Slow passage through resonance

Youngyong Park and Younghae Do*
Department of Mathematics, Kyungpook National University, Daegu 702-701, South Korea

Juan M. Lopez
School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287, USA and Department of Mathematics, Kyungpook National University, Daegu 702-701, South Korea
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The slow passage problem through a resonance is considered. As a model problem, we consider a damped harmonically forced oscillator whose forcing frequency is slowly ramped linearly in time. The setup is similar to the familiar slow passage through a Hopf bifurcation problem, where for slow variations of the control parameter, oscillations are delayed until the parameter has exceeded the critical value of the static-parameter problem by an amount that is the difference between the Hopf value and the initial value of the parameter. In sharp contrast, in the resonance problem there is an early onset of resonance, setting in when the ramped forcing frequency is midway between its initial value and the natural frequency for resonance in the unforced problem; we term this value the jump frequency. Numerically, we find that the jump frequency is independent of the system's damping coefficient, and so we also consider the undamped problem, which is analytically tractable. The analysis of the undamped problem confirms the numerical results found in the damped problem that the maximal amplitude obtained at the jump frequency scales as $A \sim \epsilon^{-1/2}$, ϵ being the ramp rate, and that the jump frequency is midway between the initial frequency at the start of the ramp and the natural frequency of the unforced problem.

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I. INTRODUCTION

There has been much interest in systems with slowly varying control parameters. Such problems naturally arise as models of systems with multiple scales. The desire to understand certain complicated multiscale dynamics by studying the bifurcation structure of the fast processes when the slow processes are treated as slowly varying control parameters motivated the early work on the slow passage problem [1–4]. Also, the control parameters in certain systems may inherently vary on a time scale much slower than the time scale of the dynamics, such as in chemical reactions where the reactant concentrations vary slowly on the intrinsic time scale of the reactors. Such systems have also been studied in the context of the slow passage problem [5]. The slow passage effect is that transition may not occur until the parameter is considerably beyond the critical value predicted from a static bifurcation analysis (delay effect), and that the delay in onset is dependent on the initial state of the system (memory effect).

Quasistatic control parameter variations are often implemented in experimental investigations of dynamical systems with the hope that for slow-enough variations of the parameter, the system will adjust adiabatically and the instantaneous state is close to the asymptotic state for the corresponding static value of the parameter [6–8]. However, as we show in this study, the system response to a slow variation of a control parameter can lead to large deviations from the expected adiabatically adjusted parameter response and in the simple model used to illustrate this in this paper, the deviation can be significant even in the limit of very slow parameter variations.

In the slow passage through resonance tongues of Mathieu's equation, which can be viewed as representing an undamped parametrically forced pendulum, it has been noted [9,10] that for very slow variations of the relative forcing frequency, the time spent in the resonance tongue is longer, and so one might expect a larger resonant response for slower ramping rates. However, numerical simulations and Wentzel-Kramers-Brillouin (WKB) analysis has shown that this is not always the case, depending on initial conditions and relative forcing amplitudes [9,10].

Resonance is the tendency of a periodically forced system to oscillate with larger amplitude at certain frequencies than at others. When the forcing frequency ω_f is approximately equal to a natural frequency ω_n of the unforced system, the system has a large amplitude response and resonance is said to have occurred. In nature, resonance occurs widely and is exploited in many devices. When a control parameter is slowly varied in time and especially as it passes through the natural frequency, it is very important to investigate the resultant dynamical behavior. In general, the system response will differ from the corresponding static system response.

In this paper we investigate the effects of slow passage through resonance associated with the linear periodically forced damped pendulum, an ODE system which is typically used to exemplify resonance phenomena, rather than the parametrically forced problem associated with Mathieu's equation. To investigate resonant phenomena from the slow-passage point of view, the forcing frequency is considered as a slowly varying-in-time control parameter, and then we study the resulting dynamical behavior.

In typical slow passage problems, after a ramped parameter passes its critical static-problem value, nonintuitive dynamical phenomena are observed [11,12]. For instance, slow passage through a Hopf bifurcation exhibits a delayed onset of oscillations, known as the delay effect. In the slow passage through resonance studied here, we find instead an early

*yhdo@knu.ac.kr

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effect, consisting of the onset of oscillations with a maximal amplitude, or the onset of resonance, prior to the parameter reaching the critical static problem value.

An important question in the slow passage problem is "when does resonant phenomena occur?" To investigate this problem, we consider several factors including damping coefficient, ramping rate, and initial frequency to find the onset condition. In Sec. II, we describe the static model equation and its static parameter dynamics and then define the slow passage problem. For a slowly varying-in-time frequency, the slow passage problem is numerically investigated in Sec. III. In Sec. IV, we derive the onset condition for resonant phenomena using an analytical solution in the limit of zero damping, and relate these analytic results to numerical results for very small damping. Conclusions are presented in Sec. V.

II. RESONANCE MODEL

To investigate the effects of a slow passage through a resonance, we consider as a paradigmatic model,

$$\ddot{x} + \gamma \dot{x} + x = \sin(\omega_f t), \tag{1}$$

where γ is a damping coefficient and ω_f is a forcing frequency. Resonance is the tendency of a system to oscillate with large amplitude at some forcing frequencies. These are known as the system's *resonant frequencies* ω_r. When damping is small, the resonant frequency is approximately equal to a *natural frequency* of the unforced system [zero right-hand side in Eq. (1)]. For $\gamma = 0$ (undamped case), the natural frequency $\omega_n = 1$, and damping adjusts the natural frequency to $\omega_n = \sqrt{1-\gamma^2/4}$. For small damping γ, when the forcing frequency ω_f is close to the natural frequency ω_n, the amplitude of the forced response (the maximal value of $|x(t)|$) is quite large even for relatively small external forces. This is called a *resonant effect*. Figure 1 shows the amplitude of the system in response to the forcing frequency for various small values of the damping coefficient. Note that the resonant frequency ω_r is equal to the natural frequency ω_n. The maximal amplitude A occurs at $\omega_f = \omega_n$ and is given by

$$A \sim \frac{1}{\gamma \omega_n}\left(1 + \frac{\gamma^2}{8}\right). \tag{2}$$

When a control parameter varies slowly in time, and especially as it passes through a critical value, it is very important to investigate the resultant dynamical behavior as, in general, it will differ from the corresponding static-parameter dynamics. Consider the forcing frequency ω_f as a slowly varying parameter in time, that is,

$$\omega_f(t) = \omega_0 + \epsilon t, \tag{3}$$

where ω_0 is an initial frequency and ϵ is a ramp rate (typically, $|\epsilon| \ll 1$). Introducing $\omega_f(t)$ into (1) results in

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\gamma y - x + \sin(\omega_f t), \quad \frac{d\omega_f}{dt} = \epsilon. \end{aligned} \tag{4}$$

Now the forcing frequency $\omega_f(t)$ is a slow varying parameter.

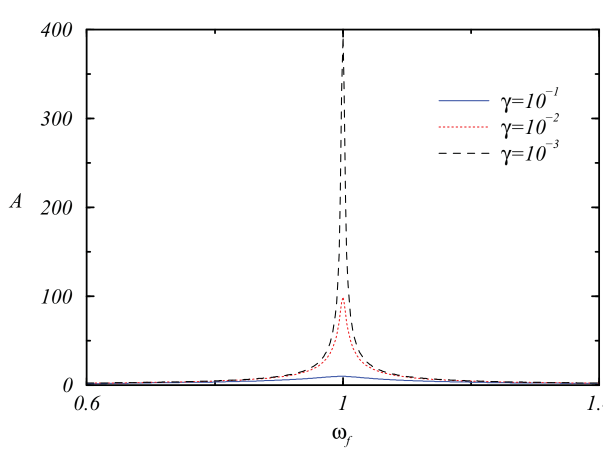

FIG. 1. (Color online) Response amplitude A vs forcing frequency ω_f for (1), for different damping coefficients γ as indicated. As a forcing frequency approaches the natural frequency $\omega_n \approx 1$, the amplitude of a phase variable x is much increased.

III. SLOW PASSAGE THROUGH RESONANCE

A. Numerical investigation

We begin by numerically integrating (4) to determine the dynamic response when the forcing frequency $\omega_f(t)$ is slowly varied near the natural frequency ω_n. We consider various fixed values of the ramp rate ϵ, damping coefficient γ, and initial forcing frequency ω_0 in Eq. (4). For $\epsilon = 10^{-6}$, $\gamma = 10^{-2}$, and $\omega_0 = 0.6$, Fig. 2 shows the behavior of a trajectory $x(t)$. Initially, the trajectory oscillates with small amplitude. As the forcing frequency approaches midway between the initial frequency and the natural frequency, the amplitude of the trajectory is suddenly amplified, giving a large resonant response at a forcing frequency relatively far from the natural frequency. Further increasing ω_f beyond the midway frequency, the amplitude of the trajectory is reduced and there is no sign of a resonant response near the natural frequency $\omega_n \approx 1$.

Comparing the envelope of $x(t)$ for the slow passage problem in Fig. 2 with the maximal amplitude A in the static response diagram (Fig. 1) for the cases with $\gamma = 10^{-2}$, we find that the slow passage response gives the same type of resonant

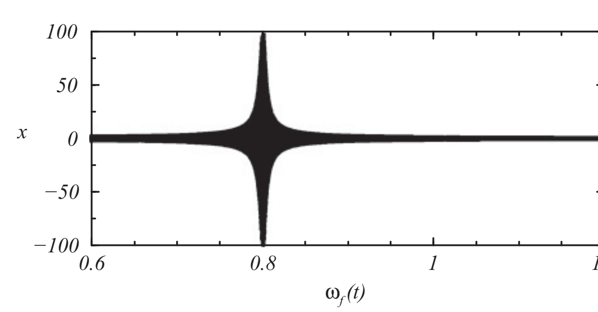

FIG. 2. Early effect. The trajectory for (4) with $\gamma = 10^{-2}$ is plotted as a function of the frequency $\omega_f(t) = \omega_0 + \epsilon t$, where $\omega_0 = 0.6$ and $\epsilon = 10^{-6}$. The maximal amplitude occurs at the jump frequency $\omega_j = 0.8$, which is dependent on the damping coefficient, before the natural frequency $\omega_n = 1$ is reached.

response, but at a frequency halfway between the initial and natural frequencies.

In typical slow passage problems, most notably the slow passage through a Hopf bifurcation, it is not until *after* the control parameter passes through its critical value that interesting dynamical phenomena is observed, that is, the delayed onset of oscillations known as the *delay effect* [1]. However, as shown in Fig. 2, resonance phenomena associated with a slow passage problem may occur *prior* to the parameter reaching its critical value; in this case the control parameter is the forcing frequency, and its critical value is the natural frequency of the static-parameter problem since resonance occurs at the natural frequency in the static problem. This is a very interesting phenomenon. We refer to it as an *early effect* to contrast it with the well-known delay effect.

An important question in slow passage problems is "at what frequency does the maximal amplitude occur or when does resonant phenomena occur?" For instance, from Fig. 2, the maximal amplitude can be obtained at the forcing frequency $\omega_f \approx 0.8$, which is not the resonant frequency. We call this special frequency a *jump frequency* ω_j in order to distinguish it from the resonant frequency in the static-parameter problem (1). In the following section, we find the onset condition for the jump frequency ω_j and investigate its dependence on other factors including the ramp rate, the damping coefficient, and the initial forcing frequency.

B. Jump frequency

To determine the onset condition for the jump frequency ω_j, the maximal amplitude of the trajectory for different initial frequencies ω_0 is investigated for various fixed ramp rates ϵ and damping coefficients γ. Figure 3 shows several trajectories with different initial forcing frequencies, and all with $\gamma = 10^{-2}$ and $\epsilon = 10^{-6}$. Numerically, we find that the maximal amplitude occurs at the midfrequency between the initial frequency ω_0 and the natural frequency $\omega_n \approx 1$. We may anticipate that ω_j

$$\omega_j = \frac{\omega_0 + \omega_n}{2} \quad \text{as} \quad \epsilon \to 0. \tag{5}$$

Analytical results supporting this in the limit of zero damping is shown in Sec. IV.

The important thing for the onset condition is that the jump frequency ω_j depends strongly on the initial forcing frequency ω_0. That is, when $\epsilon \to 0$, the jump frequency ω_j converges to the midfrequency between the initial frequency and the resonant frequency. Next, we consider how sensitive ω_j is to the ramp rate ϵ.

C. Ramp rate

For a fixed initial frequency ω_0, we now consider how the ramp rate affects the dynamical behavior in a slow passage through resonance problem. Figure 4(a) shows, on a logarithmic scale, the maximal amplitudes of trajectories for different ramp rates ϵ; there is a critical ramp rate ϵ_c for a given damping coefficient γ. That is, for $\epsilon \ll \epsilon_c$, there is no change in maximal amplitude A [the flat part in Fig. 4(a)], but for $\epsilon \gg \epsilon_c$, the maximal amplitude A depends on ϵ according to

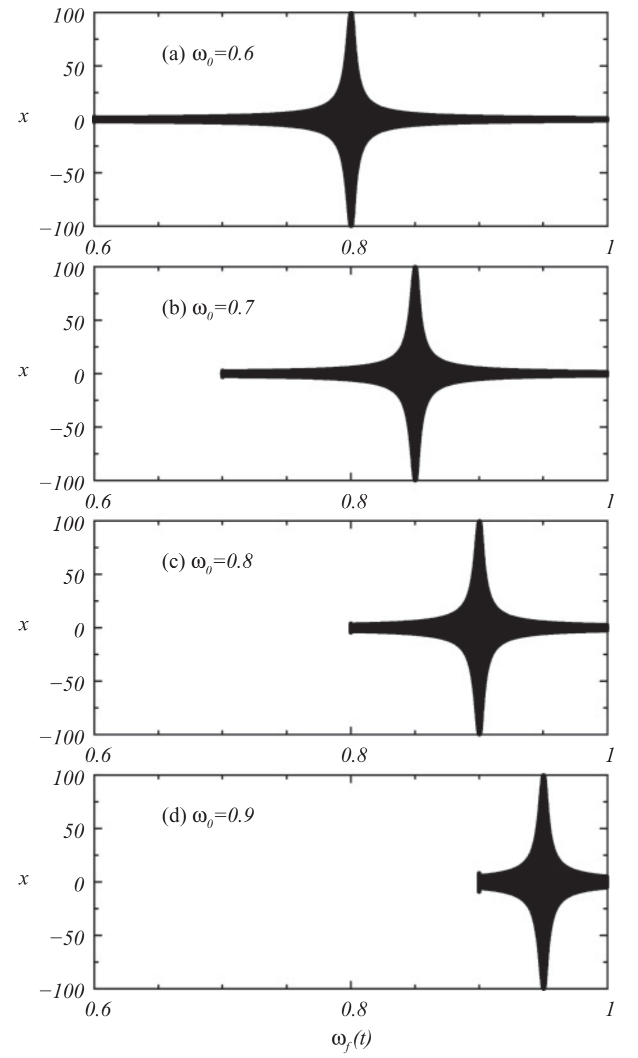

FIG. 3. Trajectories for (4) are plotted as a function of the frequency $\omega_f(t) = \omega_0 + \epsilon t$, for different initial frequencies ω_0 and $\epsilon = 10^{-6}$ and $\gamma = 10^{-2}$.

$$A \sim \left\{ \begin{array}{ll} c_\gamma & \text{for} \quad \epsilon \ll \epsilon_c, \\ \epsilon^{-1/2} & \text{for} \quad \epsilon \gg \epsilon_c, \end{array} \right\}, \tag{6}$$

where c_γ are constants that depend on γ. As the damping coefficient γ goes to zero, the maximal amplitude converges to the maximal amplitude curve for the undamped system, which has scaling $A \sim \epsilon^{-1/2}$.

For $\epsilon \ll \epsilon_c$, the resonant response amplitude A is independent of ϵ. This can be simply interpreted as being due to the forcing frequency remaining in the neighborhood of the jump frequency long enough for the trajectory to build up to a full resonant response amplitude. If $\epsilon \gg \epsilon_c$, then $\omega_f(t)$ passes past ω_j too quickly for the trajectory to achieve maximal amplitude. While for $\epsilon \ll \epsilon_c$ the response amplitude is essentially the same as for the static-parameter problem with $\omega_f \approx \omega_n$, the frequency ω_j at which this resonant response occurs during a slow passage is, in general, very different than ω_n (depending strongly on ω_0, but not on ϵ or γ). This

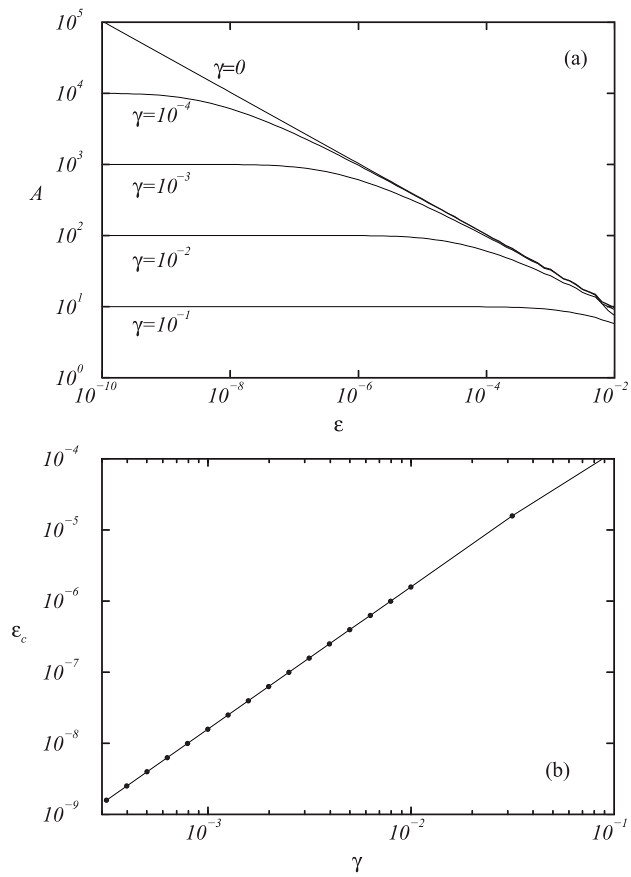

FIG. 4. (a) Maximal amplitude A vs ramp rates ϵ for different damping coefficients γ, as indicated. (b) Critical ramp rate vs damping coefficient.

has important consequences when trying to use quasistatic parameter variations to determine static-parameter dynamics: It does not matter how slowly the parameter is varied; the jump frequency will differ from the natural frequency. One could conduct a series of experiments with different ω_0 and bracket ω_n to within some precision, but this is not typically done. Furthermore, these results imply that there is a *maximal ramp rate* ϵ_c for which resonant response amplitudes comparable to those in the static-parameter problem can be recovered.

As shown in Fig. 4(a), we also observe that the critical ramp rate is affected by the damping coefficient γ. For a given damping coefficient, the critical ramp rate can be conveniently defined as the ramp rate at which the maximal amplitude attains a constant value. Numerically, we find from Fig. 4(b) that the critical ramp rate depends on the damping coefficient in the following manner:

$$\epsilon_c \sim \gamma^{\alpha}, \tag{7}$$

where we find numerically that $\alpha \approx 2$. This implies that $\gamma \sim \sqrt{\epsilon_c}$. From (2), we deduce the relation between maximal amplitude and ramping rate:

$$A \sim \frac{1}{\gamma} \sim \frac{1}{\sqrt{\epsilon_c}}.$$

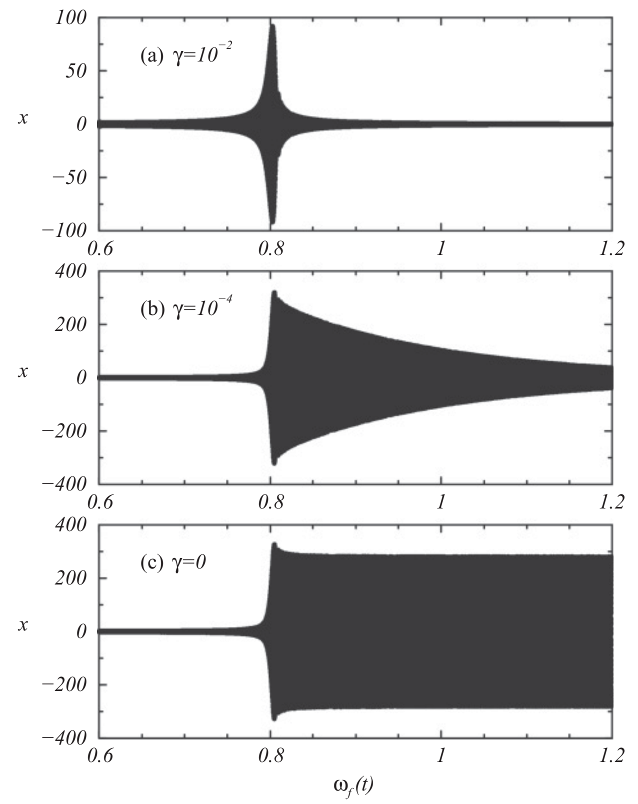

FIG. 5. Trajectories for a fixed ramp rate $\epsilon = 10^{-5}$ and damping coefficients γ, as indicated.

D. Damping limit

Typically, damping tends to reduce the amplitude of oscillations in an oscillatory system. In this section, we investigate the influence of damping on the jump frequency in the slow passage through resonance problem. For fixed ramp rate and initial forcing frequency, Fig. 5 shows trajectory behaviors for different damping coefficients. Note that the jump frequency is independent of the damping, but the amplitude of the response is damping dependent. For larger damping ($\gamma = 10^{-2}$), after the forcing frequency has passed the jump frequency value, the trajectory oscillations are strongly damped. As γ is decreased (to, say, $\gamma = 10^{-3}$), the oscillations are more slowly damped, leaving a long damped oscillatory tail as the forcing frequency

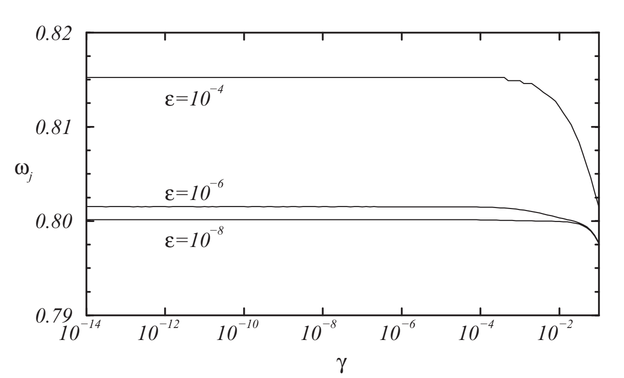

FIG. 6. Jump frequency ω_j versus damping coefficient γ for a fixed initial frequency $\omega_0 = 0.6$ and various ramping rates $\epsilon = 10^{-4}$, 10^{-6}, and 10^{-8}, as indicated.

continues to increase. In the undamped case, after ω_f exceeds ω_j, the trajectory acquires a constant amplitude oscillation. In Fig. 6, the jump frequency is numerically computed as $\gamma \to 0$ for a fixed initial frequency $\omega_0 = 0.6$ and various ramping rates $\epsilon = 10^{-4}$, 10^{-6} and 10^{-8}. The figure shows that the jump frequency becomes independent of the damping coefficient for $\gamma \lesssim 10^{-4}$. This implies that to theoretically predict a jump frequency, it is sufficient to consider an undamped system, for which we are able to obtain analytic results.

IV. ANALYTICAL RESULTS

Consider the zero damping limit of (4):

$$\ddot{x} + x = \sin[\omega_f(t)t], \tag{8}$$

where $\omega_f(t) = \omega_0 + \epsilon t$ and ω_0 is an initial forcing frequency. This equation has a solution of the form

$$x(t) = x_h(t) + x_p(t), \tag{9}$$

where $x_h(t)$ is the general solution of the corresponding homogeneous equation and $x_p(t)$ is a particular solution of the full nonhomogeneous equation. Here

$$x_h(t) = c_1 \cos(t) + c_2 \sin(t), \tag{10}$$

and

$$\begin{aligned} x_p(t) = \frac{\sqrt{2\pi}}{4\sqrt{\epsilon}}\{&S[T_1(t)]\sin[T_3(t)] - C[T_1(t)]\cos[T_3(t)] \\ &+ S[T_2(t)]\sin[T_4(t)] + C[T_2(t)]\cos[T_4(t)]\}, \end{aligned} \tag{11}$$

where $S(t) = \int_0^t \sin(\frac{\pi}{2}s^2)ds$ and $C(t) = \int_0^t \cos(\frac{\pi}{2}s^2)ds$ are Fresnel integrals and

$$T_1(t) = \frac{2\epsilon t - 1 + \omega_0}{\sqrt{2\epsilon\pi}}, \quad T_2(t) = \frac{2\epsilon t + 1 + \omega_0}{\sqrt{2\epsilon\pi}},$$

$$T_3(t) = t - \frac{1}{\epsilon}\left(\frac{1 - w_0}{2}\right)^2, \quad T_4(t) = t + \frac{1}{\epsilon}\left(\frac{1 + w_0}{2}\right)^2.$$

From a straightforward computation of $\dot{x_p}$ and $\ddot{x_p}$, one may easily check that x_p is a solution of (4).

Letting

$$t_j = \frac{(1 - \omega_0)}{2\epsilon}, \tag{12}$$

then t_j is the time when $\omega_f(t)$ arrives at the jump frequency ω_j; that is, $\omega_f(t_j) = \omega_j$. To approximate $x(t)$ with small initial conditions, that is, $|x(0)|, |\dot{x}(0)| \ll 1$, we may use the following properties of the Fresnel integrals:

$$S(x) \sim \tfrac{1}{2}, \quad C(x) \sim \tfrac{1}{2}, \quad \text{for} \quad x \gg 0,$$
$$S(x) \sim -\tfrac{1}{2}, \quad C(x) \sim -\tfrac{1}{2}, \quad \text{for} \quad x \ll 0.$$

For $0 \leqslant t \ll t_j$,

$$S[T_1(t)] \sim -1/2, \quad C[T_1(t)] \sim -1/2,$$
$$S[T_2(t)] \sim 1/2, \quad C[T_2(t)] \sim 1/2,$$

and for $t \gg t_j$,

$$S[T_1(t)] \sim 1/2, \quad C[T_1(t)] \sim 1/2,$$
$$S[T_2(t)] \sim 1/2, \quad C[T_2(t)] \sim 1/2.$$

This implies that

$$x_p(t) \sim \begin{cases} \frac{\sqrt{\pi}}{2\sqrt{\epsilon}} \cos\left(-\frac{1+\omega_0^2}{4\epsilon} + \frac{\pi}{4}\right)\cos\left(t + \frac{\omega_0}{2\epsilon}\right) & \text{for} \quad t \ll t_j, \\ \frac{\sqrt{\pi}}{2\sqrt{\epsilon}} \sin\left(-\frac{1+\omega_0^2}{4\epsilon} + \frac{\pi}{4}\right)\sin\left(t + \frac{\omega_0}{2\epsilon}\right) & \text{for} \quad t \gg t_j. \end{cases} \tag{13}$$

To obtain a full solution $x(t) = x_p(t) + c_1\cos(t) + c_2\sin(t)$, we consider small initial conditions to study resonant effects in the slow passage problem, that is, $|x(0)|, |\dot{x}(0)| \ll 1$, due to the nature of the unforced system having concentric periodic trajectories encircling the origin. We have

$$c_1 \sim \frac{\sqrt{\pi}}{2\sqrt{\epsilon}}\cos\left(-\frac{1+\omega_0^2}{4\epsilon} + \frac{\pi}{4}\right)\cos\left(\frac{\omega_0}{2\epsilon}\right), \tag{14}$$

and

$$c_2 \sim \frac{\sqrt{\pi}}{2\sqrt{\epsilon}}\cos\left(-\frac{1+\omega_0^2}{4\epsilon} + \frac{\pi}{4}\right)\sin\left(\frac{\omega_0}{2\epsilon}\right). \tag{15}$$

Substitution of the constants c_1, c_2 and a particular solution $x_p(t)$ into the full solution gives

$$x(t) \sim \begin{cases} 0 & \text{for} \quad t \ll t_j, \\ -\frac{\sqrt{\pi}}{2\sqrt{\epsilon}}\cos\left[t - \frac{1}{\epsilon}\left(\frac{1-\omega_0}{2}\right)^2 + \frac{\pi}{4}\right] & \text{for} \quad t \gg t_j. \end{cases} \tag{16}$$

Therefore, the maximal amplitude A is proportional to $\epsilon^{-1/2}$ and resonance occurs at ω_j. For large initial conditions, we find that the amplitude of oscillation is deamplified at the jump frequency [13].

V. CONCLUSIONS

We have studied the effects of a slow variation in a control parameter on the response of a system as it passes through a resonance. Our main result is the discovery of an early effect, whereby resonance phenomena (i.e., a large amplitude response) occurs prior to the slowly varying parameter reaching its critical value in the corresponding static-parameter problem. In the problem studied, the control parameter is the forcing frequency and its critical value is the natural frequency of the static-parameter model. This early effect is in sharp contrast to the well-known delay effect observed in the slow passage through a Hopf bifurcation. This nonintuitive slow passage response can have far reaching dynamical consequences in many practical situations. For example, when parameters are slowly varied to reach an end state in a controlled fashion, unexpected large scale oscillations may be triggered with unforeseen consequences.

We have shown that the onset condition for the early effect is when the slowly varying forcing frequency attains a value that is midway between an initial frequency and the natural frequency. We have called this the jump frequency. The onset of the early effect is strongly dependent on the initial frequency. Through numerical investigations of the influence of other system parameters, including the ramping rate and the damping coefficient, we have also found that there is a critical ramping rate ϵ_c, where for $\epsilon \ll \epsilon_c$, the resonant response amplitude reaches the maximal amplitude found in the corresponding static-parameter model. The value of this critical ramping rate depends on the damping coefficient; when the forcing frequency varies slowly enough, the trajectory has time to resonate at the jump frequency before it is damped out. For $\epsilon \gg \epsilon_c$, the forcing frequency is ramped past the jump frequency too fast for a trajectory to achieve maximal amplitude. This means that ϵ must be less than ϵ_c in order for the ramped response amplitude to be comparable to that of the static-parameter problem. However, even with $\epsilon \ll \epsilon_c$, the resonant response does not necessarily occur at the natural frequency, but at the jump frequency which is midway between the natural frequency and the initial frequency. Hence, in general, the slow passage with resonance problem will not recover the static-parameter dynamics irrespective of how slowly the parameter is ramped.

Based on the numerical observations that the jump frequency is independent of the damping, we considered the undamped problem which is analytically tractable and confirmed the numerical results that the maximal amplitude obtained at the jump frequency scales as $A \sim \epsilon^{-1/2}$ and that the jump frequency is midway between the initial frequency at the start of the ramp and the natural frequency of the unforced problem.

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