Lorentz-Abraham-Dirac versus Landau-Lifshitz radiation friction force in the ultrarelativistic electron interaction with electromagnetic wave (exact solutions)

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When the parameters of electron-extreme power laser interaction enter the regime of dominated radiation reaction, the electron dynamics changes qualitatively. The adequate theoretical description of this regime becomes crucially important with the use of the radiation friction force either in the Lorentz-Abraham-Dirac form, which possesses unphysical runaway solutions, or in the Landau-Lifshitz form, which is a perturbation valid for relatively low electromagnetic wave amplitude. The goal of the present paper is to find the limits of the Landau-Lifshitz radiation force applicability in terms of the electromagnetic wave amplitude and frequency. For this, a class of the exact solutions to the nonlinear problems of charged particle motion in the time-varying electromagnetic field is used.

I. INTRODUCTION

As with no other problem in classical electrodynamics the problem of the radiation friction effects on charged particle dynamics has been attracting attention for more than a century [1–4]. The radiation friction imposes constraints on the highest attainable energy of charged particles accelerated by standard accelerators [5] and in space [6], in particular, on the energy of the ultrahigh energy cosmic rays [7]. The effects of radiation reaction on electrons in a magnetically confined plasma lead to the phase space contraction [8]. Laser light being coherent and of ultrahigh intensity provides special conditions for experimentally studying the radiation friction effects. The radiation generated by present day [9,10] lasers approaches limits when the radiation friction force will change the scenario of the electromagnetic (EM) wave interaction with matter, that is, at $I > I_{rad} = 10^{23}$ W/cm^2. The electron dynamics become dissipative with fast conversion of the EM wave energy to hard EM radiation, which for typical laser parameters is in the γ-ray range [11–13]. There are discussions of the modification of the electron acceleration in the laser wake field acceleration regime [14] and the ion acceleration in the radiation pressure dominated regime [15] due the radiation friction, which are mainly obtained with computer simulations [11,16]. If the laser intensity substantially exceeds I_{rad}, novel physics of abundant electron-positron pair creation will come into play [17] (see also [13] and [18]) when the electron (positron) interaction with the EM field is principally determined by the radiation friction effects. The persistent interest towards the radiation friction effects stems from all these reasons [19,20].

In order to self-consistently find the trajectory of the emitting electron, the so called Minkovsky equations [3]

should be solved with the radiation friction force taken into account

$$m_e c \frac{du^\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu + g^\mu, \qquad (1)$$

$$\frac{dx^\mu}{ds} = u^\mu. \qquad (2)$$

Here $u_\mu = (\gamma, \mathbf{p}/m_e c)$ is the four velocity, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the EM field tensor with A_μ being the EM four vector and $\mu = 0, 1, 2, 3$, and

$$s = c \int \frac{dt}{\gamma}. \qquad (3)$$

The radiation friction force in the Lorentz-Abraham-Dirac (LAD) form [21–23] is given by

$$g^\mu = \frac{2e^2}{3c} \left[\frac{d^2 u^\mu}{ds^2} - u^\mu \left(\frac{du^\nu}{ds} \right) \left(\frac{du_\nu}{ds} \right) \right]. \qquad (4)$$

As is well known, Eq. (1) with the radiation friction force in the LAD form (4) possesses unphysical self-accelerating solutions (e.g. see Refs. [2,3]). When the radiation friction force is taken to be in the Landau-Lifshitz (L-L) form,

$$g^\mu = \frac{2e^3}{3m_e c^3} \left\{ \frac{\partial F^{\mu\nu}}{\partial x^\lambda} u_\nu u_\lambda - \frac{e}{m_e c^2} [F^{\mu\lambda} F_{\nu\lambda} u^\nu \right.$$
$$\left. - (F_{\nu\lambda} u^\lambda)(F^{\nu\kappa} u_\kappa) u^\mu] \right\}, \qquad (5)$$

the electron motion equations do not have pathological solutions, although they are not always consistent with energy-momentum conservation for an abruptly changing electromagnetic field [24].

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In the three-dimensional form it can be written as [2]

$$\mathbf{f} = \frac{2e^3}{3m_e c^3 \left(1 - \frac{v^2}{c^2}\right)^{1/2}} \left(\left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown)\right] \mathbf{E} + \frac{1}{c} \left\{ \mathbf{v} \times \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown)\right] \mathbf{B} \right\} \right) + \frac{2e^4}{3m_e^2 c^4} \left\{ \mathbf{E} \times \mathbf{B} + \frac{1}{c}[\mathbf{B} \times (\mathbf{B} \times \mathbf{v})] + \frac{1}{c}\mathbf{E}(\mathbf{v} \cdot \mathbf{E}) \right\}$$

$$- \frac{2e^4}{3m_e^2 c^5 \left(1 - \frac{v^2}{c^2}\right)} \mathbf{v} \left[\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)^2 - \frac{1}{c^2}(\mathbf{v} \cdot \mathbf{E})^2 \right]. \tag{6}$$

The L-L radiation friction force being a perturbation is valid provided there exists a frame of reference, where it is small compared to the Lorentz force $eF^{\mu\nu}u_\nu$ (as noted in Ref. [2]). Proving this frame of reference existence and finding the range of validity of the friction force in the L-L form is far from trivial. Below, using several exact analytical solutions to the electron motion equations in the EM field for the radiation friction force in the LAD and L-L forms, we discuss the validity range of the later approximation.

The electron motion equations with the LAD friction force admit an exact solution for the stationary problem describing the electron motion in the rotating electric field

(see Refs. [12,13,25]). This problem can also be solved for the case of the L-L force. Generalizing the electromagnetic field configuration, we consider the electric and magnetic field to be a superposition of components, which are rotating with frequency ω, homogeneous in space, and time-independent

$$\mathbf{E} = -\mathbf{e}_1 E_1 - \mathbf{e}_2[Dx_2 + E\cos(\omega t)] - \mathbf{e}_3[Dx_3 + E\sin(\omega t)], \tag{7}$$

$$\mathbf{B} = \mathbf{e}_1 B_1 + \mathbf{e}_2 J x_3 - \mathbf{e}_3 J x_2, \tag{8}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the unit vectors along the 1, 2, 3 axis. The EM field tensor is equal to

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & Dx_2 + E\cos(\omega t) & Dx_3 + E\sin(\omega t) \\ -E_1 & 0 & -Jx_2 & Jx_3 \\ -Dx_2 - E\cos(\omega t) & Jx_2 & 0 & -B_1 \\ -Dx_3 - E\sin(\omega t) & -Jx_3 & B_1 & 0 \end{pmatrix}. \tag{9}$$

In the case when E_1, B_1, J, and D vanish, the electric field can be realized in the antinodes, where the magnetic field vanishes, of a standing EM wave formed by two counterpropagating circularly polarized EM waves. Such an EM field configuration plays an important role in theoretical considerations of various nonlinear effects in quantum electrodynamics (e.g., see Refs. [13,17,26]). This EM configuration corresponds also to the circularly polarized EM wave propagating in the underdense plasma for the frame of reference moving with the wave group velocity [27,28]. In this frame of reference, the EM wave frequency is equal to the Langmuir frequency $\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$, where n_0 is the plasma density and the wave has no magnetic field component. The static component of the magnetic field B_1 can be generated in laser plasmas due to the inverse Faraday effect. Its effect on the charged particle motion has been studied in Ref. [25]. The radial component of the electric field $\mathbf{e}_2 Dx_2 + \mathbf{e}_3 Dx_3$, and azimuthal component of the magnetic field $\mathbf{e}_2 Jx_3 - \mathbf{e}_3 Jx_2$, correspond to a plasma wave in the boosted frame of reference with E_1 being the longitudinal component of the wake field.

It is convenient to write the electron momentum $\mathbf{p} = \mathbf{e}_1 p_1(t) + \mathbf{e}_2 p_2(t) + \mathbf{e}_3 p_3(t)$ and coordinates $\mathbf{x} = \mathbf{e}_1 x_1(t) + \mathbf{e}_2 x_2(t) + \mathbf{e}_3 x_3(t)$ as a combination of vectors, which are

nonrotating and rotating with angular frequency ω,

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} \equiv \frac{1}{m_e c} \begin{pmatrix} p_1 \\ p_{||} \\ p_\perp \end{pmatrix} = \frac{1}{m_e c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & \sin(\omega t) \\ 0 & -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

$$\times \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{10}$$

and

$$\begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin(\omega t) & -\cos(\omega t) \\ 0 & \cos(\omega t) & \sin(\omega t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{11}$$

II. STATIONARY SOLUTION OF THE ELECTRON EQUATIONS OF MOTION WITH THE RADIATION FRICTION FORCE IN THE LORENTZ-ABRAHAM-DIRAC FORM

The stationary solution of Eqs. (1) and (2), for which the vectors $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ do not depend on

time, with the radiation friction force in the LAD form, (4) can be cast as

$$0 = a_1 - \varepsilon_{\text{rad}}\tilde{u}_1 \gamma \left(\gamma^2 - 1 - \tilde{u}_1^2\right), \qquad (12)$$

$$\tilde{u}_2 = \left(d - b - j\frac{\tilde{u}_1}{\gamma}\right)\frac{\tilde{u}_2}{\gamma} + \varepsilon_{\text{rad}}\tilde{u}_3\gamma\left(\gamma^2 - \tilde{u}_1^2\right), \quad (13)$$

$$\tilde{u}_3 = \left(d + b - j\frac{\tilde{u}_1}{\gamma}\right)\frac{\tilde{u}_3}{\gamma} + a - \varepsilon_{\text{rad}}\tilde{u}_2\gamma\left(\gamma^2 - \tilde{u}_1^2\right), \quad (14)$$

where we use the relationship between $\tilde{x}_i$ and $\tilde{u}_i$, with $i = 1, 2, 3$, given by Eq. (2), which is

$$\tilde{x}_1 = s\gamma\tilde{u}_1, \quad \tilde{x}_2 = \frac{\tilde{u}_3 c}{\gamma\omega}, \quad \tilde{x}_3 = -\frac{\tilde{u}_2 c}{\gamma\omega}. \qquad (15)$$

Here the dimensionless parameter

$$\varepsilon_{\text{rad}} = \frac{2e^2\omega}{3m_e c^3} \qquad (16)$$

characterizes the radiation damping effect, $a_1 = eE_1/m_e\omega c$, $a = eE/m_e\omega c$, $d = eD/m_e\omega c$, $j = eJ/m_e\omega c$, and $b = eB_1/m_e\omega c$ are normalized longitudinal and transverse components of the electric and magnetic field, and γ is the electron relativistic Lorentz factor equal to $(1 + \tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2)^{1/2}$. The parameter ε_{rad} can also be written as $\varepsilon_{\text{rad}} = 4\pi r_e/3\lambda$ or $\varepsilon_{\text{rad}} = 2\omega t_e/3$, where $r_e = e^2/m_e c^2$ is the classical electron radius, $t_e = r_e/c$, and $\lambda = 2\pi c/\omega$.

At first we analyze the most simple case with $B_1 = J = D = 0$. The stationary solution to Eqs. (1) and (2), for which the vectors $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ do not depend on time, with the radiation friction force in the LAD form (4) can be cast as

$$0 = a_1 - \varepsilon_{\text{rad}}\tilde{u}_1\gamma\left(\gamma^2 - 1 - \tilde{u}_1^2\right), \qquad (17)$$

$$\tilde{u}_2 = \varepsilon_{\text{rad}}\tilde{u}_3\gamma\left(\gamma^2 - \tilde{u}_1^2\right), \qquad (18)$$

$$\tilde{u}_3 = a - \varepsilon_{\text{rad}}\tilde{u}_2\gamma\left(\gamma^2 - \tilde{u}_1^2\right). \qquad (19)$$

Multiplying Eq. (17) by $\tilde{u}_1$, Eq. (18) by $\tilde{u}_2$, and Eq. (19) by $\tilde{u}_3$, and adding them, we obtain

$$a_1\tilde{u}_1 + a\tilde{u}_2 = \varepsilon_{\text{rad}}\gamma^3\left(\gamma^2 - 1 - \tilde{u}_1^2\right), \qquad (20)$$

which is the zero component of Eq. (1). The left-hand side of this equation is proportional to the work produced by the electric field in the units of time and the right-hand side is proportional to the energy dissipation rate due to the radiation losses.

Multiplying Eq. (18) by $\tilde{u}_3$ and Eq. (19) by $\tilde{u}_2$, and adding them, we obtain

$$\tilde{u}_2^2 + \tilde{u}_3^2 = a\tilde{u}_3. \qquad (21)$$

A. Electron in the rotating electric field

If, in addition, the longitudinal component of electric field vanishes, $a_1 = 0$ with $\tilde{u}_1 = 0$, we obtain from Eqs. (12)–(14)

$$p_{\parallel} = \varepsilon_{\text{rad}} p_{\perp} \gamma^3, \qquad (22)$$

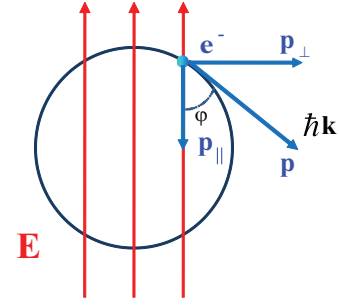$$p_{\perp} = m_e ca - \varepsilon_{\text{rad}} p_{\parallel} \gamma^3, \qquad (23)$$

FIG. 1. (Color online) Electron moving in the rotating electric field emits EM radiation. Due to this the angle between the electron momentum and electric field, φ, is not equal to $\pi/2$.

where the components of the electron momentum parallel and perpendicular to the electric field defined by Eq. (10) are equal to

$$p_{\parallel} = \frac{(\mathbf{p} \cdot \mathbf{E})}{|\mathbf{E}|} = m_e c\tilde{u}_2, \qquad (24)$$

$$p_{\perp} = (p^2 - p_{\parallel}^2)^{1/2} = m_e c\tilde{u}_3, \qquad (25)$$

respectively (see Fig. 1). In this case Eq. (21) yields a relationship between $p_{\parallel}$ and $p_{\perp}$:

$$p_{\parallel}^2 + p_{\perp}^2 = m_e cap_{\perp}. \qquad (26)$$

The electron γ factor γ is equal to $(1 + \tilde{u}_2^2 + \tilde{u}_3^2)^{1/2} \equiv (1 + p_{\parallel}^2/m_e^2 c^2 + p_{\perp}^2/m_e^2 c^2)^{1/2}$. As we see from the relationship

$$p_{\parallel} = [p_{\perp}(m_e ca - p_{\perp})]^{1/2} \qquad (27)$$

it follows that the component of the electron momentum perpendicular to the electric field is always equal or less than $a\, m_e c$. Multiplying Eq. (24) by $p_{\perp}$ and the Eq. (25) by $p_{\parallel}$ and subtracting them, we find

$$a\frac{p_{\parallel}}{m_e c} = \varepsilon_{\text{rad}}\gamma^3(\gamma^2 - 1), \qquad (28)$$

which corresponds to the energy balance equation (20) for $a_1 = \tilde{u}_1 = 0$ in the limit $a \gg 1$.

If the EM field amplitude is relatively small, that is, $1 \ll a \ll \varepsilon_{\text{rad}}^{-1/3}$, Eqs. (26) and (28) yield for the components of the electron momentum perpendicular and parallel to the electric field

$$p_{\perp} \approx m_e c\left(a - \varepsilon_{\text{rad}}^2 a^7\right), \qquad (29)$$

$$p_{\parallel} \approx m_e c\varepsilon_{\text{rad}} a^4. \qquad (30)$$

In the opposite limit, when $a \gg \varepsilon_{\text{rad}}^{-1/3}$, we obtain

$$p_{\perp} \approx \frac{m_e c}{(\varepsilon_{\text{rad}} a)^{1/2}}, \qquad (31)$$

$$p_{\parallel} \approx m_e c\left(\frac{a}{\varepsilon_{\text{rad}}}\right)^{1/4}. \qquad (32)$$

In Fig. 2(a) we show the dependence of $p_{\perp}$ and $p_{\parallel}$ on the EM field amplitude a for the dimensionless parameter $\varepsilon_{\text{rad}} = 10^{-8}$, obtained by a numerical solution of Eq. (23). Here the horizontal axis is normalized by $\varepsilon_{\text{rad}}^{-1/3}$ and the vertical axis is normalized by $(a_m/\varepsilon_{\text{rad}})^{1/4}$.
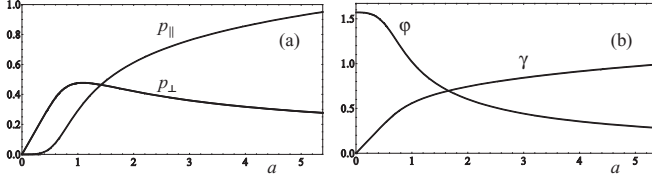
FIG. 2. (a) Dependence of the components of the electron momentum perpendicular $p_\perp$ and parallel $p_\parallel$ to the electric field [normalized by $m_e c(a_m/\varepsilon_{\rm rad})^{1/4}$] on the normalized EM field amplitude $a\varepsilon_{\rm rad}^{1/3}$, and (b) dependence of φ and electron γ-factor, normalized by $(a_m/\varepsilon_{\rm rad})^{1/4}$, on $a\varepsilon_{\rm rad}^{1/3}$ for $a_m = 2500$ and $\varepsilon_{\rm rad} = 10^{-8}$.

As we see, the dependences of the components of the electron momentum perpendicular and parallel to the electric field correspond to the asymptotics given by Eqs. (30) and (32). The perpendicular momentum reaches the maximum at $a \approx \varepsilon_{\rm rad}^{-1/3}$ and then decreases. The parallel momentum component monotonically increases with the EM amplitude growth.

It is also convenient to represent the momentum components in the complex form

$$p_\parallel + i p_\perp = \mathfrak{p}\exp(i\varphi) \qquad (33)$$

with $\mathfrak{p} = (p_\perp^2 + p_\parallel^2)^{1/2}$ and φ being the momentum value and the phase between the rotating electric field and the momentum vector. Equations (22,23) can be rewritten as

$$a^2 = (\gamma^2 - 1)(1 + \varepsilon_{\rm rad}^2 \gamma^6), \qquad (34)$$

$$\tan\varphi = \frac{1}{\varepsilon_{\rm rad}\gamma^3}, \qquad (35)$$

where the electron γ-factor is equal to $(1 + \mathfrak{p}^2)^{1/2}$. These equations are the same as Eqs. (6) in Ref. [13].

In Fig. 2(b) we present the electron γ-factor normalized by $(a_m/\varepsilon_{\rm rad})^{1/4}$ and the angle φ versus the EM field amplitude a for $\varepsilon_{\rm rad} = 10^{-8}$. The angle φ changes from $\pi/2$ at $a = 0$ when the electron momentum is perpendicular to the electric field vector, to 0 at $a \to \infty$ when the electron momentum becomes antiparallel to the electric field. The horizontal axis is normalized in the same way as in Fig. 2(a).

B. Electron dynamics in the superposition of rotating and radial electric fields

Electron dynamics in the superposition of rotating and radial electric fields corresponds to the case of the electron direct acceleration by the laser pulse propagating inside the self-focusing channel [29]. Its realization provides the conditions for substantial enhancement of the betatron radiation allowing for photon emission in the γ-ray energy range [30].

In the frame of reference moving with the laser pulse group velocity v_g, the equations of the electron motion are Eqs. (12) and (14) with $b = j = a_1 = 0$:

$$(\gamma - d)\tilde{u}_2 = \varepsilon_{\rm rad}\tilde{u}_3\gamma^4, \qquad (36)$$

$$(\gamma - d)\tilde{u}_3 = a\gamma - \varepsilon_{\rm rad}\tilde{u}_2\gamma^4. \qquad (37)$$

For variables $\mathfrak{p}$ and φ defined by Eq. (33) we can rewrite Eqs. (36) and (37) as

$$a^2 = (\gamma^2 - 1)\left[\left(1 - \frac{d}{\gamma}\right)^2 + \varepsilon_{\rm rad}^2\gamma^6\right], \qquad (38)$$

or

$$d = \gamma - \gamma\sqrt{\frac{a^2}{\gamma^2 - 1} - \varepsilon_{\rm rad}^2\gamma^6} \qquad (39)$$

and

$$\tan\varphi = \frac{\gamma - d}{\varepsilon_{\rm rad}\gamma^4} \qquad (40)$$

with $\gamma = (1 + \mathfrak{p}^2)^{1/2}$.

In Fig. 3(a) we show dependences of the normalized electron energy $K = (\gamma - 1)/(\gamma_m - 1)$, where $\gamma_m = [1 + (p_m/m_e c)^2]^{1/2}$ with $p_m/m_e c = 700$, $d = 250$, $\varepsilon_{\rm rad} = 10^{-8}$, and angle φ on the electric field amplitude a. As we see, asymptotically at $\gamma \to \infty$ their behavior is the same as in the above discussed case corresponding to Eqs. (30), (32), and (35) and illustrated by Fig. 2. In the relatively low-energy region the dependence of electron momentum on the electric field amplitude shows the hysteresis behavior as seen in Fig. 3(a). In the region $a_{m1} < a < a_{m2}$, with

$$a_{m1} \approx \varepsilon_{\rm rad}d^4, \qquad (41)$$

$$a_{m2} \approx d, \qquad (42)$$

there are three values of the electron momentum corresponding to one value of a. At $a \approx a_{m1}$ the electron energy is approximately equal to $\gamma_{m1} \approx d$ and for $a \approx a_{m2}$ we have $\gamma_{m2} \approx d^{1/3}$, provided $4\varepsilon_{\rm rad}d^3 \ll 1$. The condition for the hysteresis to occur is

$$\varepsilon_{\rm rad}d^{2/3} \leqslant 0.276. \qquad (43)$$

The hysteresis is distinctly seen in Fig. 3(b) showing typical behavior for nonlinear resonance [31], the nonlinear resonance dependence of the quiver energy K on the parameter d, which is equal to the square of the ratio of the electron oscillation frequency in the radial electric field to the frequency of the EM wave. This corresponds to the nonlinear regime of the "betatron resonance" studied in Ref. [29].

For the electron moving inside the self-focusing channel a typical value of the parameter d is of the order of

$$d = \frac{\omega_{pe}^2}{\omega^2}\gamma_g, \qquad (44)$$

where $\gamma_g = (1 - v_g^2/c^2)^{-1/2}$, that is, it is approximately equal to γ_g. Here we take into account that in the boosted frame of reference the EM wave frequency is equal to the Langmuir frequency. As a result we can find the electron energy accelerated by the "betatron resonance" mechanism in the laboratory frame of reference to be equal to

$$\mathcal{E}_e \approx m_e c^2 \gamma_g^2. \qquad (45)$$

This corresponds to K_d in Fig. 3(b). For example, for the plasma density of the order of $10^{19}\,{\rm cm}^{-3}$ and the laser wavelength $\approx 1\,\mu{\rm m}$, the electron energy is about 50 MeV.
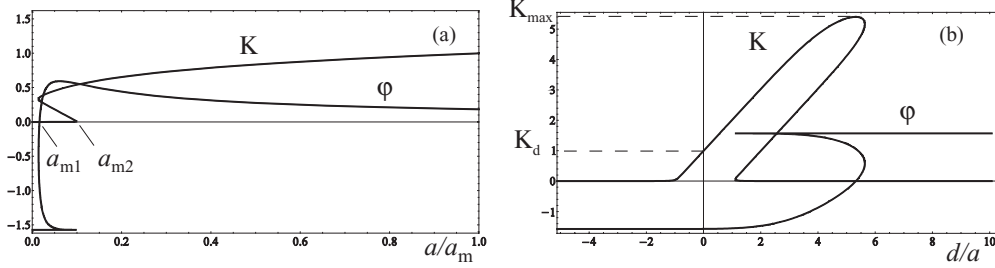
FIG. 3. Solution of the electron equation of motion with the radiation friction force in the LAD form in the case of the superposition of rotating and radial electric fields. (a) Normalized electron energy $K = (\gamma - 1)/(\gamma_m - 1)$ and the angle φ vs the electric field amplitude a/a_m with $a_m = 2500$. (b) Normalized electron energy $K = (\gamma - 1)/(\gamma_m - 1)$ and the angle φ vs parameter d/a for $a = 50$. Here $\gamma_m = [1 + (p_m/m_e c)^2]^{1/2}$ with $p_m/m_e c = 700$, $d = 250$, and $\varepsilon_{\text{rad}} = 10^{-8}$.

The transverse component of the electron momentum is of the order of $m_e c \gamma_g \approx 5$ MeV.

The maximal energy, which an electron can achieve in the resonance case according to Eq. (39) in the boosted frame of reference is approximately equal to $\tilde{\mathcal{E}}_e = m_e c^2 \gamma_{\max} = m_e c^2 (a/\varepsilon_{\text{rad}})^{1/4}$, which corresponds to $K_{\max}$ in Fig. 3(b). In the laboratory frame of reference we obtain

$$\mathcal{E}_e \approx m_e c^2 \gamma_{\max} \gamma_g. \tag{46}$$

For $a \approx 10$ and $\gamma_g = 10$ it gives $\mathcal{E}_e \approx 500$ MeV. The energy of emitted photons is of the order of $\approx 0.3\hbar\omega\gamma_{\max}^3$. This is of the order of 300 keV. The emitted γ-rays are collimated within the angle $\approx 1/\gamma_g$.

III. STATIONARY SOLUTION OF THE ELECTRON EQUATIONS OF MOTION WITH THE RADIATION FRICTION FORCE IN THE LANDAU-LIFSHITZ FORM

Here we analyze the electron motion equations with the radiation friction force taken in the L-L form (5) for a stationary electron orbit. We retain the leading order terms in the limit $\gamma \gg 1$, which is the last term in Eq. (6).

A. Electron motion in the rotating homogeneous electric field

We look for the solutions describing a stationary electron orbit in a rotating homogeneous electric field, that is, E_1, D, B_1, J vanish in Eqs. (7)–(9). From Eq. (6) we obtain for

the $\tilde{u}_2 = p_{||}/m_e c$ and $\tilde{u}_3 = p_\perp/m_e c$ momentum components the algebraic equations

$$\tilde{u}_2 = \varepsilon_{\text{rad}} \frac{\tilde{u}_3}{\gamma} a^2 (1 + \tilde{u}_3^2), \tag{47}$$

$$\tilde{u}_3 = a - \varepsilon_{\text{rad}} \frac{\tilde{u}_2}{\gamma} a^2 (1 + \tilde{u}_3^2). \tag{48}$$

Using the variables $\mathfrak{p}$ and φ defined by Eq. (33) we can present these equations in the form

$$a^2 = \frac{\gamma^2}{2\varepsilon_{\text{rad}}^2(\gamma^2 - 1)}\left[1 - \sqrt{1 - 4\varepsilon_{\text{rad}}^2(\gamma^2 - 1)^2}\right] - (\gamma^2 - 1)^2, \tag{49}$$

$$\tan\varphi = \frac{2\varepsilon_{\text{rad}}(\gamma^2 - 1)}{\gamma - \sqrt{\gamma^2 - 4\varepsilon_{\text{rad}}^2\gamma^2(\gamma^2 - 1)^2}}. \tag{50}$$

In the range of the EM field amplitude, $1 \ll a \ll \varepsilon_{\text{rad}}^{-1}$, solution to these equations has the same asymptotic dependences as given by Eqs. (23) and (26). However, when the EM field amplitude approaches the value of $\varepsilon_{\text{rad}}^{-1}$, the solution qualitatively changes. According to Eq. (50), the electron momentum decreases as also shown in Fig. 4. In Fig. 4(a) we present the components of the electron momentum parallel and perpendicular to the instantaneous electric field as functions of the electric field amplitude. Figure 4(b) shows the dependences of the angle φ and the electron γ-factor on the electric field. The momentum and γ-factor are normalized by $(a_m/\varepsilon_{\text{rad}})^{1/4}$, and the dimensionless electric field amplitude by ε_{rad}.
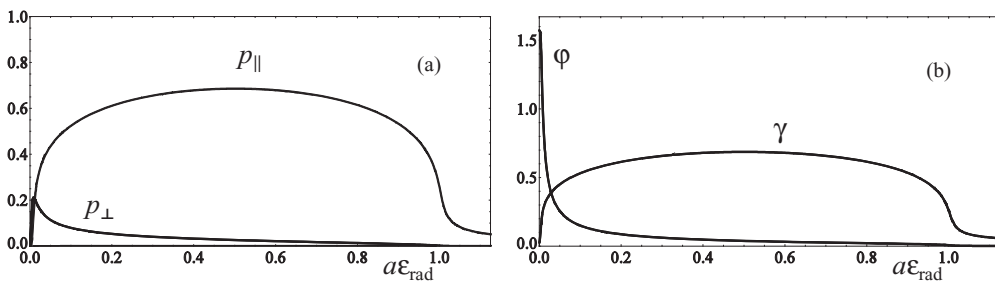

FIG. 4. Solution of the electron equation of motion with the radiation friction force in the L-L form in the case of rotating homogeneous electric field: (a) Dependence of the components of the electron momentum [normalized by $m_e c(a_m/\varepsilon_{\text{rad}})^{1/4}$] perpendicular $p_\perp$ and parallel $p_{||}$ to the electric field on the normalized EM field amplitude $a\varepsilon_{\text{rad}}$, and (b) dependence of φ and the electron γ-factor [divided by $(a_m/\varepsilon_{\text{rad}})^{1/4}$], on $a\varepsilon_{\text{rad}}$ for $a_m = 1500$ and $\varepsilon_{\text{rad}} = 7.5 \times 10^{-4}$.
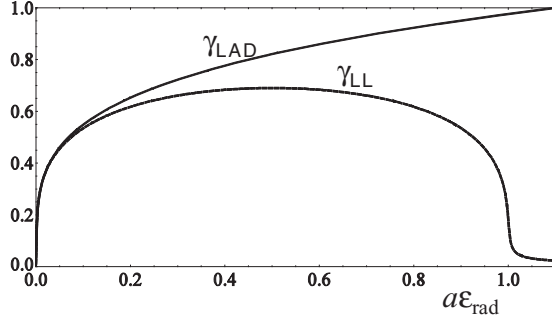
FIG. 5. Solution of the electron equation of motion in a rotating homogeneous electric field for $a_m = 1500$ and $\varepsilon_{\rm rad} = 7.5 \times 10^{-4}$. Dependences of the electron γ-factors on the electric field: $\gamma_{\rm LAD}$ and $\gamma_{\rm LL}$ correspond to the radiation friction force taken in the LAD and L-L form, respectively. The normalization is the same as in Fig. 4.

In Fig. 5 we present the results of the solution of the electron motion equation in a rotating homogeneous electric field. Here the dependences of the electron γ factors on the electric field, $\gamma_{\rm LAD}$ and $\gamma_{\rm LL}$, correspond to the radiation friction force taken in the LAD and L-L form, respectively. The normalization is the same as in Fig. 4.

B. Electron dynamics in the superposition of rotating and radial electric fields

The equations of the electron motion in the superposition of rotating and radial electric fields ($b = j = a_1 = 0$) with the radiation friction force in the L-L form are

$$(\gamma - d)\tilde{u}_2 = \varepsilon_{\rm rad}\tilde{u}_3[(d\gamma + a\tilde{u}_3)^2 + a^2 - d^2], \qquad (51)$$

$$(\gamma - d)\tilde{u}_3 = a\gamma - \varepsilon_{\rm rad}\tilde{u}_2[(d\gamma + a\tilde{u}_3)^2 + a^2 - d^2]. \qquad (52)$$

We can rewrite Eqs. (51) and (52) as

$$a^2 = d^2 - \frac{(d - \gamma + \gamma^3)^2}{\gamma^2} + \frac{\gamma^2 - \sqrt{\gamma^4[1 - 4\varepsilon_{\rm rad}^2(\gamma^2 - 1)^2]}}{2\varepsilon_{\rm rad}^2(1 - \gamma^2)}, \qquad (53)$$

$$\tan\varphi = \frac{2\varepsilon_{\rm rad}(\gamma - d)(\gamma^2 - 1)}{\gamma^2[1 - \sqrt{1 - 4\varepsilon_{\rm rad}^2(\gamma^2 - 1)^2}]}, \qquad (54)$$

where φ is defined by Eq. (33).

As in the above considered case described by the LAD model, the dependence of electron momentum on the electric field amplitude shows similar behavior, provided $a \gg 1$, $\gamma \gg 1$, and $2\varepsilon_{\rm rad}\gamma^2 \ll 1$.

IV. DISCUSSION

High-order derivatives with respect to time in the electron motion equations with the radiation friction force in the LAD form stem from the 3D geometry of the electromagnetic field interaction with a point charge, when the electrostatic energy diverges for the charge radius tending to zero leading to the so-called classical mass renormalization [3,4]. In a 1D electrodynamics model with the point charge role played by an infinitely thin foil there are no such difficulties (see analysis of this case in Appendix A).

As follows from consideration of the above presented exact solutions to the electron equations of motion with the radiation friction force taken in the LAD and L-L form, in the limit of relatively low electric field amplitude they show the same behavior (as seen in Fig. 5). When the electric field is strong, that is, the normalized field amplitude a approaches the value of $\varepsilon_{\rm rad}^{-1}$, the solutions are drastically different.

The condition $a = \varepsilon_{\rm rad}^{-1}$ corresponds to the electric field equal to the critical electric field of classical electrodynamics,

$$E_{\rm cr} = \frac{m_e^2 c^4}{e^3}. \qquad (55)$$

This electric field can produce a work equal to $m_e c^2$ over the distance of the classical electron radius r_e.

The radiation friction force in the Landau-Lifshitz form assumes the smallness of the EM field amplitude compared to the critical field of classical electrodynamics. Another parameter which should be small is the ratio of the EM field inhomogeneity scale length to the classical electron radius r_e. The time dependent EM fields should be slowly evolved on a time scale compared to $t_e = r_e/c$, as discussed in Refs. [24,32] devoted to the problem of classical electrodynamics applicability.

Obviously, the limit of the EM field amplitude of the order of $E_{\rm cr}$ (and of the space and time scales of the order of r_e and $t_e = r_e/c$) is of pure academic interest, because quantum mechanical effects become important at electric field amplitudes substantially lower (and on spatial scales of the order of the electron Compton wavelength $\hbar/m_e c$). The critical electric field of quantum electrodynamics (QED),

$$E_S = \frac{m_e^2 c^3}{e\hbar}, \qquad (56)$$

is a factor $\alpha = e^2/\hbar c \approx 1/137$ smaller. Here α is the fine structure constant. For the electron motion in colliding EM waves, the QED effects due to the recoil from the photon emission should be incorporated into the description of the electron interaction with the EM field for an even smaller EM wave amplitude. As shown in Ref. [12], the recoil from the photon emission comes into play when the photon momentum $\hbar \mathbf{k}_m$ becomes of the order of the electron momentum $\mathbf{p}$. In other words, the photon with the energy larger than the electron cannot be emitted. Here the wave vector of the photon emitted by the electron $\mathbf{k}_m$ is of the order of $\omega\gamma^3/c$ and $p \approx m_e c\gamma$. This yields the γ-factor at which the photon recoil should be taken into account, equal to $\gamma_{QM} = (m_e c^2/\hbar\omega)^{1/2}$. As noted above, for $a \gg \varepsilon_{\rm rad}^{-1/3}$ there is a relationship between the electron energy and the EM field amplitude, which has a form $\gamma = (a/\varepsilon_{\rm rad})^{1/4}$. For $(a/\varepsilon_{\rm rad})^{1/4} = \gamma_{QM}$ we have $a = a_{QM} = \varepsilon_{\rm rad}m_e^2 c^4/\hbar^2\omega^2 = 2e^2 m_e c/\hbar^2\omega$. The electric field $m_e c\omega a_{QM}/e$ is of the order of αE_S, which is equal to $\alpha^2 E_{\rm cr}$, below which both the LAD and L-L forms for the radiation friction force give the same result. This confirms a conclusion made in Ref. [33], where on the basis of numerical calculations of an electron motion in a very strong laser pulse it was found that in the classical regime the L-L damping equation is very adequate.

Although conclusions following from the above presented consideration do not have the character of a rigorously proved mathematical theorem, they give an indication of the range of

the L-L forms for the radiation friction force, which can be written in terms of the normalized EM wave amplitude:

$$a < a_{QM}. \tag{57}$$

We note here that the question as to whether or not the EM field amplitude is small compared to E_{cr} should be answered based on the examination of the field in the electron rest frame of reference. For example, if a relativistic electron bunch interacts with the EM field, the radiation friction force in the L-L form can predict electron behavior different from that described with the LAD radiation friction force at substantially lower electric field amplitude.

Quantitatively this limit is described in terms of the Lorentz and gauge invariant parameter

$$\chi = \frac{[(F^{\mu\nu}u_\nu)^2]^{1/2}}{E_{cr}}. \tag{58}$$

It is of the order of the ratio $E = E_{cr}$ in the electron rest frame of reference. It can be expressed via the electric and magnetic fields and electron momentum as

$$\chi = \frac{[(m_e c\gamma \mathbf{E} + \mathbf{p} \times \mathbf{B})^2 - (\mathbf{p} \cdot \mathbf{E})^2]^{1/2}}{m_e c \, E_{cr}}. \tag{59}$$

For the case of an electron interacting with a laser pulse, using the solution presented in Ref. [2], we find that the components of the electron momentum along $p_1 = (\mathbf{p} \cdot \mathbf{k})/|\mathbf{k}|$ and perpendicular $\mathbf{p_a} = \mathbf{p} - \mathbf{k}p_1/|\mathbf{k}|$ to the direction of the laser pulse propagation can be found from the equations

$$\mathbf{p_a} = \mathbf{p_{a,0}} + m_e c(\mathbf{a} - \mathbf{a}_0) \tag{60}$$

and

$$\{m_e^2 c^2 + [\mathbf{p_{a,0}} + m_e c(\mathbf{a} - \mathbf{a}_0)]^2 + p_1^2\}^{1/2} - p_1$$
$$= [m_e^2 c^2 + (\mathbf{p_{a,0}} - m_e c\mathbf{a}_0)^2 + p_{1,0}^2]^{1/2} - p_{1,0}. \tag{61}$$

Here $\mathbf{k} = |\mathbf{k}|\mathbf{e}_1$ is the wave vector of the EM wave. These expressions use the conservation of generalized momentum, assuming the radiation friction effects are negligibly small (see analysis of the radiation damping effects on the parameter χ in Appendix B). $p_{1,0}$ and $\mathbf{p_{a,0}}$ are the initial components of the electron momentum, that is, before collision with the laser pulse for $\mathbf{a}_0 = 0$.

For a plane EM wave propagating along the x axis with the electric $\mathbf{E} = -c^{-1}\partial_t \mathbf{A}$ and magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}(x - ct)$ is the EM vector potential, prime denotes differentiation with respect to the variable $x - ct$, and $\mathbf{a} = e\mathbf{A}/m_e c^2$, the invariant χ takes the form

$$\chi = \frac{E}{E_{cr}}\left(\gamma - \frac{p_1}{m_e c}\right). \tag{62}$$

Substituting expression (61) into Eq. (62) we obtain

$$\chi = \frac{E}{E_{cr}}\left\{ \frac{[m_e^2 c^2 + (\mathbf{p_{a,0}} - m_e c\mathbf{a}_0)^2 + p_{1,0}^2]^{1/2} - p_{1,0}}{m_e c} \right\}. \tag{63}$$

As we see, for an ultrarelativistic electron $p_{1,0} \gg m_e c$ colliding with the laser pulse in the copropagating configuration, that is, $p_{1,0} > 0$, $\mathbf{p_{a,0}} = 0$, and $\mathbf{a}_0 = 0$, the parameter χ is negligibly small:

$$\chi \approx (E/E_{cr})(m_e c/2p_{1,0}). \tag{64}$$

In the case of the head-on collision of an ultrarelativistic electron with the laser pulse, when $p_{1,0} < 0$, $\mathbf{p_{a,0}} = 0$, and $\mathbf{a}_0 = 0$, the parameter χ is a factor $(2p_{1,0}/m_e c)^2$ larger and is approximately equal to

$$\chi \approx (E/E_{cr})(|2p_{1,0}|/m_e c). \tag{65}$$

If the electron appears inside the laser pulse as result of a gas ionization or due to the electron-positron pair creation (see Refs. [13,17,34]), the initial electron momentum is negligibly small, $p_{1,0} \approx 0$ and $\mathbf{p_{a,0}} \approx 0$, the constant $\mathbf{a}_0$ corresponds to the EM field in the point and instant of time where and when the electron is created. In this case the invariant χ is equal to

$$\chi = (E/E_{cr})\left(1 + \mathbf{a}_0^2\right)^{1/2} \approx aa_0/a_{cr} \tag{66}$$

with $a_{cr} = eE_{cr}/m_e\omega_0 c = 1/(\omega_0 t_e)$.

The above defined parameter χ is again a factor $1/\alpha \approx 137$ smaller than the known quantum electrodynamics parameter $\chi_e = [(F^{\mu\nu}u_\nu)^2]^{1/2}/E_S$, which gives the ratio of the EM field amplitude to the QED critical field E_S in the electron rest frame of Ref. [35]. For a 1 μm 10-PW laser pulse focused to a few microns focus spot with the dimensionless amplitude $a = 3 \times 10^2$ the parameter χ becomes equal to unity for the electron energy of about of 40 GeV and the QED parameter χ_e is of the order of unity for the electron energy of about 300 MeV.

APPENDIX A: RADIATION FRICTION IN 1D ELECTRODYNAMICS

In a 1D electrodynamics model with the point charge role played by an infinitely thin foil [36] there are no difficulties with high-order derivatives with respect to time in the electron motion equations with the radiation friction force. This model has been extensively used in studying the problem of relativistic thin plasma layer transparency, particularly for the purposes of the laser pulse shaping [37] (see also the experimental paper [38]), in the high order harmonics generation [39], in the laser ion acceleration [15,40], and in the generation of coherent extremely high intensity x-ray pulses by relativistic mirrors [41].

Using the results of Refs. [36,37], we consider the case of normal incidence of a plane electromagnetic wave on an infinitely thin foil. The foil is located in the plane $x = 0$. The interaction of the wave with the foil is described by Maxwell's equations for the vector potential $\mathbf{A}(x,t)$ which yield

$$\partial_{tt}\mathbf{A} - c^2\partial_{xx}\mathbf{A} = 4\pi c\delta(x)\mathbf{J}(\mathbf{A}) + \dot{\delta}(t)\mathbf{A}(x,0) + \delta(t)\dot{\mathbf{A}}(x,0), \tag{A1}$$

where $\delta(x)$ is the Dirac delta function and a dot denotes differentiation with respect to time. The first term on the

right-hand side of Eq. (A1) describes the electric current in the foil and the delta function $\delta(x)$ represents its sharp localization. The electric current $\mathbf{J}(\mathbf{A})$ is a function of the vector potential $\mathbf{A}(0,t)$ at $x = 0$. The last two terms on the right-hand side of Eq. (A1) are equivalent to the initial conditions: $\mathbf{A}(x,0) = \mathbf{A}_0(x)$ and $\partial_t \mathbf{A}(x,0) = \dot{\mathbf{A}}_0(x)$. Here the functions $\mathbf{A}_0(x)$ and $\dot{\mathbf{A}}_0(x)$ define the incident electromagnetic wave $\mathbf{A}_0(x,t)$. Convolution of the Green function for the one-dimensional wave equation $G(x,t;s,\tau) = \theta[(t - \tau) - |x - s|/c]/2$, with the terms in the right-hand side of Eq. (A1) yields

$$\mathbf{A}(x,t) = \mathbf{A}_0(x,t) + 2\pi \int_0^{t-|x|/c} \mathbf{J}[\mathbf{A}(0,\tau)]d\tau. \quad \text{(A2)}$$

Assuming $x = 0$ on both sides of Eq. (A2) and taking the derivative with respect to time, we obtain

$$\dot{\mathbf{A}}(0,t) = \dot{\mathbf{A}}_0(0,t) + 2\pi \mathbf{J}[\mathbf{A}(0,t)]. \quad \text{(A3)}$$

On the right-hand side $\dot{\mathbf{A}}_0(0,t)$ is a known function and the electric current $\mathbf{J}$ is assumed to be a given function of $\mathbf{A}(0,t)$. In this way a nonlinear boundary problem for a system of partial differential equations is reduced to the ordinary differential equation for the field inside the foil (A3). Solving this equation we find the vector potential $\mathbf{A}(0,t)$ inside the foil. Substituting it into Eq. (A2) we obtain the expression that describes transmitted and reflected waves.

Taking into account the generalized electron momentum conservation $\mathbf{p} - e\mathbf{A}/c = \text{constant}$, and the relationship between the electric current and the electron velocity $\mathbf{J} = -en_e l \mathbf{v} = -en_e l c \mathbf{p}/(m_e c^2 + p^2)^{1/2}$, where n_e and l are the electron density and the foil thickness, we find that the 1D equation (A3) for the stationary motion of a thin foil interacting of its interaction with a rotating electric field can be written in the form

$$\tilde{u}_2 = \varepsilon_0 \tilde{u}_3 / \gamma, \quad \text{(A4)}$$

$$\tilde{u}_3 = a - \varepsilon_0 \tilde{u}_2 / \gamma. \quad \text{(A5)}$$

Here ε_0 is the dimensionless parameter [37],

$$\varepsilon_0 = \frac{2\pi e^2 n_e l}{m_e \omega_0 c}, \quad \text{(A6)}$$

proportional to the surface electric charge of the foil $en_e l$.

Solving this system of algebraic equations we obtain

$$\tilde{u}_2 = \frac{\varepsilon_0}{2^{1/2} a} \frac{\left\{ \left[4a^2 + \left(1 - a^2 + \varepsilon_0^2 \right)^2 \right]^{1/2} - \left(1 - a^2 + \varepsilon_0^2 \right) \right\}^{1/2}}{\left\{ \left[4a^2 + \left(1 - a^2 + \varepsilon_0^2 \right)^2 \right]^{1/2} + \left(1 + a^2 - \varepsilon_0^2 \right) \right\}^{1/2}}, \quad \text{(A7)}$$

$$\tilde{u}_3 = \frac{1}{2a} \left\{ \left[4a^2 + \left(1 - a^2 + \varepsilon_0^2 \right)^2 \right]^{1/2} - \left(1 - a^2 + \varepsilon_0^2 \right) \right\}. \quad \text{(A8)}$$

In the limit of a relatively weak EM field when $\varepsilon_0 \gg a \gg 1$ solutions to Eqs. (A7) and (A8) have the asymptotics

$$\tilde{u}_2 = \frac{\varepsilon_0}{1 + \varepsilon_0^2} a - \frac{\varepsilon_0 \left(1 - \varepsilon_0^2 \right)}{2 \left(1 + \varepsilon_0^2 \right)^3} a^3 + O(a^4), \quad \text{(A9)}$$

$$\tilde{u}_3 = \frac{1}{1 + \varepsilon_0^2} a - \frac{\varepsilon_0^2}{\left(1 + \varepsilon_0^2 \right)^3} a^3 + O(a^4). \quad \text{(A10)}$$
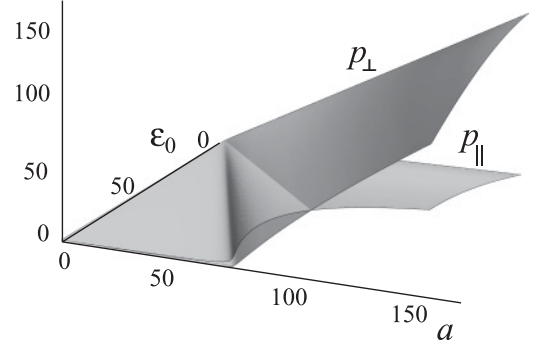

FIG. 6. Normalized momentum components $p_\parallel / m_e c = \tilde{u}_2$ and $p_\perp / m_e c = \tilde{u}_3$ vs the rotating field amplitude a and parameter ε_0.

In the opposite limit for $a \gg \varepsilon_0 \gg 1$ the asymptotics are

$$\tilde{u}_2 = \varepsilon_0 - \frac{\varepsilon_0 \left(1 + \varepsilon_0^2 \right)}{2a^2} a + O(a^{-4}), \quad \text{(A11)}$$

$$\tilde{u}_3 = a - \frac{\varepsilon_0^2}{a} + \frac{\varepsilon_0^2}{a^3} + O(a^{-4}). \quad \text{(A12)}$$

Dependences of $\tilde{u}_2 = p_\parallel / m_e c$ and $\tilde{u}_3 = p_\perp / m_e c$ on the rotating field amplitude a and ε_0 are shown in Fig. 6.

As we can see, in the dissipative range of parameters, which corresponds to a relatively low EM field amplitude $\varepsilon_0 \gg a$, the component of the electron momentum $p_\parallel$ parallel to the electric field is much larger than the perpendicular component $p_\perp$. In the limit of a strong electric field $\varepsilon_0 \ll a$ we have $p_\parallel \ll p_\perp$, that is, the electron momentum is almost perpendicular to the instantaneous direction of the electric field, contrary to the case of a 3D point electric charge, when the dissipative regime with $p_\parallel \gg p_\perp$ requires the condition $a \gg \varepsilon_{\text{rad}}$.

APPENDIX B: DEPENDENCE OF RADIATION FRICTION EFFECTS ON THE PARAMETER χ IN THE ULTRARELATIVISTIC ELECTRON INTERACTION WITH AN EM PULSE

Incorporating the radiation friction effects in the Landau-Lifshitz form into the electron equation of motion [Eqs. (1) and (6)],

$$\frac{d\mathbf{p}}{dt} = -e\mathbf{E} - \frac{e}{m_e c \gamma}(\mathbf{p} \times \mathbf{B})$$
$$- \frac{2e^4}{3m_e^4 c^7} \frac{\mathbf{p}}{\gamma} [(m_e c \gamma \mathbf{E} + \mathbf{p} \times \mathbf{B})^2 - (\mathbf{p} \cdot \mathbf{E})^2]. \quad \text{(B1)}$$

Here we retained the main order terms in the radiation friction force. If $\varepsilon_{\text{rad}} a \gamma^2 \gg 1$, which for a 10 PW laser with $a = 300$ corresponds to $\gamma \approx 500$, the interaction becomes purely dissipative and Eq. (B1) can be reduced to

$$\frac{dp_1}{dt} = -\varepsilon_{\text{rad}} \omega a^2 (-2t) \frac{p_1^2}{m_e c}. \quad \text{(B2)}$$

In this equation we assume the head-collision case of the electron interaction with the laser pulse, for which $x \approx -ct$ and $a(x - ct) \approx a(-2t)$. Its solution is given by

$$p_1(t) = -\frac{p_{1,0} m_e c}{m_e c + \varepsilon_{\text{rad}} \omega p_{1,0} \int_0^t a^2 (-2t') dt'}. \quad \text{(B3)}$$

For a pulse envelope $a(t)$ of the Gaussian form $a(t) = a_0 \exp(-t^2/2\tau_{\text{las}}^2)$, Eq. (B3) takes the form

$$p_1(t) = -\frac{p_{1,0} m_e c}{m_e c + \varepsilon_{\text{rad}} a_0^2 \omega \tau_{\text{las}} p_{1,0} (\pi/32)^{1/2} [\text{erf}(2^{1/2} t/\tau_{\text{las}}) - 1]}, \tag{B4}$$

where $\text{erf}(x)$ is the error function equal to [42]

$$\text{erf}(x) = \frac{\pi^{1/2}}{2} \int_0^x \exp(-t^2) dt. \tag{B5}$$

The dependence given by Eq. (B4) shows that for large enough $\omega \tau_{\text{las}} p_{1,0} a_0^2$ the electron momentum tends to the limit of

$$p_1(t) \rightarrow \left(\frac{\pi}{8}\right)^{1/2} \frac{m_e c}{\varepsilon_{\text{rad}} \omega \tau_{\text{las}} a_0^2}, \tag{B6}$$

which is independent of the initial momentum value $p_{1,0}$. In this limit the parameter χ is equal to

$$\chi = \left(\frac{\pi}{8}\right)^{1/2} \frac{1}{\omega \tau_{\text{las}} a_0}. \tag{B7}$$

Although for $\omega \tau_{\text{las}} \gg 1$ and $a_0 \gg 1$ it is substantially smaller than unity, the QED parameter χ_e being a factor $\alpha^{-1} \approx 137$ larger can be larger than unity as shown in Ref. [18].

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