Moving nonradiating kinks in nonlocal ϕ^4 and ϕ^4-ϕ^6 models

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We explore the existence of moving nonradiating kinks in nonlocal generalizations of ϕ^4 and ϕ^4-ϕ^6 models. These models are described by nonlocal nonlinear Klein-Gordon equation, $u_{tt} - \mathcal{L}u + F(u) = 0$, where $\mathcal{L}$ is a Fourier multiplier operator of a specific form and $F(u)$ includes either just a cubic term (ϕ^4 case) or cubic and quintic (ϕ^4-ϕ^6 case) terms. The general mechanism responsible for the discretization of kink velocities in the nonlocal model is discussed. We report numerical results obtained for these models. It is shown that, contrary to the traditional ϕ^4 model, the nonlocal ϕ^4 model does not admit moving nonradiating kinks but admits solitary waves that do not exist in the local model. At the same time the nonlocal ϕ^4-ϕ^6 model describes moving nonradiating kinks. The set of velocities allowed for these kinks is discrete with the highest possible velocity c_1. This set of velocities is unambiguously determined by the parameters of the model. Numerical simulations show that a kink launched at the velocity c higher than c_1 starts to decelerate, and its velocity settles down to the highest value of the discrete spectrum c_1.

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I. INTRODUCTION

The nonlocal nonlinear Klein-Gordon equation

$$u_{tt} - \mathcal{L}u + F(u) = 0, \quad F(u) = U'(u), \tag{1}$$

where $\mathcal{L}$ is a Fourier multiplier operator, $\widehat{\mathcal{L}u}(\omega) = l(\omega)\widehat{u}(\omega)$, arises in many physical applications as a generalization of the traditional (local) nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + F(u) = 0, \quad F(u) = U'(u). \tag{2}$$

In particular, Eq. (1) appears in Josephson electrodynamics [1–6], in models of ferromagnets [7] and in lattice models [8–10]. Nonlocality in Eq. (1) can appear due to various physical reasons such as specific spatial dispersion of material, long-range interparticle interactions, or due to complex geometry of the model. The examples for the nonlinear term $F(u)$ and the corresponding potential $U(u)$ are

$$F(u) = -\sin\pi u, \quad U(u) = \frac{1}{\pi}(\cos\pi u - 1) \tag{3}$$

(nonlocal sine-Gordon model),

$$F(u) = -u + u^3, \quad U(u) = \tfrac{1}{4}u^4 - \tfrac{1}{2}u^2 + \tfrac{1}{4} \tag{4}$$

(nonlocal ϕ^4 model), or

$$F(u) = -u(1-u^2)(1+\alpha u^2),$$
$$U(u) = \frac{3+\alpha}{12} - \frac{1}{2}u^2 + \frac{1-\alpha}{4}u^4 + \frac{\alpha}{6}u^6 \tag{5}$$

(nonlocal $\phi^4 - \phi^6$ model). In all these models, except (5) for $\alpha < 0$, the potentials $U(u)$ are bounded from below, so temporal evolution governed by Eq. (1) should not expose a blow-up phenomenon. In the local case all the potentials (3)–(5) allow for bistable equilibria $u = \pm 1$ as well as waves of switching between these equilibria (kinks or antikinks).

In the context of *Josephson electrodynamics* Eq. (1) comes with the sine nonlinearity (3), and the operator $\mathcal{L}$ has the form

$$\mathcal{L}u = \frac{d}{dx}\int_{-\infty}^{\infty} G(x-x')u_{x'}(x')\,dx', \tag{6}$$

where the kernel $G(\xi)$ is determined by the dispersive properties of superconductor leads and the geometry of the Josephson junction (a list of the examples of possible kernels can be found in Ref. [6]). It has been found that this nonlocal model has new types of propagating nonradiating excitations. They are, for instance, kinks and antikinks with topological charge greater then one (e.g., 4π- and 6π-kinks). The velocities of these kinks belong to the discrete spectrum of allowed values; otherwise the kinks begin to radiate and lose energy. In the context of nonlocal Josephson electrodynamics the radiation of moving kinks has been discussed in various papers [11–14].

In the context of *lattice models* Eq. (1) arises as a continuous approximation describing the dynamics of a chain of particles placed in the external potential $U(u)$. The nonlocal term in this case also can be written in the form (6), where the kernel $G(\xi)$ accounts for the interactions between the particles in the chain. The kernel function can decay by a power law [10] or exponentially. The latter case includes lattice models with the so-called Kac-Baker interaction potential, $V_{ij} \sim \exp(-\gamma|i-j|)$, $\gamma > 0$, where i and j are the numbers of particles in the chain. The models of this kind have been discussed in many papers since the early 1980s [8,15–19]. The studies have addressed the properties of kinklike excitations in the presence of long-range interactions, the estimation of Peierls-Nabarro barrier, and thermodynamics of the system. The case of cubic nonlinearity (4) has been studied in papers of the Woafo group [16–18]. The lattice kink was approximated by its continuous counterpart subjected to a "dressing" procedure. It was concluded that the Peierls-Nabarro barrier vanishes in the limit $\gamma \to 0$. However, it was noted later (see Ref. [8]) that the results of Woafo and co-authors are valid in the limit of small kink velocities only, because their study did not take into account the higher derivative term in the continuous approximation. This is the term that is responsible for the radiation of waves by moving kinks that causes kink deceleration.

In fact, the phenomenon of kink radiation and deceleration due to nonlocality in the dispersive term (6) does take place in many situations. However, it may be promising to consider this issue from the viewpoint of *the principle of kink* *velocity discretization* in nonlocal models, which is, in our opinion, of a quite general nature. It implies that switching on the nonlocality results in the discretization of allowed kink velocities. In some cases the spectrum of the allowed velocities consists of zero velocity only, and therefore all moving kinklike excitations lose energy through radiation and eventually stop. This takes place in a nonlocal ϕ^4 model; see Sec. III. However, there are cases when this spectrum includes nonzero kink velocities as this takes place in the case of the nonlocal sine-Gordon equation (see Refs. [20,21]) or in a ϕ^4-ϕ^6 model (see Sec. IV). Here we should remind readers that the discrete nonlinear Klein-Gordon model corresponding to

$$\mathcal{L}u = \alpha[u(x+h) - 2u(x) + u(x-h)], \quad \alpha > 0$$

can also be rewritten in the form (6) with "triangular" kernel (see Ref. [20]). Therefore the discretization of 4π-kink velocities in the Frenkel-Kontorova model discovered in Ref. [22] and studied in detail in Ref. [23] also agrees with this general principle.

The phenomenon of kink velocity discretization is closely related to the existence of a branch of "slow" phonon velocities, $|c| < 1$. This branch corresponds to real roots of the dispersion relation

$$-c^2\omega^2 + U''(0) + \omega^2\tilde{G}(\omega) = 0.$$

Here c is the velocity of a linear wave, and $\tilde{G}(\omega)$ is the Fourier transform of $G(\xi)$. Contrary to the local case, Eq. (1) can admit "slow" phonon linear waves, and the velocity of the kink can fall into the spectrum of their velocities. It is known that in this case bound states of a nonlinear wave (the kink) and linear oscillations can appear. These states ("delocalized solitons" or "nanopterons"; see Ref. [24]) do not satisfy the condition of exact localization in space. This condition can be restored for some values of the velocity when the oscillatory component vanishes. Nonlinear waves with the velocity "embedded" in a continuous spectrum of phonon velocities has been discussed in various contexts such as optical and hydrodynamical ones. At the end of the 1990s the examples of such waves were generalized, and the concept of an "embedded soliton" was suggested [25–27]. In these terms the main result of our paper is as follows: *When switching from local to nonlocal Klein-Gordon equation the kinks acquire features of embedded solitons*.

Detailed study of the kink velocity spectrum for the particular example of Eq. (1) with the sine nonlinearity was reported in Refs. [20,21]. In these papers the explanation of kink velocity discretization was based on the ideas of the theory of dynamical systems. In the present paper we apply the same approach for the case of a more general nonlinearity. We admit that the following assumptions about the potential $U(u)$ hold:

$$(NL0): \quad U(u) \in C^2(\mathbb{R}),$$
$$(NL1): \quad U(u) = U(-u),$$
$$(NL2): \quad U'(1) = U'(-1) = 0,$$
$$(NL3): U(u) \geqslant 0 \ \text{ for } \ u \in \mathbb{R} \ \text{ and } \ U(1) = U(-1) = 0,$$
$$(NL4): \quad U''(\pm 1) \neq 0.$$

Prototypical examples of $U(u)$ are the nonlinearities (3)–(5). The kink solutions of Eq. (1) satisfy the boundary conditions

$$\lim_{x\to-\infty} u(t,x) = -1, \quad \lim_{x\to\infty} u(t,x) = 1, \tag{7}$$

while for the antikinks the boundary conditions are

$$\lim_{x\to-\infty} u(t,x) = 1, \quad \lim_{x\to\infty} u(t,x) = -1.$$

In this paper we restrict our analysis to the kernel corresponding to Kac-Baker nonlocal interactions

$$G_\lambda(\xi) = \frac{1}{2\lambda}\exp\left(-\frac{|\xi|}{\lambda}\right). \tag{8}$$

The parameter λ measures the "strength" of the nonlocality. The kernel $G_\lambda(\xi)$ is normalized in such a way that $G_\lambda(\xi) \to \delta(\xi)$ as $\lambda \to 0$, therefore at $\lambda = 0$ Eq. (1) degenerates into Eq. (2). The traveling waves of constant profile $u(t,x) = u(x - ct)$ satisfy the equation

$$c^2u_{zz} + F(u) = \frac{d}{dz}\int_{-\infty}^{\infty} e^{-\frac{|z-z'|}{\lambda}} u_{z'}(z')\,dz'. \tag{9}$$

Generalization of these results for a nonlocality of a more general kind can be done using the approach developed for the nonlocal sine-Gordon equation in Ref. [20].

The paper is organized as follows. In Sec. II we discuss the general mechanism for kink velocity discretization for the case when the assumptions (NL0)–(NL4) hold. Also in Sec. II we describe a numerical method to seek kink velocities that is consistent with this mechanism. Section III is devoted to the nonlocal ϕ^4 model. We argue that nonradiating moving kinks do not exist in this case, and we present a family of new solitary wave solutions with a continuous spectrum of velocities. Section IV is devoted to the nonlocal ϕ^4-ϕ^6 model. We show that in this case nonradiating moving kinks do exist, and the set of their velocities is discrete. Section V contains summary and discussion.

II. KINK SOLUTIONS FOR EQ. (1): GENERAL CASE

A. The local case

In the local limit $\lambda = 0$ the traveling wave solutions satisfy the equation

$$(1 - c^2)u_{zz} = F(u). \tag{10}$$

All types of solutions for Eq. (10) can be easily described by means of the phase plane analysis. One can show that the assumptions (NL0)–(NL4) ensure the existence of *unique kink* and *unique antikink* solutions for any velocity $c^2 < 1$. The implicit form of kink-antikink solutions is

$$\pm\sqrt{\frac{1-c^2}{2}}(t - t_0) = \int_0^u \frac{d\zeta}{\sqrt{U(\zeta)}}. \tag{11}$$

The signs "+" and "−" correspond to the kink and antikink, respectively. The existence of the kink-antikink follows from the fact that under the conditions (NL0)–(NL4) the integral in right-hand side of (11) converges for $|u| < 1$ and diverges for $u = \pm 1$.

B. The nonlocal case: Reduction to the system of ordinary differential equations

If $\lambda \neq 0$, then the analysis of Eq. (9) can be done using the following trick [20,21]. Let us introduce an auxiliary function

$$q(z) = \frac{1}{2\lambda}\int_{-\infty}^{\infty} \exp\left(-\frac{|z-z'|}{\lambda}\right) u_{z'}(\xi)\, dz'. \tag{12}$$

Then $q(z)$ satisfies the equation

$$-\lambda^2 q_{zz} + q = u_z.$$

Therefore for a certain class of solutions, Eq. (9) can be replaced with the system

$$c^2 u_{zz} + F(u) = q_z, \tag{13}$$

$$-\lambda^2 q_{zz} + q = u_z. \tag{14}$$

Evidently, if $u(z)$ is a solution of Eq. (9), then $[u(z),u_z(z),q(z),q_z(z)]$, where $q(z)$ is given by (12), satisfies (13)–(14). The converse, in general, does not hold: The general solution of (14) is

$$q(z) = \frac{1}{2\lambda}\int_{-\infty}^{+\infty} \exp\left(-\frac{|z-z'|}{\lambda}\right) u_{z'}(z')\, dz' + C_+ e^{z/\lambda} + C_- e^{-z/\lambda} \tag{15}$$

and contains two arbitrary constants C_+ and C_-. If both of them do not vanish, then the solution of (13)–(14) has no relation to (9). One can prove complete equivalence of (13)–(14) and (9) in the case of a bounded nonlinearity [28]. However, in general, the equivalence of (13)–(14) and (9) can be achieved for some class of solutions only. Obviously, the equivalence takes place for the class of *bounded* solutions, which includes the kinks and the antikinks.

The cases $c = 0$ and $c \neq 0$ for the system (13)–(14) should be treated separately.

C. The nonlocal case: Resting kinks, $c = 0$

In the case $c = 0$ the system (13)–(14) can be reduced to the equation

$$u_z^2 = \frac{C + \lambda^2 F^2(u) + U(u)}{[1 + \lambda^2 F'(u)]^2}, \tag{16}$$

where C is an arbitrary constant. The kink and antikink solutions correspond to the case $C = 0$. Simple phase plane arguments allow us to conclude that under the assumptions (NL0)–(NL4) the kink-antikink solution exists and is unique if λ obeys the inequality

$$\lambda^2 < -\frac{1}{\min\limits_{[-1;1]} F'(u)}. \tag{17}$$

D. The nonlocal case: Moving kinks, $c \neq 0$

Let us now assume that $c \neq 0$, $\lambda \neq 0$ and $U(u)$ satisfies the assumptions (NL0)–(NL4). Then we can introduce the phase space (u,u',q,q'), where $u' = u_z$, $q' = q_z$. In this phase space Eqs. (13)–(14) determine a dynamical system. The first integral of (13)–(14) is

$$I = \frac{c^2}{2}u_z^2 + \frac{\lambda^2}{2}q_z^2 - \frac{1}{2}q^2 + U(u). \tag{18}$$

Equilibrium points of this dynamical systems include the points $O_{-1}(u = -1, u' = q = q' = 0)$, $O_1(u = 1, u' = q = q' = 0)$, which are situated in zero level of the first integral and the point $O_0(u = u' = q = q' = 0)$. Direct calculation of eigenvalues for the equilibrium points $O_{\pm 1}$ results in the following assertion:

Statement 1. The equilibrium points $O_{\pm 1}$ are the saddle-centers.

It follows from Statement 1 that both the points O_1 and O_{-1} have a pair of incoming trajectories, $\gamma_{1,2}^-(O_{\pm 1})$, and a pair of outgoing trajectories, $\gamma_{1,2}^+(O_{\pm 1})$. All of them are situated in the zero level of the first integral I. Merging one of the outgoing trajectory, $\gamma^+(O_{-1})$, and one of the incoming trajectory, $\gamma^-(O_1)$ [or vice versa, $\gamma^+(O_1)$ and $\gamma^-(O_{-1})$] yields heteroclinic trajectories connecting the equilibrium points O_{-1} and O_1. These trajectories correspond to the kink or antikink solutions. We call these heteroclinic connections $(-1 \leftrightarrow 1)$ *trajectories*.

The following statements are valid:

Statement 2. On the trajectories $\gamma^-(O_{\pm 1})$ and $\gamma^+(O_{\pm 1})$ the component q does not change sign.

Indeed, since the trajectories $\gamma^{\pm}(O_{\pm 1})$ are situated in the zero level of the first integral I, it follows from (18) that if at some point z_0 the component $q(z)$ vanishes, then $F[u(z_0)] = u'(z_0) = q'(z_0) = q(z_0) = 0$. These relations cannot take place at any point of $\gamma^{\pm}(O_{\pm 1})$.

Statement 3. There are no $(-1 \leftrightarrow 1)$-trajectories for $c^2 > 1$.

Indeed, if $[u(z),u'(z),q(z),q'(z)]$ is a $(-1 \leftrightarrow 1)$ trajectory, then there exist $z = z_0$, such that $q_{zz}(z_0) = 0$. It follows from Eq. (14) that $q(z_0) = u_z(z_0)$. If $c^2 > 1$ then Eq. (18) yields

$$\frac{1}{2}(c^2 - 1)u_z^2(z_0) + \frac{\lambda^2}{2}q_z^2(z_0) + U[u(z_0)] = 0.$$

This means that if $c^2 > 1$, then $U[u(z_0)] = 0$ and $u'(z_0) = q'(z_0) = q(z_0) = 0$. These relations cannot take place at any point of the $(-1 \leftrightarrow 1)$ trajectory.

Statement 4. The plane $P(u = 0, q' = 0)$ is the symmetry plane for $(-1 \leftrightarrow 1)$ trajectories in the sense that:

(a) *All $(-1 \leftrightarrow 1)$ trajectories pass through the plane $P(u = 0, q' = 0)$.*

(b) *If an outgoing trajectory of the equilibrium point $O_{\pm 1}$ crosses the plane P, it is a $(-1 \leftrightarrow 1)$ trajectory.*

In order to prove Statement 4 let us note the following. Since the system (13)–(14) is invertible, if

$$\gamma_1^+(O_{-1}) = [u_1(z),u_1'(z),q_1(z),q_1'(z)],$$
$$\gamma_2^+(O_{-1}) = [u_2(z),u_2'(z),q_2(z),q_2'(z)]$$

are the outgoing trajectories of O_{-1} then the incoming trajectories of O_{-1} are

$$\gamma_1^-(O_{-1}) = [u_1(-z), -u_1'(-z), -q_1(-z), q_1'(-z)],$$
$$\gamma_2^-(O_{-1}) = [u_2(-z), -u_2'(-z), -q_2(-z), q_2'(-z)].$$

Due to Statement 2, the q component conserves sign at $\gamma_{1,2}^{\pm}(O_{-1})$. At one of the outgoing trajectories [$\gamma_1^+(O_{-1})$, for instance] the q component is positive, and at another, $\gamma_2^+(O_{-1})$, it is negative. Then q component is positive also at $\gamma_2^-(O_{-1})$

and negative at $\gamma_1^-(O_{-1})$. By virtue of the symmetry $u \to -u$ the incoming and outgoing trajectories of O_1 are

$$\gamma_1^+(O_1) = [-u_1(z), -u_1'(z), -q_1(z), -q_1'(z)],$$
$$\gamma_2^+(O_1) = [-u_2(z), -u_2'(z), -q_2(z), -q_2'(z)],$$
$$\gamma_1^-(O_1) = [-u_1(-z), u_1'(-z), q_1(-z), -q_1'(-z)],$$
$$\gamma_2^-(O_1) = [-u_2(-z), u_2'(-z), q_2(-z), -q_2'(-z)].$$

The q component is positive at $\gamma_1^-(O_1)$ and $\gamma_2^+(O_1)$ but negative at $\gamma_2^-(O_1)$. Therefore $(-1 \leftrightarrow 1)$ trajectory can arise if $\gamma_k^+(O_{-1})$ merges with $\gamma_k^-(O_1)$, or $\gamma_k^-(O_{-1})$ merges with $\gamma_k^+(O_1)$, $k = 1,2$. In both these cases at some $z = z_0$ the following equalities hold:

$$u(z_0 + z) = -u(z_0 - z), \quad q(z_0 + z) = q(z_0 - z),$$
$$u'(z_0 + z) = u'(z_0 - z), \quad q'(z_0 + z) = -q'(z_0 - z). \quad (19)$$

Formulas (19) imply the point (a) of Statement 4. Due to the symmetry relation between $\gamma_k^{\pm}(O_{-1})$ and $\gamma_k^{\mp}(O_1)$, $k = 1,2$, if $\gamma_k^{\pm}(O_{-1})$ crosses the plane P, $\gamma_k^{\mp}(O_1)$ also crosses P in the same point, and they merge. This proves the point (b).

An immediate corollary of Statement 4 is that *all kink solutions of Eq. (9) are odd, up to a shift with respect to variable z*. An intersection of one-dimensional (a trajectory) and two-dimensional (2D) (a plane) manifolds in four-dimensional phase space is a situation of co-dimension one. Therefore, Statement 4 implies that the merging of incoming and outgoing trajectories corresponds to a situation of a codimensional one. If λ is fixed, then the merging of these trajectories can take place for selected values of c only. Statement 4 allows also offering *a strategy* of numerical search of $(-1 \leftrightarrow 1)$-trajectories. The idea consists of numerical tracing of the outgoing trajectory $\gamma^+(O_{-1})$ seeking intersections of this trajectory with the plane $P(u = 0, q' = 0)$. Practical implementation of this is as follows. Let $\gamma^+(O_{-1}) = [u(z), u_z(z), q(z), q_z(z)]$ and z_n, $n = 1,2,\ldots$, be the values such that $u(z_n) = 0$. It is obvious that z_n, $n = 1,2,\ldots$ depend on c and λ. Let us now introduce the functions

$$Q_n(c,\lambda) \equiv q_z(z_n; c,\lambda), \quad n = 1,2,\ldots .$$

Let λ to be fixed. Statement 4 implies that the velocities c of moving kinks for Eq. (9) are the zeros of the functions $Q_n(c,\lambda)$, $n = 1,2,\ldots$.

In some cases the following statement allows us to restrict the analysis to studying the function $Q_1(c,\lambda)$ only:

Statement 5. Let $\gamma^+(O_{-1}) = [u(z), u_z(z), q(z), q_z(z)]$ be an outgoing trajectory of O_{-1} and z_n, $n = 1,2,\ldots$, are such that $u(z_n) = 0$. Let

$$z_1 < z_2 < \cdots < z_n < \cdots .$$

Then if $q(z)$ on $\gamma^+(O_{-1})$ is positive, then one has

$$q_z(z_1) < q_z(z_2) < \cdots < q_z(z_n) < \cdots ,$$

and if it is negative, then

$$q_z(z_1) > q_z(z_2) > \cdots > q_z(z_n) > \cdots .$$

Statement 5 follows immediately from Statement 2 and Eq. (14). Statement 5 implies that if $Q_1(c,\lambda)$ has no zeros and its sign coincides with the sign of $q(z)$ on $\gamma^+(O_{-1})$, then $Q_n(c,\lambda)$, $n > 1$, also has no zeros.

Now let us apply the above results for the study of the kinks in the ϕ^4 and the ϕ^4-ϕ^6 models.

III. KINKS AND SOLITARY WAVES IN NONLOCAL ϕ^4 MODEL

Travelling waves in nonlocal ϕ^4 model with the kernel (8) satisfy the equation

$$c^2 u_{zz} - u + u^3 = \frac{\partial}{\partial z} \int_{-\infty}^{\infty} e^{-\frac{|z-z'|}{\lambda}} u_{z'}(z')\, dz'. \quad (20)$$

A corresponding system of equations (13)–(14) reads

$$c^2 u_{zz} - u + u^3 = q_z, \quad (21)$$
$$-\lambda^2 q_{zz} + q = u_z. \quad (22)$$

There is no complete equivalence between (21)–(22) and (20) (see Ref. [28]); however, they are equivalent in the class of bounded solutions.

A. Kinks

a. Local limit: In the local limit, $\lambda = 0$, Eq. (20) transforms into the equation

$$(1 - c^2) u_{zz} + u - u^3 = 0. \quad (23)$$

It has a unique kink (antikink) solution

$$u(z) = \pm \tanh\left[\frac{z - z_0}{\sqrt{2(1 - c^2)}}\right]. \quad (24)$$

The sign "+" in Eq. (24) corresponds to the kink and "−" to the antikink, and z_0 is an arbitrary parameter. The kink (antikink) (24) can propagate at any velocity $c^2 < 1$.

b. Nonlocal case, Resting kink: In the case of $c = 0$ the unique kink (antikink) solution exists for $0 \leqslant \lambda < 1$. It can be written in the implicit form

$$\pm \frac{z - z_0}{\sqrt{2}} = \varkappa \ln\left(\frac{\varkappa u + \sqrt{1 + 2\lambda^2 u^2}}{\sqrt{1 - u^2}}\right) - \frac{3\sqrt{2}\lambda}{2} \ln(\sqrt{2}\lambda u + \sqrt{1 + 2\lambda^2 u^2}), \quad (25)$$

where $\varkappa = \sqrt{1 + 2\lambda^2}$. The plus sign in Eq. (25) corresponds to the kink, and minus to the antikink. The solution (25) agrees with formula (23) of Ref. [15] when $\sigma = \lambda^2/(1 - \lambda^2)$ and $v = 0$.

c. Nonlocal case, No nonradiating kinks: To study the existence of moving kink solutions we applied the method described in Sec. II D. Let λ be fixed. Then the spectrum of kink velocities is given by the zeros of functions $Q_n(c,\lambda)$, $n = 1,2,\ldots$. Due to Statement 3 the functions $Q_n(c,\lambda)$, $n = 1,2,\ldots$ need to be computed for $c < 1$ only.

Our conclusion is that there is *no kink-antikink solution of Eq. (20) for all $\lambda > 0$ and $c \neq 0$*. Let us illustrate the typical situation by the case with $\lambda = 0.3$. The plot of the function $Q_1(c,\lambda)$ is depicted in Fig. 1(A). The function $Q_1(c,\lambda)$ is positive for $0.3 < c < 1$. However, the region of small c needs more delicate treatment, since the values of $Q_1(c,\lambda)$ in this

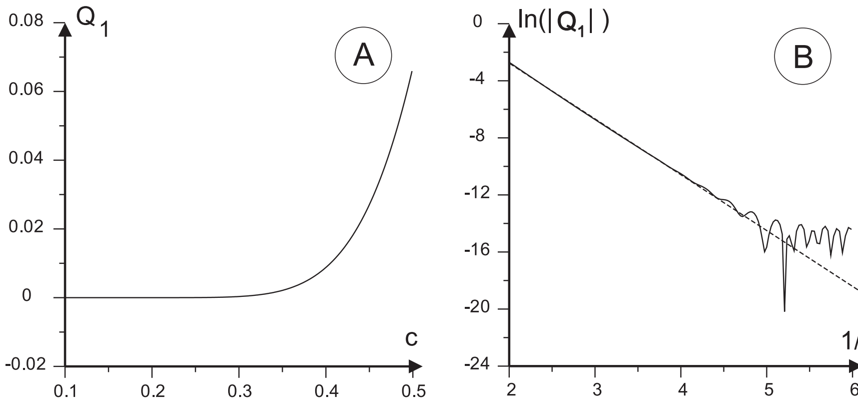

FIG. 1. (A) The dependence of the value $Q_1 \equiv q_z|_{u=0}$ on c; (B) the dependence of $\ln|Q_1(c,\lambda)|$ on $1/c$; for both cases $\lambda = 0.3$.

region are tiny. Figure 1(B) shows the plot of $\ln|Q_1(c,\lambda)|$ as a function of $1/c$ under the same value $\lambda = 0.3$. It follows from Fig. 1(B) that for $Q_1 > 10^{-5}$ (the controlled accuracy of the computations), the plot can be fitted well by a straight line. This allows us to assume for $c \ll 1$ the following dependence:

$$Q_1(c,\lambda) \sim C_1 e^{-C_2/c}, \quad C_2 > 0, \tag{26}$$

which is typical of separatrix splitting phenomena. Asymptotic dependence (26) implies that $Q_1(c,\lambda) > 0$ for small c too.

Summarizing, the hypothesis that $Q_1(c,\lambda)$ is positive for $0 < c < 1$, $\lambda = 0.3$ has been supported by numerical investigations. It follows from Statement 5 that all $Q_n(c,\lambda)$, $n = 2,3,\ldots$ as functions of c also have no zeros if $\lambda = 0.3$. So we can conclude that nonlocal ϕ^4 model does not admit the kink-antikink traveling waves of constant profile for $\lambda = 0.3$. A similar situation takes place for other values of $\lambda > 0$, which we have considered.

d. Propagation of kinklike excitation, Numerical study: In the paragraph above the problem has been studied from the mathematical point of view assuming the *exact* conservation of the traveling kink shape and the absence of radiation. From a physical point of view this condition is too restrictive. One cannot exclude the existence of weakly radiating kinklike excitations with a huge lifetime that can cover long distances. Therefore the question arises: How the dynamics governed by the nonlocal ϕ^4 model "feels" that the nonradiating moving kinks are not possible.

Figure 2 shows the results of the following numerical experiment. The profiles of the resting kink were calculated using the formula (25) for different values of λ and furnished with moderate velocity $c = 0.2$. The curves in Fig. 2 show the motion of the kink center, the point where $u = 0$. It follows from Fig. 2 that for $\lambda \lesssim 0.6$ the kink propagates nearly uniformly, and its velocity is a little smaller than the initial velocity $c = 0.2$. At greater values of λ the kink decelerates, and its motion may become unpredictable. So one can conclude that under some restrictions the kinks are movable entities also within the framework of the nonlocal ϕ^4 model. However, strong nonlocality suppresses the kink mobility.

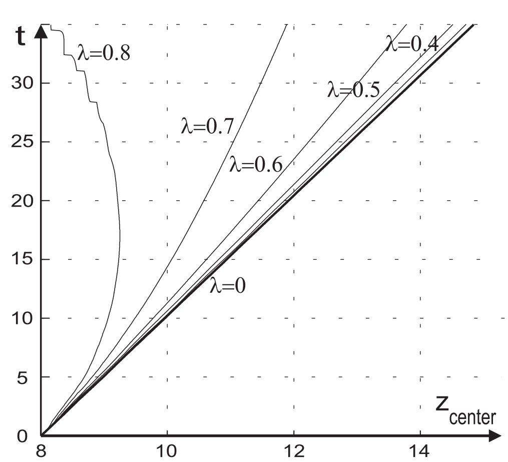

FIG. 2. The propagation of kinklike excitations in nonlocal ϕ^4 model. A resting kink calculated using formula (25) for various values of λ is furnished with moderate velocity $c = 0.2$. The curves are the motion of the kink front center, i.e., the point where $u = 0$.

B. Solitary waves for nonlocal ϕ^4 model: Continuous spectrum of velocities

Apart from kinks, the nonlocal ϕ^4 model also admits nonlinear excitations in the form of solitary waves. The solutions obeys the boundary conditions

$$\lim_{z\to-\infty} u(z) = 0, \quad \lim_{z\to+\infty} u(z) = 0$$

and disappears when passing to the local limit $\lambda \to 0$. In terms of the dynamical system (21)–(22) they correspond to homoclinic loops of equilibrium point O_0. Even solitary waves correspond to symmetric homoclinic loops, which are invariant with respect to transformation $u' \to -u'$, $q \to -q$. Eigenvalues of O_0, $\Lambda_{1,2,3,4}$, satisfy the equation

$$c^2\lambda^2\Lambda^4 - (c^2 + \lambda^2 - 1)\Lambda^2 + 1 = 0. \tag{27}$$

Therefore, O_0 may be an elliptic point (two pairs of pure imaginary eigenvalues $\Lambda_1 = \overline{\Lambda_2}$, $\Lambda_3 = \overline{\Lambda_4}$), saddle (two pair of real eigenvalues, $\Lambda_1 = -\Lambda_2$, $\Lambda_1 = -\Lambda_2$), or saddle-focus (complex quadruple, $\Lambda_1 = -\Lambda_2 = \overline{\Lambda_3} = -\overline{\Lambda_4}$). The type of the point O_0 depends on the parameters c and λ. The left panel of Fig. 3 shows corresponding areas in the parameter plane (λ,c).

If O_0 is a saddle or a saddle-focus equilibrium point, then there exist 2D stable $W^s(O_0)$ and 2D unstable $W^u(O_0)$ manifolds of O_0. The homoclinic loop of O_0 lies in the intersection $W^u(O_0) \cap W^s(O_0)$. Symmetric homoclinic loops pass through the plane $R(u' = q = 0)$. Therefore, a strategy for numerical search for these loops consists in tracing $W^u(O_0)$ until its intersection with the plane R. This intersection is a phenomenon of codimension zero. Then the studies

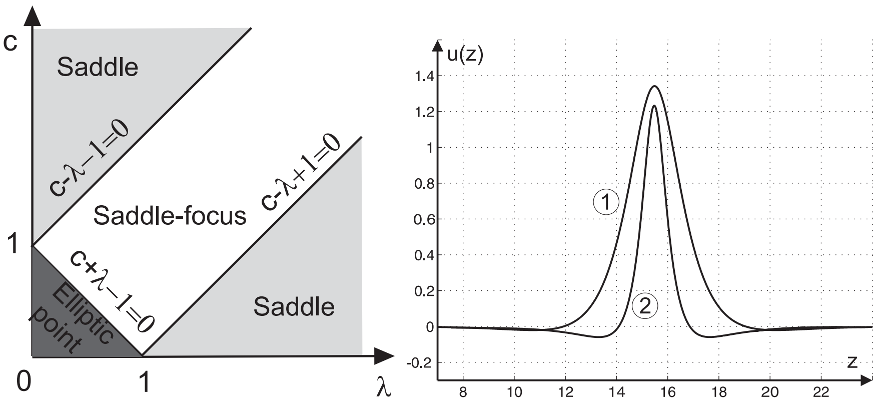

FIG. 3. Left panel: Areas in the parameters plane (λ,c) where equilibrium point O_0 is elliptic point, saddle, and saddle-focus correspondingly. Right panel: The profiles of the solitary waves for $\lambda = 1.5$ and $c = 0.35$ (plot 1, saddle case), and $c = 0.9$ (plot 2, saddle-focus case).

of saddle and saddle-focus cases differ only in suitable parametrization of $W^u(O_0)$ in the vicinity of the equilibrium point O_0.

Numerical investigation shows the presence of solitary wave solutions in both saddle and saddle-focus cases. Two of them are depicted in the right panel of Fig. 3.

Here we would like to make the following comments:

(1) The spectrum of the solitary wave velocities is *continuous*. This is a consequence of the fact that the intersection of $W^u(O_0)$ and R is a phenomenon of codimension zero; i.e., no additional parameters are needed to ensure this intersection.

(2) Contrary to the kink velocities, the velocities of the solitary waves can be greater then 1.

(3) The asymptotics of the solitary wave "wings" is monotonic when O_0 is a saddle, and it has damped oscillations when O_0 is a saddle-focus.

(4) The solitary waves are *unstable*. This fact is caused by the instability of the zero equilibrium state for Eq. (1).

IV. KINKS IN THE ϕ^4-ϕ^6 MODEL

Let us now discuss the results for the case of nonlinearity (5). The system of equations (13)–(14) in this case reads

$$c^2 u_{zz} - u(1-u^2)(1+\alpha u^2) = q_z, \quad (28)$$

$$-\lambda^2 q_{zz} + q = u_z, \quad z = x - ct. \quad (29)$$

A. Existence of nonradiating kinks

Let $\lambda > 0$. Numerical results for kink solutions in this case can be outlined as follows. For *the resting kinks* the two cases should be separated:

Case $\alpha < 0$. According to (17) a resting kink exists for $0 < \lambda < 1$ if $-1.5 < \alpha < 0$ and $0 < \lambda < \sqrt{-1/(2+2\alpha)}$ if $\alpha < -1.5$.

At the same time for $\alpha < 0$ the spectrum of allowed kink velocities consists of *zero velocity only*; i.e., the resting kink is the unique kink solution. The behavior of $Q_1(c,\lambda)$ in this case is similar to the case of the ϕ^4 model.

Case $\alpha > 0$. According to (17) a resting kink exists for $0 < \lambda < 1$ if $0 < \alpha < 1$ and for

$$0 < \lambda < \sqrt{\frac{20\alpha}{9 + 2\alpha + 9\alpha^2}}$$

if $\alpha > 1$.

In addition, in the ϕ^4-ϕ^6 model the spectrum of kink velocities also *includes nonzero velocities*. This means that the equation has solutions in the form of moving but nonradiating kinks. The dependence of the highest possible velocity of the nonradiating kink, $c = c_1$, on the nonlocality parameter λ for $\alpha = 0.5$ is shown in Fig. 4. There are two regions in this plot where the accuracy of computation drastically falls: the region of small λ and the region of small c. The continuation of the graph in this regions (dashed line) is conjectural. It worth noting that the kink front becomes sharper when approaching the limit of small λ.

Apart from the highest kink velocity $c = c_1$ there are other allowed kink velocities. The corresponding kink shapes for two neighboring velocities, $c_1 \approx 0.437$ and $c_2 \approx 0.243$, are shown in Fig. 5. One should observe that the difference of kink shapes for c_1 and c_2 is quite insignificant.

B. Propagation of nonradiating kinks in the ϕ^4-ϕ^6 model

According to the results of Sec. IV A there exist some "priviledged" values of kink velocities that provide radiationless kink propagation. In order to see whether the dynamical problem for the ϕ^4-ϕ^6 model "feels" these values of velocities, the following numerical experiment was performed. At the first stage, the kink profile and the velocity $c = c_1$ were calculated

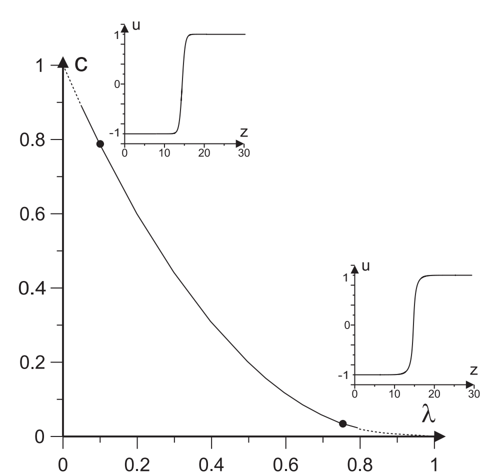

FIG. 4. The dependence of highest possible velocity of nonradiating kink, $c = c_1$, on the nonlocality parameter λ for $\alpha = 0.5$. The kink shapes are shown for $\lambda = 0.75$ and 0.1.

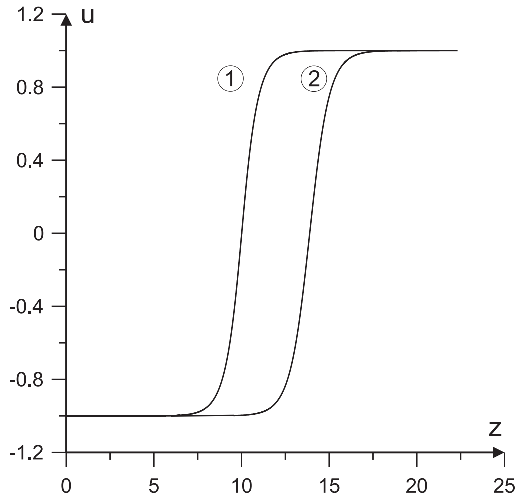

FIG. 5. The nonradiating kinks corresponding to $\lambda = 0.3$ and $\alpha = 0.5$ with different velocities (1) $c_1 \approx 0.437$ and (2) $c_2 \approx 0.243$.

(below the results for $\alpha = 0.4$, $\lambda = 0.3$ are shown; in this case $c_1 \approx 0.432$). The Cauchy problem for Eq. (1) with these initial conditions was solved [see Fig. 6(A)], and no radiation was emitted by the propagating kink. At the second step the Cauchy problem for Eq. (1) was solved taking the same profile furnished with greater velocity $c = 0.9$ as initial condition [see Fig. 6(B)]. The dynamics of the kink center is shown in Fig. 6(C). One can see that the initial velocity decreases; the velocity averaged over time $\Delta t = 180$ on the interval between the points 1 and 2 is ñ≈ 0.449, which is close to the velocity c_1.

V. CONCLUSION

To summarize, we have studied the existence of moving nonradiating kink solutions for nonlocal generalizations of nonlinear Klein-Gordon equations. The basic examples are ϕ^4 and ϕ^4-ϕ^6 models, corresponding to cubic and cubic-quintic nonlinearities. The nonlocality is represented by the Fourier multiplier operator of special form. The main results of the paper are as follows:

(1) The continuous spectrum of kink velocities that exists in the local model disappears when passing to the nonlocal model. The spectrum of kink velocities in the nonlocal model is *discrete*. The mechanism for this discrete spectrum arising can be explained in terms of phase-space analysis and works for a wide class of nonlinearities. The method for "demonstrative computation" of the velocity spectrum has been suggested.

(2) In the case of nonlocal ϕ^4 model the computation shows *the absence* of nonradiating moving kinks. In another words, the spectrum of possible kink velocities includes zero velocity only. However, if the nonlocality is weak, kinks can travel for long distances, and the radiation is relatively weak. At the same time strong nonlocality suppress the kink mobility. Apart from kinks, the nonlocal ϕ^4 model supports traveling solitary waves, which are unstable entities.

(3) In the case of the nonlocal ϕ^4-ϕ^6 model the nonradiating moving kinks *do exist*. The set of velocities for these kinks is discrete. Numerical simulations show that the evolution described by ϕ^4-ϕ^6 model "feels" these values of velocities. Specifically, a kink supplied with high initial velocity loses

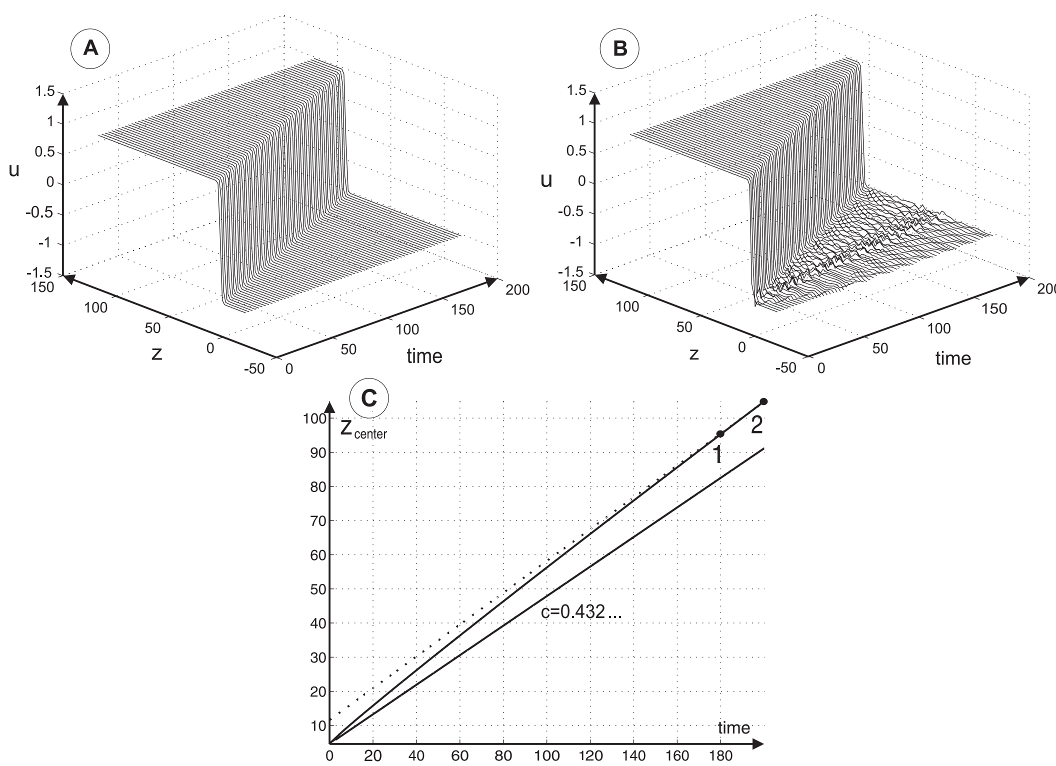

FIG. 6. Nonlocal ϕ^4-ϕ^6-model, $\alpha = 0.4$, $\lambda = 0.3$. (A) Radiationless propagation of the kink with velocity $c = c_1 \approx 0.432$; (B) propagation of the kink with the same shape as in (A) but supplied with velocity $c = 0.9$; (C) motion of the kink center (the point where $u = 0$) corresponding to the propagation in panels (A) and (B). The average velocity between the points 1 and 2 is ≈ 0.449.

energy by radiation, and its velocity settles down to the highest value of the discrete spectrum $c = c_1$.

In the light of these results some issues for the further study should be outlined. One of them is as follows: What features of nonlinearity are decisive for the existence or nonexistence of moving and nonradiating kink solutions? Why does the situation change qualitatively by adding the quintic term into the ϕ^4 model? We suppose that the comprehensive study of the weak nonlocality limit can make the situation clearer. Another question is related to the generalization of the approach to the kernels of a nonlocal operator of a more general kind. Following Ref. [20], it is straightforward to extend the approach to the kernels represented by a finite sum of exponents with positive weights (E-kernels, in terms of Ref. [20]). In the case of N exponents the system (13)–(14) will be replaced by an ordinary differential equations system of $N + 1$ equations of the second order. Further extensions of this approach imply the study of ordinary differential equations systems in an infinite dimensional phase space. This is a quite interesting problem from the mathematical point of view.

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