unity if $\sigma_m\sigma_n = -1$.

Moreover, the term with $j = 3$ (the so-called cubic nonlinearity) generates terms proportional to $|\zeta_n|^2\zeta_m \propto e^{-i\omega_m\tau}$, which leads to the nonlinear shift of the rotator frequencies for an arbitrary relationship between frequencies[8] $\omega_{1,\ldots,M}$.

Using the averaging method [18] and assuming that the frequency-synchronism conditions in Eq. (10) are fulfilled, we retain only terms proportional to $e^{-i\omega_m\tau}$(resonant terms) on the right-hand side of Eq. (8). Replacing the variables $\zeta_m = c_m \exp(-i\int_{-\infty}^{\tau}\omega_m d\tau)$, which excludes "fast" motions, we finally transform system (8) to the following form:

$$\dot{c}_m + \kappa_m c_m + i\delta_m c_m - i\sum_{n=1}^{M}\gamma_{mn}|c_n|^2 c_m = i\beta_m (c_m^*)^{N_m-1}\prod_{n\neq m}^{M}(\hat{\Theta}_{mn}c_n)^{N_n}. \quad (12)$$

It is significant that the coefficients $\beta_m$ on the right-hand side of Eq. (12) are coupled by certain relations.[9] Indeed, since the total "energy" $W \propto \sum_{m=1}^{M}|c_m|^2$ is preserved in the original physical system in the absence of dissipation for any set of frequencies $\omega_m$ (the corresponding example is given in Appendix A), then with the parametric synchronism conditions (10) taken into account, different coefficients $\beta_m$ and $\beta_{n\neq m}$ should be coupled by the relationship $\beta_m/\omega_m N_m = \hat{\Theta}_{mn}\beta_n^*/\omega_n N_n$ (see, e.g., [22]). Thus, the coefficients $\beta_m$ can be represented in the following form:

$$\beta_m = \omega_m N_m \hat{\Theta}_{m1} w_1^*, \quad (13)$$

where $w_1 = \beta_1/\omega_1 N_1$ (cumbersome expressions for the quantities $w_1$ and $\gamma_{mn}$ are given in Appendix B for an important particular case). In the absence of dissipation (when $\kappa_{1,\ldots M} = 0$) and with relations (13) taken into account, the so-called Manly-Rowe relationships determining the laws of preservation of quantum numbers in the parametric processes (see [20,22,24]) follow from Eq. (12):

$$\frac{|c_m|^2}{\omega_m N_m} - \sigma_m\sigma_n\frac{|c_n|^2}{\omega_n N_n} = \Lambda_{mn} = \text{const}. \quad (14)$$

In the absence of dissipation, the fulfillment of integral (14) allows the system of equations (12) to be transformed to a 1D Hamiltonian system. *Exactly the same* system is obtained from Eq. (12) with relation (13) taken into account in the presence of dissipation if *at least one* of the dissipation coefficients $\kappa_m$ can be assumed to be negligibly small. However, such a system should be supplemented with equations for the quantities $\Lambda_{mn}$, which are not constants in a dissipative system. Indeed, we choose $\kappa_1 = 0$ and $\sigma_1 = +1$ for definiteness. After the replacement of the variables $\frac{|c_1|^2}{\omega_1 N_1} = p, q = \sum_{m=1}^{M} N_m\sigma_m\varphi_m - \varphi_w$, and $\frac{|c_{m\neq 1}|^2}{\omega_m N_m} = \sigma_m(p - \Lambda_{1m})$, where $|c_m|e^{i\varphi_m} = c_m$, $w_1 = |w|e^{i\varphi_w}$, from Eqs. (12) we obtain a canonical system specified by the Hamiltonian

$$H(p,q) = \delta(\tau,\Lambda_m)p + \gamma(\tau)\frac{p^2}{2} + 2\Omega(\tau)p^{\frac{N_1}{2}}\prod_{m=2}^{M}[\sigma_m(p-\Lambda_m)]^{\frac{N_m}{2}}\cos q, \quad (15a)$$

and equations for the parameters $\Lambda_m$:

$$\frac{d\Lambda_m}{d\tau} = 2\kappa_m(p - \Lambda_m), \quad m = 2,\ldots,M. \quad (15b)$$

In Eqs. (15a) and (15b), we used the following notation:

$$\Lambda_m \equiv \Lambda_{1m} \quad (\text{for } m = 2, \ldots, M),$$

$$\delta = \delta_0(\tau) + \tilde{\delta}(\Lambda_m), \quad \delta_0 = \sum_{m=1}^{M} -\sigma_m N_m\delta_m - \dot{\varphi}_w,$$

$$\tilde{\delta} = -\sum_{p=1}^{M}\sum_{m=2}^{M}\gamma_{pm}\sigma_p\sigma_m\Lambda_m\omega_m N_m N_p,$$

$$\gamma = \sum_{p=1}^{M}\sum_{m=1}^{M}\gamma_{pm}\sigma_p\sigma_m\omega_m N_m N_p, \quad \Omega = |w|\prod_{m=1}^{M}(\omega_m N_m)^{\frac{N_m}{2}}.$$

The quantities $\delta_0(\tau)$ and $\Omega(\tau)$ can, in general, be "slow" functions of the independent variable.

The system described by the Hamiltonian (15a) is the generalization of the so-called second universal model of nonlinear resonance [16,17], which in the case $\Lambda_m = \text{const}$ is widely used in celestial mechanics, the theory of motion of particles in resonant HF fields, and the theory of wave interaction. In particular, this system corresponds to various processes of transformation, scattering, and/or parametric decay in the wave systems [15,25,26]. The independent variable $\tau$ has the meaning of a coordinate in problems of propagation and interaction of stationary waves, and the meaning of time in problems of mode dynamics in resonators.

[7]The resonant interaction between modes coupled by condition (10) can be provided by degrees of nonlinearity less than $q$ (for example, a simple case $j = 2$ is sufficient), if the "intermediate" excitation of nonresonant combinative harmonics is taken into account. However, this fact affects the final result only quantitatively (see also Appendix A).

[8]The same effect, of course, takes place for all odd powers. However, in the case of weak nonlinearity, it stands to reason to allow for only the cubic term in this meaning.

[9]Essentially, that is due to symmetry of the matrix elements of the interaction operator for "multiphoton" processes (see [20,22–24]).

### C. Four-wave mixing of radiation in the electromagnetically induced transparency regime in an ensemble of three-level atoms.

We are speaking of the effect which was intensely studied in recent years and is promising for recording and transporting quantum and optical information [27–36]. This effect is based on the interaction of the pump wave, the probe wave, and the Stokes satellite of the pump with frequencies $\omega_{3,2,1}$, respectively, which are coupled by the synchronism condition $\omega_1 + \omega_2 = 2\omega_3$. The probe wave and the pump wave (frequencies $\omega_{2,3}$) also satisfy the conditions of one- and two-photon resonance with a three-level quantum system (see Fig. 1). In such a three-level medium, the absorption of the nonresonant Stokes wave can usually be neglected. The corresponding system of equations for stationary waves can also be reduced to a 1D Hamiltonian system with self-consistent equations for the parameters (see [36]):

$$H = \chi\left\{\frac{2\,(2\Lambda_3 - \Lambda_2 - 2p)\sqrt{p(p-\Lambda_2)}}{2\Lambda_3 - \Lambda_2 - p}\cos q + 3p + \frac{1}{4}\frac{(3p - \Lambda_2 - 2\Lambda_3)(4D - 5 - 2\Lambda_3 + \Lambda_2)}{2\Lambda_3 - \Lambda_2 - p}\right\}, \quad (16a)$$

$$\frac{d}{d\tau}\Lambda_3 = -\frac{1}{2}\frac{d}{d\tau}\Lambda_2 = -\kappa_{\rm EIT}\frac{(p-\Lambda_2)(2\Lambda_3 - \Lambda_2 - 2p)}{(2\Lambda_3 - \Lambda_2 - p)^2}. \quad (16b)$$

The independent variable in Eqs. (16a) and (16b) has the meaning of a coordinate. The canonical variables are $p = \frac{n_{\omega_1}}{n^0_{\omega_3}}$ and $q = \varphi_1 + \varphi_2 - 2\varphi_3$, where $\varphi_j$ are the phases of complex amplitudes, $n_{\omega_j}$ are the photon flux densities in the corresponding modes, and the normalization $n^0_{\omega_3}$ corresponds to the boundary value of the pump photon flux density. The parameters are $\Lambda_2 = (n_{\omega_1} - n_{\omega_2})/n^0_{\omega_3}$, $\Lambda_3 = (n_{\omega_3}/2 + n_{\omega_1})/n^0_{\omega_3}$, $\chi$ is the coupling coefficient for the four-wave mixing, $\kappa_{\rm EIT}$ is the standard coefficient of absorption of the probe wave in the regime of electromagnetically induced transparency, $D = \frac{1}{\nu_0}(\omega_{21} - \omega_2 + \omega_3 + \frac{5}{4}\nu_0)$, $\nu_0 = \frac{2|\Omega^0_R|^2}{\omega_{21}}$, $\omega_{21}$ is the frequency of the transition 1-2 (see Fig. 1), and $\Omega^0_R$ is the boundary value of the pumping wave Rabi frequency.

By virtue of the complicated nonlinearity, which is typical of the regime of electromagnetically induced transparency and is not reduced to the power-law dependence, the system of equations (16a) and (16b) differs notably from the standard form of Eqs. (15a) and (15b). This appears, in particular, in the specific relation between the nonlinear synchronism detuning and the pump depletion effects. However, within the framework of the fixed pump intensity approximation, when $p,\Lambda_2 \ll \Lambda_3 \approx 1$, this system can be represented in the form of Eqs. (15a) and (15b) for $M = 2$ and $N_1 = N_2 = 1$.

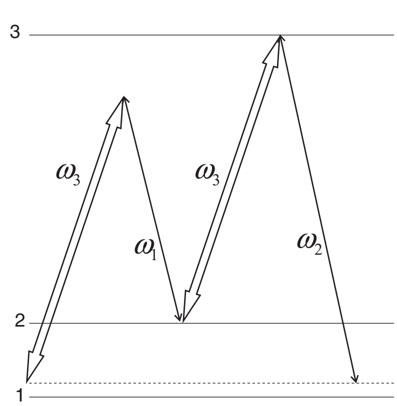


FIG. 1. Four-wave mixing in the EIT scheme. $\omega_3$ and $\omega_2$ are the pump and probe resonant frequencies, respectively; $\omega_1$ is the frequency of the Stokes satellite of the pump.

## IV. APPLICATION OF THE METHOD OF AVERAGED EQUATIONS TO THE GENERALIZED NONLINEAR RESONANCE MODEL

### A. Averaged equations for the generalized nonlinear resonance model

We now apply the results of Sec. II for the generalized nonlinear resonance model specified by Hamiltonian (15a), which was presented in Sec. III. Within the framework of this model, we have the "external" parameters $\delta_0(\tau)$ and $\Omega(\tau)$ and the parameters $\Lambda_m$ determined by Eq. (15b).

Averaged equations (5a) and (5b) can be significantly simplified in this case if instead of the energy $E$, we choose the action $I_0$ as the parameter indicating the unperturbed phase trajectory, i.e., if we represent the unperturbed phase trajectory in the form $p = p(q, I_0, \Lambda_2, \ldots, \Lambda_M, \delta_0, \Omega,)$. As a result, we arrive at averaged equations in the following form:

$$\frac{dI_0}{d\tau} = -\sum_{m=2}^{M}\left(\kappa_m\frac{\partial I_T}{\partial\Lambda_m}\right), \quad (17a)$$

$$\frac{d\Lambda_m}{d\tau} = \kappa_m\left(\frac{\partial I_T}{\partial I_0} - 2\Lambda_m\right), \quad m = 2,\ldots,M, \quad (17b)$$

where $I_T(I_0,\Lambda_2,\ldots,\Lambda_M,\delta_0,\Omega) = \oint p^2 dq$. Note that by virtue of exact equation (15b) in accordance with definition (14), we always have $\frac{d\Lambda_m}{d\tau} \geqslant 0$ for $\sigma_m = +1$ and $\frac{d\Lambda_m}{d\tau} \leqslant 0$ for $\sigma_m = -1$ in relationship (17b).

Near the "local" equilibrium state, for which[10] $p = p_0(\Lambda_2,\ldots,\Lambda_M,\delta_0,\Omega)$, the system of equations (17a) and (17b) takes an especially clear form. In a relatively small vicinity of the elliptic stationary point, we obtain $I_T = \oint p^2 dq \approx 2p_0 I_0$, which reduces Eqs. (17a) and (17b) to a simple form:

$$\frac{dI_0}{d\tau} = -\mu I_0, \quad (18a)$$

$$\frac{d\Lambda_m}{d\tau} = 2\kappa_m(p_0 - \Lambda_m), \quad (18b)$$

[10]The angular coordinate $q$, as follows from the expression for Hamiltonian (15a), at the stationary point is equal to 0 or $\pi$.

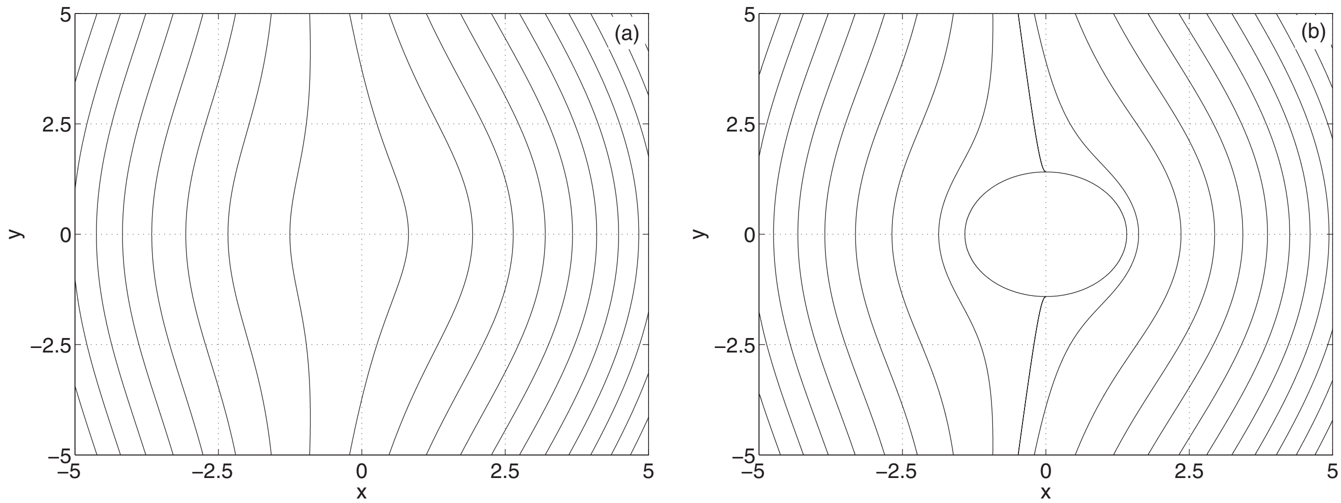


FIG. 2. Phase plane for the system with Hamiltonian (20a) and constant parameter $\Lambda$ ($\kappa = 0$). Regime of decay instability: $\frac{\delta_0}{2\Omega} = 0.1$, $\gamma = 0$, (a) $\Lambda = -1$, (b) $\Lambda = 1$.

where $\mu = 2\sum_{m=2}^{M} \kappa_m \frac{dp_0}{d\Lambda_m}$. From the expression for the coefficient $\mu$ it is apparent that the influence of the variation of the "nonstandard" parameters $\Lambda_m$ on the phase-trajectory trapping effect in the vicinity of a local elliptic stationary point is determined by the dependence of the position of this stationary point on the value of the parameters $\Lambda_m$. Certainly; the adiabatic variation of the "standard" parameters $\delta_0(\tau)$ and $\Omega(\tau)$ does not affect the trapping stability.

It is useful to mention that the function $I_T$, which governs the system of equations (17a) and (17b), acquires a simple geometric meaning when the phase plane is represented in the coordinates

$$x = \sqrt{2p}\cos q, \quad y = \sqrt{2p}\sin q. \tag{19}$$

Such a representation was used in [6,7,15] to simplify the form of the phase portrait of the universal nonlinear resonance model. In these variables, the quantity $I_T$ is determined by the expression $I_T = \iint_{S_0}(x^2+y^2)dxdy$, where $S_0$ is the area encircled by the phase trajectory. Thus, the quantity $I_T$ on the plane $(x,y)$ corresponds to the inertia moment of a plane figure bounded by a closed phase trajectory relative to the origin of coordinates. The action $I_0$ in representation Eq. (19) preserves the former geometric meaning: $I_0 = \iint_{S_0} dxdy$.

### B. Autoresonance (phase locking) regime of dissipative instability

The efficiency of the approach developed in this paper can be demonstrated by the example of studying the influence of dissipation on the decay parametric interaction of waves. Consider the frequency synchronization condition $\omega_1 + \omega_2 = N\omega_3$. If the wave absorption at the frequency $\omega_1$ can be neglected, then this process corresponds to Hamiltonian (15a) for $M = 3$, $N_{1,2} = 1, N_3 = N$, $\sigma_2 = +1$, and $\sigma_3 = -1$. Confining ourselves to the approximation of a strong undepleted pump at frequency $\omega_3$, when $\Lambda_3 - p \approx \Lambda_3 = \text{const}$, after the replacement $\Omega\Lambda_3^{\frac{N}{2}} \to \Omega$ we arrive at Hamiltonian (15a) with $M = 2, \sigma_2 = +1$, and $N_{1,2} = 1$. Putting for simplicity $\gamma_{ij} = 0$ for all $\gamma_{ij}$ except $\gamma_{11}$ and choosing for definiteness $\gamma > 0$, from Eq. (15a) we obtain the following expression for the Hamiltonian:

$$H(p,q,\Lambda) = \delta_0 p + \gamma\frac{p^2}{2} + 2\Omega\sqrt{p(p-\Lambda)}\cos q, \tag{20a}$$

which should be supplemented with an equation for the parameter

$$\dot{\Lambda} = 2\kappa(p-\Lambda) \tag{20b}$$

(here, $\Lambda \equiv \Lambda_2$ and $\kappa \equiv \kappa_2$). The canonical momentum $p$ has the meaning of the number of quanta[11] $n_{\omega_1}$ in the mode at the frequency $\omega_1$. The number of quanta in the mode at the frequency $\omega_2$ is equal to $n_{\omega_2} = p - \Lambda$. The difference of quantum numbers $\Lambda = n_{\omega_1} - n_{\omega_2}$ is the integral in a nondissipative system. Areas of definition for the generalized momentum are $p \in [\Lambda,\infty]$ or $p \in [0,\infty]$ for $\Lambda > 0$ or $\Lambda < 0$, respectively, in any case[12] $\dot{\Lambda} \geqslant 0$. If the nonlinear loss of synchronism and the dissipation are absent (i.e., $\gamma = \kappa = 0$), then a decay instability exists in this system in the parameter region $|\delta_0| < 2\Omega$. In this case, the phase trajectories are open and go to infinity. The corresponding phase plane is shown in Fig. 2 for the coordinates $(x,y)$ introduced by replacement of the variables given by Eq. (19). Outside the instability region, when $|\delta_0| > 2\Omega$, or with the nonlinear loss of synchronism taken into account, when $\gamma \neq 0$, the stabilization effect takes place. In this case, there is a unique equilibrium state on the

[11]In the study of propagation and interaction of stationary waves, the canonic momentum has the meaning of a quantum flux.

[12]The system of equations (7a) and (7b) considered earlier is similar to Eqs. (20a) and (20b), but there is a radical difference. For the two-level quantum system, we always have $p \in [0,\Lambda]$ and $\dot{\Lambda} \leqslant 0$. These conditions have a clear physical meaning: the lower-level population is less than the total population which decreases all the time.

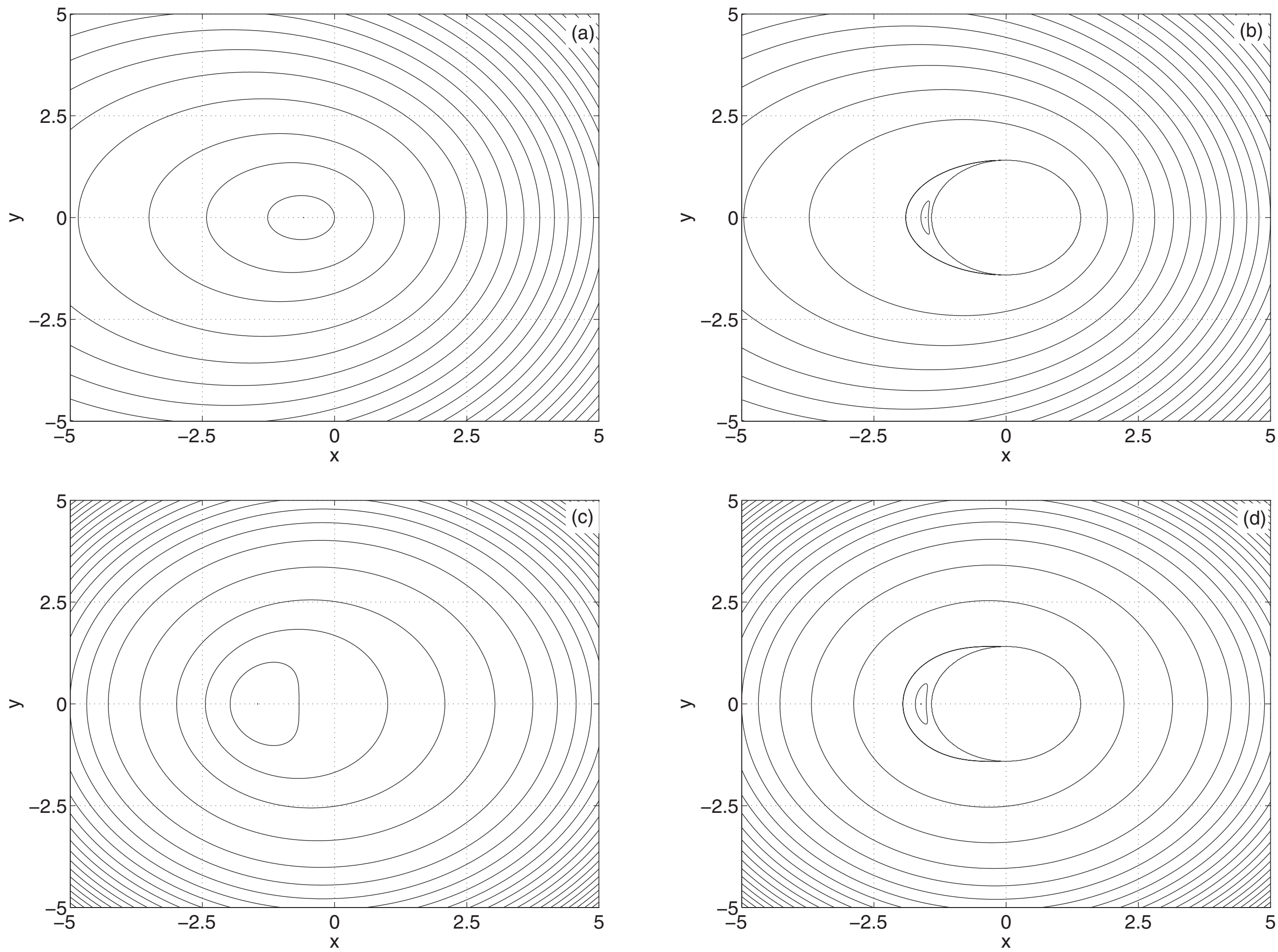

FIG. 3. Phase plane for the system with Hamiltonian (20a) and constant parameter $\Lambda$ ($\kappa = 0$). Stabilization of decay instability (a, b) due to linear synchronism detuning $\frac{\delta_0}{2\Omega} = 2$, $\gamma = 0$, (a) $\Lambda = -1$, (b) $\Lambda = 1$, and (c, d) due to nonlinear synchronism detuning $\delta_0 = 0$, $\frac{\gamma}{2\Omega} = 1$, (c) $\Lambda = -1$, (d) $\Lambda = 1$.

phase plane (see Fig. 3), for whose coordinates the following expression can easily be obtained:

$$q_0 = \pi, \quad \delta_0 + \gamma p_0 = \Omega \frac{2p_0 - \Lambda}{\sqrt{p_0(p_0 - \Lambda)}}. \tag{21}$$

The trajectories surround the equilibrium state or the separatrix. The separatrix exists for positive $\Lambda$.

However, the stabilization effect disappears if the absorption of the wave with frequency $\omega_2$ is taken into account. Indeed, assuming $\kappa \neq 0$, we use the relationships in Eqs. (17a), (17b), (18a), and (18b) for analysis of the evolution of the system, where we put $M = 2$, $\Lambda_2 \equiv \Lambda$, and $\kappa_2 \equiv \kappa$. Since $\dot{\Lambda} \geqslant 0$, the quantity $\Lambda$ monotonically increases. For the initial value $\Lambda < 0$, the absolute value of the parameter $|\Lambda|$ drops to zero at first, but then begins to rise once the positive values of $\Lambda$ are reached. In accordance with Eq. (21), the quantity $p_0$ rises, i.e., the local position of the equilibrium state moves adiabatically smoothly toward the periphery of the phase plane. From qualitative considerations (confirmed by a numerical calculation) it follows that all trajectories that initially remote from the separatrix will be finally captured in it. For trajectories within the separatrix, the inertia moment $I_T(\Lambda, I_0)$ of a plane figure always increases with increasing $\Lambda$ for a fixed value of $I_0$ under the condition $\Lambda > 0$, i.e., in this case, $\frac{\partial I_T}{\partial \Lambda} > 0$. It follows from Eq. (17a) that the action $I_0$ decreases in this case [in a relatively small vicinity of the equilibrium state, the action exponentially decreases since the coefficient $\mu > 0$ as it follows from relationships (18a) and (18b)].

Thus, the phase trajectory is trapped in the vicinity of the "local" equilibrium state, which moves to the periphery of the phase plane. Such an evolution of the dynamic system obviously corresponds to the autoresonance (phase locking) regime. In a standard variant, the realization of this effect requires the slow variation of the "external" parameters of the system. Usually, such a parameter is the linear frequency detuning [5–14]. In the present case, however, allowance for the dissipation of one of the interacted waves is sufficient, i.e., the autoresonance regime of dissipative instability takes place.

For sufficiently long interaction, from relationships (21) and (18b) the following asymptotic forms can be obtained for the quantum numbers in the modes $n_{\omega_1} \approx p_0$, $n_{\omega_2} \approx p_0 - \Lambda$:

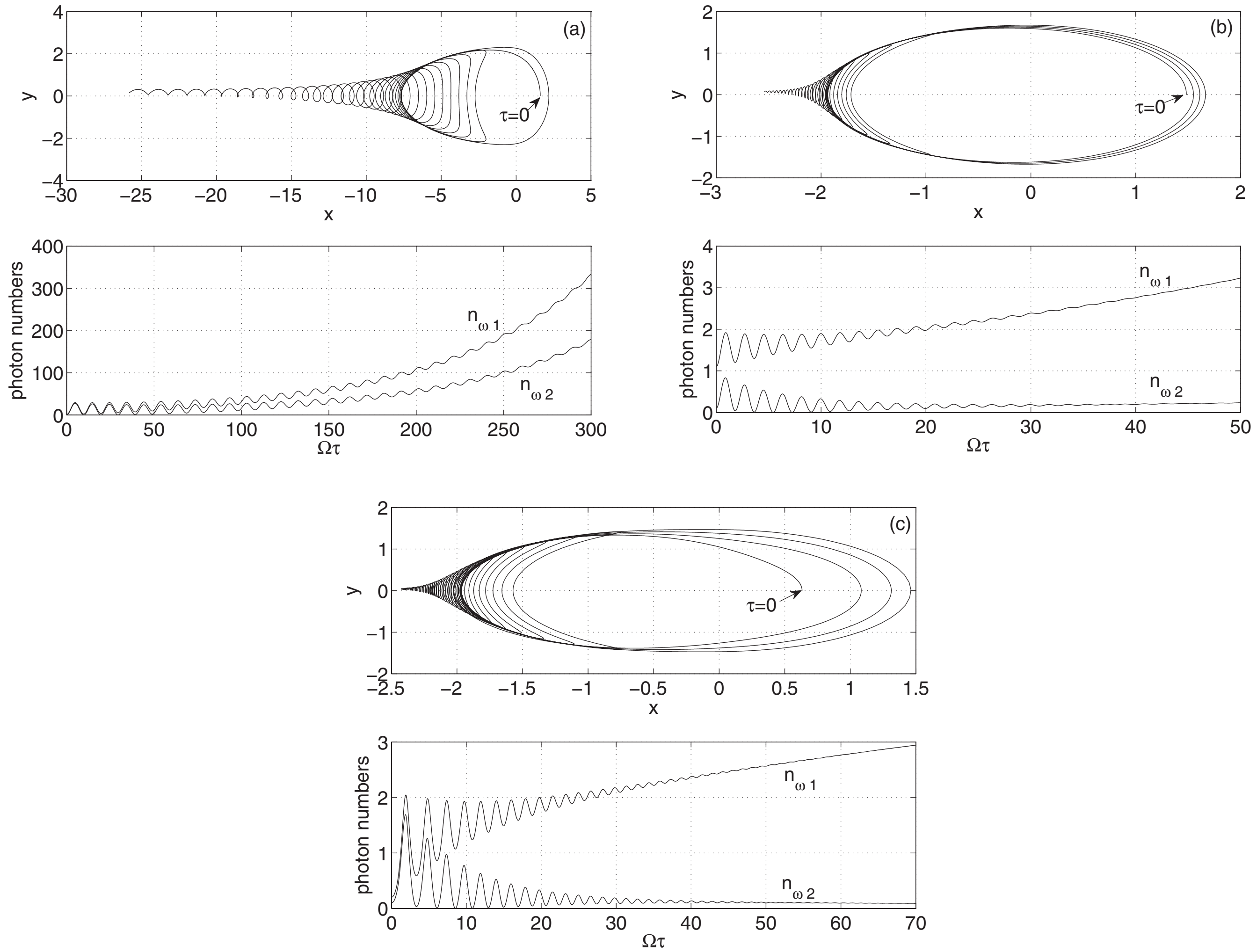


FIG. 4. Upper panels display examples of phase trajectories for the system with Hamiltonian (20a) and varying parameter $\Lambda$ due to dissipation ($\kappa \neq 0$). The corresponding time dependences of photon numbers in two interacted modes are displayed in the bottom panels. The autoresonance regime of dissipative instability is demonstrated with different asymptotical behavior: (a) $\frac{2\kappa}{\Omega} = 0.01$, $\frac{\delta_0}{2\Omega} = 1.05$, $\gamma = 0$, $\Lambda(\tau = 0) = 1$, $p(\tau = 0) = 1.3$, (b) $\frac{2\kappa}{\Omega} = 0.2$, $\frac{\delta_0}{2\Omega} = 2$, $\gamma = 0$, $\Lambda(\tau = 0) = 1$, $p(\tau = 0) = 1.1$, and (c) $\frac{2\kappa}{\Omega} = 0.2$, $\delta_0 = 0$, $\frac{\gamma}{2\Omega} = 1$, $\Lambda(\tau = 0) = 0.1$, $p(\tau = 0) = 0.2$.

If $\gamma n_{\omega_1} \ll |\delta_0|, |\delta_0| > 2\Omega, (|\delta_0| - 2\Omega) \ll |\delta_0|$ then $n_{\omega_1} \approx n_{\omega_2} \approx p_0 \gg \Lambda$,

$$\begin{pmatrix} n_{\omega_1} \\ n_{\omega_2} \end{pmatrix} \propto \begin{pmatrix} \frac{1}{2}\left[\frac{1}{\sqrt{(\delta_0/2\Omega)^2 - 1}} + 1\right] \\ \frac{1}{2}\left[\frac{1}{\sqrt{(\delta_0/2\Omega)^2 - 1}} - 1\right] \end{pmatrix} \times \exp\left(\kappa\tau\left[\frac{1}{\sqrt{(\delta_0/2\Omega)^2 - 1}} - 1\right]\right); \tag{22a}$$

if $\gamma n_{\omega_1} \ll |\delta_0|, |\delta_0| \gg 2\Omega$ then $n_{\omega_1} \approx p_0 \approx \Lambda$,

$$\begin{pmatrix} n_{\omega_1} \\ n_{\omega_2} \end{pmatrix} \propto \begin{pmatrix} 1 \\ (\Omega/\delta_0)^2 \end{pmatrix} \times 2(\Omega/\delta_0)^2 \kappa\tau; \tag{22b}$$

if $\gamma n_{\omega_1} \gg |\delta_0|$ and $\Lambda\gamma \gg \Omega$ then $n_{\omega_1} \approx p_0 \approx \Lambda$,

$$\begin{pmatrix} n_{\omega_1} \\ n_{\omega_2} \end{pmatrix} \approx \frac{\Omega}{\gamma} \times \begin{pmatrix} 2\sqrt{\kappa\tau} \\ 1/(2\sqrt{\kappa\tau}) \end{pmatrix}. \tag{22c}$$

The dependences in Eqs. (22a) and (22b) describe the instability effect in the parameter region in which the system is stable in the absence of dissipation. Equation (22c) describes the dissipative instability under conditions of the nonlinear synchronism detuning (but without allowance for the depletion of the pump which was assumed fixed for the sake of simplicity). Thus, the allowance for dissipation leads to the extension of the instability to beyond the range $|\delta_0| < 2\Omega$ and impedes its stabilization due to the nonlinear resonance detuning effect. Figure 4 presents the numerical calculation data illustrating the "trapping" of such a system into the autoresonance regime. The calculations were made for the initial strict system with Hamiltonian (20a) and varying parameter obeying Eq. (20b). They confirm the qualitative conclusions and asymptotic results of Eqs. (22a)–(22c) obtained from truncated equations. This result illustrates that discrepancy with the exact solution, in particular the dimension of the "trapping area," is defined by some small parameter for a sufficiently long time.

Dependences (22a)–(22c) obtained above can be used, in particular, for analysis of the bichromatic radiation generation during four-wave mixing in the regime of electromagnetically induced transparency in an ensemble of three-level atoms [27–29,32,35,36]. Expressions (22a) and (22b) describe the regime of dissipative instability of a bichromatic mode formed by the probe wave and Stokes satellite of the pump, which was predicted in [35]. Expression (22c) in the case of electromagnetically induced transparency can be used only to a certain degree of convention. It was mentioned in Sec. III that the parameter region in which the nonlinear loss of parametric synchronism can correctly be taken into account, neglecting the pump depletion effect at the same time, is virtually absent in this regime. Nevertheless, the numerical calculations performed in [36] demonstrate that the effect of the probe-wave intensity decrease accompanied by a weak increase in the Stokes pump satellite, which corresponds to Eq. (22c), indeed takes place at a certain stage of development of the four-wave mixing process under conditions of electromagnetically induced transparency.

## V. CONCLUSIONS AND DISCUSSION

Thus, a wide range of problems dealing with resonant interaction of dissipative oscillatory-wave systems can be reduced to a 1D Hamiltonian system with self-consistent parameters. The dynamics of these parameters is determined by the first-order equations dependent on the canonical variables. For the reduction of the equations describing an actual physical system to such a form, it is necessary, as a rule, that dissipation be neglected in at least one "partial" oscillatory system. A modified adiabatic invariant (different, in the general case, from the integral of action) always exists within the framework of such a model. It can be found as an integral of a system of ordinary differential equations of order equal to the number of varying parameters. However, in the general case, the search for the modified adiabatic invariant is a fairly complicated problem. Another method of analysis is based on the self-consistent averaged equations for the action and the system parameters. This approach was used to study the decay parametric instability of waves with allowance for dissipation. The autoresonance (phase locking) regime was detected for such a system.

## ACKNOWLEDGMENTS

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## APPENDIX A: EXCITATION OF POLARIZATION IN A NONLINEAR ANISOTROPIC CRYSTAL

To derive a system of equations describing the parametric interaction of monochromatic waves in a nonlinear medium, it is necessary to consider the effect of excitation of the nonlinear polarization. Use the model of a nonlinear anisotropic crystal without the spatial dispersion, in which the polarization of unit volume $\boldsymbol{P}$ is given by the relationship

$$\hat{L}\left(\frac{\partial}{\partial t},\frac{\partial^2}{\partial t^2}\right)\boldsymbol{P}=\boldsymbol{E}+\boldsymbol{U}(\boldsymbol{P}). \quad \text{(A1)}$$

Here $\hat{L}$ is the linear operator, whose correct form for quantum systems was discussed in [38], $\boldsymbol{E}$ is the electric field, and $\boldsymbol{U}(\boldsymbol{P})$ is a phenomenological nonlinear term. We show in what follows that the relationships in Eq. (13) follow from this sufficiently general formula if the operator $\hat{L}$ is supposed to be Hermitian.

Let $U(0)=0$ and $\frac{\partial \boldsymbol{U}}{\partial P_x}|_{\boldsymbol{P}=0}=\frac{\partial \boldsymbol{U}}{\partial P_y}|_{P=0}=\frac{\partial \boldsymbol{U}}{\partial P_z}|_{\boldsymbol{P}=0}=0$, i.e., assume that the linear terms of expansion of the function $\boldsymbol{U}(\boldsymbol{P})$ with respect to its argument are included in the linear operator $\hat{L}$. For simplicity, we assume that the nonlinear effects themselves do not couple $x$, $y$, and $z$ components of the polarization.[13] In this case, the function $\boldsymbol{U}(\boldsymbol{P})$ can be represented in the form

$$\boldsymbol{U}(\boldsymbol{P})=\sum_{n=2}^{\infty}\left(x_0\alpha_x^{(n)}P_x^n+y_0\alpha_y^{(n)}P_y^n+z_0\alpha_z^{(n)}P_z^n\right), \quad \text{(A2)}$$

where $\alpha_{x,y,z}^{(n)}$ are the corresponding coefficients of expansion into a Taylor series, and $\boldsymbol{z}_0,\boldsymbol{x}_0$, and $\boldsymbol{y}_0$ are the unit vectors of the coordinate axes.

Consider the electromagnetic field in the form of a set of monochromatic components

$$\boldsymbol{E}=\mathrm{Re}\sum_{m=1}^{M}\boldsymbol{E}_m e^{-i\omega_m t}. \quad \text{(A3)}$$

Let us find the polarization of the medium at frequencies $\omega_1,\ldots,\omega_M$. Assuming that the field $\boldsymbol{E}$ is small in a sense, we will make use of the perturbation method. Within the framework of the linear approximation, the operator $\hat{L}$ determines the tensor of linear polarizability of the medium $\hat{\chi}_m$, i.e., the vector $\boldsymbol{P}_{\boldsymbol{m}}=\hat{\chi}_m\boldsymbol{E}_m e^{-i\omega_m t}$ is the solution of the equation $\hat{L}\boldsymbol{P}_{\boldsymbol{m}}=\boldsymbol{E}_m e^{-i\omega_m t}$. The nonlinear solution can be presented as

$$\boldsymbol{P}_m=\mathrm{Re}(\hat{\chi}_m\boldsymbol{E}_m+\delta\boldsymbol{P}_m)e^{-i\omega_m t}.$$

The nonlinear polarization of the medium appears at the frequency $\omega_m$ for any relationship between the frequencies $\omega_1,\ldots,\omega_M$ due to the dependence of the function $\boldsymbol{U}(\boldsymbol{P})$ on polarization in any odd power. Thus, taking into account the cubic nonlinearity, i.e., $\delta P^{(3)}\propto E^3$, is usually sufficient for the description of this effect [20,21]. Moreover, under the condition of frequency synchronism Eq. (10), nonlinear polarization at the given frequency $\omega_m$ appears if the dependence of the polarization on the electric field in power $q$, $\delta P^{(q)}\propto E^q$, is taken into account, where the number $q$ is determined by Eq. (10). To obtain the nonlinear terms of the corresponding power, one should allow for the terms of expansion of the vector $\boldsymbol{U}$ over powers of $P_{x,y,z}$ in order $q$.[14] Thus we obtain

$$\delta\boldsymbol{P}_m=\delta\boldsymbol{P}_m^{(3)}+\delta\boldsymbol{P}_m^{(q)}, \quad \text{(A4)}$$

$$\delta P_{m;p}^{(3)}=\sum_{n\neq m}^{M}\chi_{m;ps}\alpha_s^{(3)}|\chi_{n;sk}E_{n;k}|^2\chi_{m;sr}E_{m;r} + 2\chi_{m;ps}\alpha_s^{(3)}|\chi_{m;sk}E_{m;k}|^2\chi_{m;sr}E_{m;r}, \quad \text{(A5a)}$$

[13] Such a relation can occur, in principle, in an anisotropic medium.

[14] Thus we get a nonlinear solution proportional to the first order of nonlinear coefficients $\alpha_{x,y,z}^{(3)}$ and $\alpha_{x,y,z}^{(q)}$. If the degree of nonlinearity is less than $q$ then the nonlinear polarization can also be excited at given frequencies but proportional to higher orders of corresponding coefficients.

$$\delta P_{m;p}^{(q)} = \chi_{m;ps}\alpha_s^{(q)} \prod_{n\neq m}^{M} (\hat{\Theta}_{mn}\chi_{n;sk}E_{n;k})^{N_n} \times (\chi_{m;sr}^{*}E_{m;r}^{*})^{N_m-1} N_m \frac{2^{-q}q!}{\prod_{n=1}^{M} N_n}. \quad \text{(A5b)}$$

Here, $\delta P_{m;p}^{(3)}$ and $\delta P_{m;p}^{(q)}$ are the Cartesian components of the vectors $\delta \boldsymbol{P}_m^{(3)}$ and $\delta \boldsymbol{P}_m^{(q)}$, $\chi_{m;ps}$ are the corresponding matrix elements of the polarizability tensor, namely, $\hat{\chi}_m \Rightarrow \chi_{m;xx},\chi_{m;xy},\chi_{m;xz}$, etc., and the indices $p,s,k$, and $r$ run through the values $x,y$, and $z$. The operator $\hat{\Theta}_{nm}$ was introduced earlier in Sec. II. To avoid a misunderstanding, we emphasize that in Eqs. (A4), (A5a), and (A5b) and elsewhere, as usual, the summation is performed over recurrent indices denoting the Cartesian coordinates; but, of course, this does not concern the indices $m$ and $n$ which stand for harmonic numbers.

It can be verified that in the case of Hermitian tensors $\hat{\chi}_m$ the following properties are valid:

$$\mathrm{Im}\big(\delta \boldsymbol{P}_m^{(3)}\boldsymbol{E}_m^{*}\big) = 0, \quad \text{(A6a)}$$

$$\frac{1}{N_m}\delta \boldsymbol{P}_m^{(q)}\boldsymbol{E}_m^{*} = \frac{1}{N_n}\hat{\Theta}_{mn}\delta \boldsymbol{P}_n^{(q)}\boldsymbol{E}_n^{*}. \quad \text{(A6b)}$$

Relationships (A6a) and (A6b) in combination with the conditions in Eq. (10) ensure the absence of the resonant energy exchange between the field $\boldsymbol{E}$ and the medium due to the nonlinear polarization current if the influence of dissipative effects on the excitation of nonlinear polarization of the medium is neglected. In this case, the energy exchange between the field and the medium due to linear dissipation can be taken into account additively.

## APPENDIX B: PARAMETRIC INTERACTION OF STATIONARY MODES IN A NONLINEAR WAVEGUIDE SYSTEM

Consider a stationary electromagnetic field formed by a set of $M$ waveguide modes. In this case, the amplitudes of the monochromatic components in Eq. (A3) can be specified by the relationships

$$\boldsymbol{E}_m = \boldsymbol{\Psi}_m(\boldsymbol{r}_\perp,\eta z)A_m(\eta z)\exp\left(i\int_{-\infty}^{z} h_m(\eta z)dz\right). \quad \text{(B1a)}$$

Here, $A_m$ are the scalar amplitudes of the waves propagating along the $z$ axis; $\boldsymbol{\Psi}_m(r_\perp,\varepsilon z) = x_0\Psi_{m;x} + y_0\Psi_{m;y} + z_0\Psi_{m;z}$ are the vector functions which specify polarizations of the partial waves and their spatial structure in the $(x,y)$ plane; and $\boldsymbol{r}_\perp = \boldsymbol{x}_0 x + \boldsymbol{y}_0 y$ is the radius vector in that plane. The small parameter $\eta$ determines the adiabatically smooth dependence of the functions $\boldsymbol{\Psi}_m$, $A_m$, and $h_m$ on the $z$ coordinate. Then use of the wave equation for electromagnetic waves

$$\nabla \times (\nabla \times \boldsymbol{E}_m) - \frac{\omega_m^2}{c^2}\hat{\varepsilon}_m \boldsymbol{E}_m = 4\pi\frac{\omega_m^2}{c^2}\delta \boldsymbol{P}_m, \quad \text{(B1b)}$$

where $\hat{\varepsilon}_m = \hat{\delta} + 4\pi\hat{\chi}_m$, $\hat{\chi}_m(\omega_m,\boldsymbol{r}_\perp,\eta z)$ is the linear susceptibility tensor at the frequency $\omega_m$ in an inhomogeneous medium, and the nonlinear term $\propto \delta \boldsymbol{P}_m$ is determined by Eqs. (A4), (A5a), and (A5b). We determine the functions $\boldsymbol{\Psi}_m$ and wave numbers $h_m$ by omitting the nonlinear term $\propto \delta \boldsymbol{P}_m$ in wave equation (B1b). Then, assuming $\frac{\partial}{\partial z} \rightarrow ih_m$, we obtain

$$\hat{D}_m \boldsymbol{\Psi}_m = 0, \quad \text{(B2a)}$$

where the operator $\hat{D}_m \equiv \hat{D}_{m;ps}$ (here, $p,s = x,y,z$) has the following form:

$$\hat{D}_{m;ps} = \begin{vmatrix} \left(h_m^2 - \frac{\omega_m^2}{c^2}\varepsilon_{m;xx}\right) & \left(\frac{\partial^2}{\partial x\partial y} - \frac{\omega_m^2}{c^2}\varepsilon_{m;xy}\right) & \left(ih_m\frac{\partial}{\partial x} - \frac{\omega_m^2}{c^2}\varepsilon_{m;xz}\right) \\ \left(\frac{\partial^2}{\partial y\partial x} - \frac{\omega_m^2}{c^2}\varepsilon_{m;yx}\right) & \left(h_m^2 - \frac{\omega_m^2}{c^2}\varepsilon_{m;yy}\right) & \left(ih_m\frac{\partial}{\partial y} - \frac{\omega_m^2}{c^2}\varepsilon_{m;yz}\right) \\ \left(ih_m\frac{\partial}{\partial x} - \frac{\omega_m^2}{c^2}\varepsilon_{m;zx}\right) & \left(ih_m\frac{\partial}{\partial y} - \frac{\omega_m^2}{c^2}\varepsilon_{m;zy}\right) & \left(-\frac{\omega_m^2}{c^2}\varepsilon_{m;zz}\right) \end{vmatrix}. \quad \text{(B2b)}$$

For the given frequency $\omega_m$ and boundary conditions in the $(x,y)$ plane, Eqs. (B2a) and (B2b) determine the "transverse" spatial structure of the vector mode and its longitudinal wave number $h_m$ (see, e.g., [39]). The adiabatically smooth dependence on the longitudinal coordinate is taken into account only as dependence of the coordinates of the tensor $\hat{\varepsilon}_m$ (and/or transverse structure of the waveguide system) on the parameter $\eta z$, i.e., Eqs. (B2a) and (B2b) themselves do not contain a derivative with respect to the variable $z$.

By analogy with [40], it is convenient to determine the wave number $h_m$ by using the "local" dispersion equation in the form $\lambda_m(\omega_m,h_m,\eta z) = 0$, where $\lambda_m$ is the eigenvalue of the operator $\hat{D}_m\boldsymbol{\Psi}_m = \boldsymbol{\Psi}_m\lambda_m$ corresponding to the given type of the waveguide eigenmode. The transverse structure of the mode corresponds to the vector eigenfunction $\boldsymbol{\Psi}_m$ of the operator $\hat{D}_m$ when the eigenvalue tends to zero, i.e., $\lambda_m \rightarrow 0$.

Then, generalizing the results of [40], one can obtain the following expression for the energy flux $S_m$ along the $z$ axis for the wave mode with number $n$:

$$S_m = s_m|A_m|^2, \quad s_m = \frac{c^2}{16\pi\omega_m}\left(\frac{\partial \mathrm{Re}\lambda_m}{\partial h_m}\right)\Bigg|_{\mathrm{Im}h_m\rightarrow 0}, \quad \text{(B3a)}$$

where

$$\lambda_m = \iint_\infty \boldsymbol{\Psi}_m^{*}\hat{D}_m\boldsymbol{\Psi}_m dxdy, \quad \text{(B3b)}$$

using the standard normalization $\iint_\infty \boldsymbol{\Psi}_m^{*}\boldsymbol{\Psi}_m dxdy = 1$. Taking into account the nonlinear term $\propto \delta\boldsymbol{P}_m$ in the wave equation (B1b), one can, by analogy with [41], obtain the following equations of the first order for complex amplitudes $A_m$ with accuracy up to quadratic terms with respect to the small parameter $\eta$:

$$s_m \frac{dA_m}{dz} + \frac{A_m}{2}\frac{ds_m}{dz} + \mu_m A_m = i\frac{\omega_m}{4}\left(-i\int_{-\infty}^{z} h_m dz\right)\iint_\infty (\delta \boldsymbol{P}_m \boldsymbol{\Psi}_m^*)dxdy, \quad \text{(B4)}$$

where

$$\mu_m = \frac{\omega_m}{4i}\mathrm{Im}\iint_\infty \boldsymbol{\Psi}_m^* \hat{\varepsilon}_m^{(aH)}\boldsymbol{\Psi}_m dxdy, \quad \text{(B5)}$$

and $\hat{\varepsilon}_m^{(aH)}$ is the anti-Hermitian component of the tensor $\hat{\varepsilon}_m$. At the limit $\delta \boldsymbol{P}_m \to 0$, from Eqs. (B4) and (B5) a standard equation for the energy flux follows: $\partial S_m/\partial z + 2\mu_m |A_m|^2 = 0$.

We now assume that the wave frequencies are related to the synchronism condition of Eq. (10). We then use the relationships in Eqs. (A4), (A5a), and (A5b) for the nonlinear polarization $\delta \boldsymbol{P}_m$, neglecting the anti-Hermitian components of the polarizability tensor $\chi_{m;ps}$. As a result, we transform Eq. (B4) to the following form:

$$s_m \frac{dA_m}{dz} + \frac{A_m}{2}\frac{ds_m}{dz} + \mu_m A_m - i\sum_{n=1}^{M} \upsilon_{mn}|A_n|^2 A_m = i\omega_m N_m W_m e^{-i\sigma_m\theta}(A_m^*)^{N_m-1}\prod_{n\neq m}^{M}(\hat{\Theta}_{mn}A_n)^{N_n}, \quad \text{(B6)}$$

where

$$\upsilon_{n\neq m} = \iint_\infty \alpha_s^{(3)}|\chi_{n;sk}\Psi_{n;k}|^2|\chi_{m;sr}\Psi_{m;r}|^2 dxdy,$$

$$\upsilon_{mm} = 2\iint_\infty \alpha_s^{(3)}|\chi_{m;sk}\Psi_{m;k}|^4 dxdy, \quad \text{(B7a)}$$

$$W_m = \frac{q!}{2^{q+2}\prod_{n=1}^{M} N_n} \times \left(\iint_\infty \alpha_s^{(q)}\prod_{n=1}^{M}(\hat{\Theta}_{nm}\chi_{n;sk}\Psi_{n;k})^{N_n}dxdy\right), \quad \text{(B7b)}$$

$$\theta = \int_{-\infty}^{z}\sum_{m=1}^{M}\sigma_m N_m h_m dz, \quad \text{(B7c)}$$

and the operator $\hat{\Theta}_{nm}$ was introduced earlier in Sec. III. We now take into account that for the effective parametric interaction, besides the temporal synchronism condition [Eq. (10)] for the wave frequencies, the corresponding condition of spatial synchronism for the wave vectors should be fulfilled (see [20]) at least at some point $z = z_0$

$$\sum_{m=1}^{M}\sigma_m N_m h_m(z_0) = 0. \quad \text{(B8)}$$

In this case, from Eqs. (B7a)–(B7c) and (B8) after the replacement of variables and parameters,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}, \quad c_m e^{i\delta_m z} = \sqrt{s_{m;z}}A_m, \quad \delta_m = (h - h_m(z_0)),$$

$$\gamma_{mn} = \frac{\upsilon_{mn}}{s_{n;z}s_{m;z}}, \quad \kappa_m = \frac{\mu_m}{\sqrt{s_{m;z}}}, \quad w_1 = \frac{W_1}{\prod_{n=1}^{M}\sqrt{s_{n;z}^{N_n}}},$$

Eq. (12) rigorously follows under the condition in Eq. (13).

A similar problem for a plane-layered medium is obviously a particular case of the above-presented analysis (see [41] in this connection).


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