Exact dynamical state of the exclusive queueing process with deterministic hopping

Chikashi Arita*
Institut de Physique Théorique, CEA Saclay, F-91191, Gif-sur-Yvette, France

Andreas Schadschneider†
Institut für Theoretische Physik, Universität zu Köln, D-50937 Köln, Germany
(Received 1 September 2011; published 28 November 2011)

The exclusive queueing process (EQP) has recently been introduced as a model for the dynamics of queues that takes into account the spatial structure of the queue. It can be interpreted as a totally asymmetric exclusion process of varying length. Here we investigate the case of deterministic bulk hopping $p = 1$ that turns out to be one of the rare cases where exact nontrivial results for the dynamical properties can be obtained. Using a time-dependent matrix product form we calculate several dynamical properties, e.g., the density profile of the system.

DOI: 10.1103/PhysRevE.84.051127 PACS number(s): 02.50.−r, 05.70.Ln

I. INTRODUCTION

The one-dimensional asymmetric exclusion process, which can be regarded as the prototypical stochastic interacting particle system [1], has been intensively studied in view of its nonequilibrium properties [2], exact solvability [3,4] and applicability to practical problems [5]. The state space for the exclusion process is the set of configurations of particles (in other words, the exclusion process has a spatial structure), and each particle can hop to its nearest-neighbor sites only if the target site is empty (excluded-volume effect).

On the other hand, the queueing process is one of the basic stochastic processes in the field of operations research [6–8]. In addition to its practical relevance it often appears as effective model, e.g., in all kinds of jamming phenomena. Usually the spatial structure of the queue is neglected, i.e., the queues are regarded as "compact." However, often this assumption is not justified, e.g., in pedestrian queues. Therefore, recently a queueing process with excluded-volume effect [*exclusive queueing process* (EQP)] has been proposed [9–11]. On a semi-infinite lattice, particles enter the system at the left site next to the leftmost occupied site and leave the system at the rightmost site. In the bulk the particles move according to the rules of the totally asymmetric exclusion process (TASEP); see Fig. 1.

The EPQ can be interpreted as a TASEP with a varying system length that allows us to analyze its stationary-state properties [9–11]. In a more recent article [12], the dynamical properties of the EQP were analyzed. Especially for the deterministic bulk hopping case, the dynamical behaviors of the average system length and the average number of particles were investigated exactly. In this article we derive more detailed results for the dynamical properties.

The stationary state of the TASEP and some of its generalizations have been solved by means of the matrix product ansatz in the recent two decades [4]. The application of the matrix product ansatz to the calculation of nonstationary states is quite challenging and has been achieved only in a few cases so far; see, e.g., Refs. [13–17]. In this article we will introduce a matrix product dynamical state for the EQP, providing an explicit representation for the matrices. We utilize it for calculating typical quantities both in queueing theory and exclusion processes.

Here we define the EQP as a discrete-time Markov process on a semi-infinite chain where sites are labeled by natural numbers from right to left (Fig. 1). A new particle enters the chain with probability α only at the left site next to the leftmost occupied site ($j = L$). If there is no particle on the chain, a new particle enters at the (fixed) rightmost site ($j = 1$) with probability α. Each particle on the chain necessarily hops to its right nearest-neighbor site if it is empty, i.e., we consider the limit of deterministic bulk hopping ($p = 1$). A particle on the rightmost site leaves the system with probability β. These transitions occur simultaneously within one time step, i.e., we apply the fully parallel update scheme. Since we restrict our consideration to the case of deterministic bulk hopping $p = 1$ (the so-called rule 184 cellular automaton), the stochasticity of the model is due to only the injection and extraction probabilities α and β.

In Refs. [10–12], the phase diagram of the EQP was derived. The parameter space is divided into two regions (Fig. 2), the convergent phase $\alpha < \frac{\beta}{1+\beta}$ and the divergent phase $\alpha > \frac{\beta}{1+\beta}$. In the convergent phase, the system approaches a stationary state that can be written in a matrix product form. On the other hand, in the divergent phase, a stationary state does not exist, and the average length of the system $\langle L_t \rangle$ and the average number of particles $\langle N_t \rangle$ increase asymptotically linearly in time t. On the "critical line" $\alpha = \frac{\beta}{1+\beta}$, both $\langle L_t \rangle$ and $\langle N_t \rangle$ exhibit diffusive behavior, i.e., they increase being asymptotically proportional to $\sqrt{t}$.

In this article, we investigate the dynamical (i.e., time-dependent) properties of the EQP in more detail. In the next section we write down the dynamical state (solution to the master equation) in a matrix product form. Using this form we investigate the waiting time, which is one of the basic quantities in queueing theory, in Sec. III. In Sec. IV we determine the density profile and the particle current profile. Concluding remarks are given in Sec. V. In the Appendix we review

*chikashi.arita@cea.fr
†as@thp.uni-koeln.de

 ©2011 American Physical Society

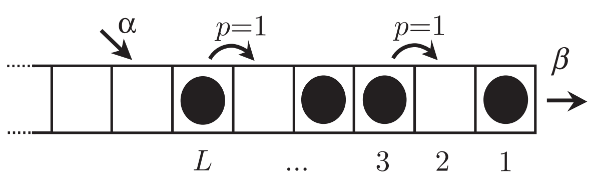

FIG. 1. Exclusive queueing process (EPQ) with deterministic bulk hopping ($p = 1$).

results on the usual (i.e., without excluded-volume effect) discrete-time queueing process.

II. EXACT DYNAMICAL STATE

For each site j we define the state variable $\tau_j = 1$ or 0 corresponding to being occupied or unoccupied, respectively. For simplicity we impose the initial condition that there is no particle in the system, i.e., an empty chain. The state space is

$$\begin{aligned}\widetilde{S} &= \{\emptyset\} \cup \{\sigma_\ell \cdots \sigma_1 | \ell \in \mathbb{N}, \sigma_j \in \{1,10\}\} \\ &= \{\emptyset, 1, 10, 11, 110, 101, 1010, 111, \ldots\},\end{aligned} \tag{1}$$

which is a subset ($\widetilde{S} \subset S$) of

$$S = \{\emptyset, 1\} \cup \{1\tau_{L-1} \cdots \tau_1 | L-1 \in \mathbb{N}, \tau_j \in \{1,0\}\}. \tag{2}$$

The element $\emptyset$ corresponds to the state where there is no particle in the system. Note that for $p = 1$, the sequence 00 never appears if the system starts from the empty chain. For simplification, we do not write the infinite number of 0's left to the leftmost particle. We denote the probability of finding a state $\tau \in \widetilde{S}$ at time t by $P_t(\tau)$, and the initial condition is written as $P_0(\emptyset) = 1$ and $P_0(\tau) = 0$ ($\tau \in \widetilde{S} \setminus \{\emptyset\}$). We denote the system "length" for the state $\tau \in \widetilde{S}$ or $\tau \in S$ by $|\tau|$, which is nothing but the position of the leftmost particle (Fig. 1). In particular we define $|\emptyset| = 0$.

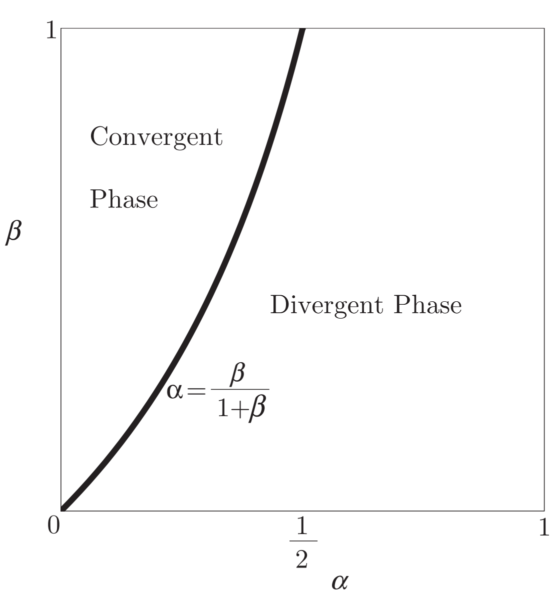

FIG. 2. Phase diagram for the EQP with $p = 1$.

For the generic choice of the parameters $0 < \alpha < 1$ and $0 < \beta < 1$, the process is irreducible and nonperiodic on $\widetilde{S}$. The master equation is simply written as

$$P_{t+1}(\emptyset) = (1-\alpha)\beta P_t(1) + (1-\alpha)P_t(\emptyset), \tag{3}$$

$$P_{t+1}(1) = (1-\alpha)P_t(10) + (1-\alpha)(1-\beta)P_t(1) + \alpha P_t(\emptyset), \tag{4}$$

$$\begin{aligned}P_{t+1}(u10) &= (1-\alpha)\beta P_t(1u1) + (1-\alpha)\beta P_t(10u1) \\ &\quad + \alpha\beta P_t(u1),\end{aligned} \tag{5}$$

$$\begin{aligned}P_{t+1}(u101) &= (1-\alpha)P_t(1u10) + (1-\alpha)P_t(10u10) \\ &\quad + \alpha P_t(u10),\end{aligned} \tag{6}$$

$$\begin{aligned}P_{t+1}(u11) &= (1-\alpha)(1-\beta)P_t(1u1) + (1-\alpha)(1-\beta) \\ &\quad \times P_t(10u1) + \alpha(1-\beta)P_t(u1)\end{aligned} \tag{7}$$

for $u \in \widetilde{S}$. In particular, for $u = \emptyset$ we set $\emptyset 10 = 10$, $1\emptyset 1 = 11$, and so on. This simple form is due to the deterministic hopping $p = 1$.

We derive an exact dynamical state, beginning with the factorization ansatz

$$P_t(\tau) = Q_t(|\tau|)Y(\tau). \tag{8}$$

The first part, Q, depends only on time and the system length, and the second part, Y, is independent of time and satisfies the following relations:

$$Y(u_1 101 u_2) = \beta Y(u_1 1 u_2), \tag{9}$$

$$Y(u_1 11 u_2) = (1-\beta)Y(u_1 1 u_2), \tag{10}$$

$$Y(u_1 10) = \beta Y(u_1 1), \tag{11}$$

$$Y(1) = 1, \tag{12}$$

$$Y(\emptyset) = 1. \tag{13}$$

One can easily see that the solution to these relations is

$$Y(\tau_L \cdots \tau_1) = \beta^{\#\{j|\tau_j=0\}}(1-\beta)^{2\#\{j|\tau_j=1\}-L-\tau_1} \tag{14}$$

for $\tau_L \cdots \tau_1 \in \widetilde{S} \setminus \{\emptyset\}$. The relations (9)–(12) also have the following matrix product representation, which is more convenient later:

$$Y(\tau_L \cdots \tau_1) = \langle W | X_{\tau_L} \cdots X_{\tau_1} | V \rangle \tag{15}$$

with

$$X_1 = D = \begin{pmatrix} 1-\beta & 0 \\ \sqrt{\beta} & 0 \end{pmatrix}, \quad X_0 = E = \begin{pmatrix} 0 & \sqrt{\beta} \\ 0 & 0 \end{pmatrix}, \tag{16}$$

$$\langle W | = (1 \ \sqrt{\beta}), \quad |V\rangle = \begin{pmatrix} 1 \\ \sqrt{\beta} \end{pmatrix}. \tag{17}$$

These are essentially the same matrices and vectors as for the matrix product *stationary* state for the EQP with $p = 1$ [11]. The first part $Q_t(L)$ gives the probability that the system length is L at time t since

$$\sum_{\substack{\tau \in \widetilde{S} \\ |\tau| = L}} Y(\tau) = \sum_{\substack{\tau \in S \\ |\tau| = L}} Y(\tau) = \langle W | D(D+E)^{L-1} | V \rangle = 1. \tag{18}$$

Note that we can replace $\widetilde{S}$ by S in the above equation thanks to $E^2 = 0$. Inserting the relations (9)–(13) into the master

Eqs. (3)–(7), we obtain

$$Q_{t+1}(0) = (1-\alpha)Q_t(0) + \beta(1-\alpha)Q_t(1), \tag{19}$$

$$Q_{t+1}(L) = \alpha Q_t(L-1) + (1-\alpha)(1-\beta)Q_t(L) + (1-\alpha)\beta Q_t(L+1). \tag{20}$$

These equations actually agree with Eqs. (65) and (66) in Ref. [12] that were derived in a different way. The solution to this recurrence formula with the initial condition

$$Q_0(0) = 1, \quad Q_0(L) = 0\ (L \in \mathbb{N}) \tag{21}$$

is given by [12]

$$Q_t(L) = C_{z^t} \frac{1-\Lambda}{1-z}\Lambda^L \tag{22}$$

with

$$\Lambda = \frac{1-(1-\alpha)(1-\beta)z - r}{2(1-\alpha)\beta z}, \tag{23}$$

$$r = \sqrt{[1-(1-\alpha)(1-\beta)z]^2 - 4(1-\alpha)\alpha\beta z^2}, \tag{24}$$

where $C_{z^t}F(z)$ denotes the coefficient of z^t in the Laurent series for the function $F(z)$, that is, $C_{z^t}F(z) = \oint \frac{dz}{2\pi i z^{t+1}}F(z)$ with a small counterclockwise path enclosing the origin of the complex plane. The average system length $\langle L_t\rangle$ at time t is derived as [12]

$$\langle L_t\rangle = \sum_{L\geqslant 0} L Q_t(L) = C_{z^t}\frac{\Lambda}{(1-z)(1-\Lambda)} \tag{25}$$

$$\simeq \begin{cases} \frac{\alpha}{\beta-\alpha-\alpha\beta} & \left(\alpha < \frac{\beta}{1+\beta}\right) \\ 2\sqrt{\frac{\beta t}{\pi(1+\beta)}} & \left(\alpha = \frac{\beta}{1+\beta}\right), \\ (\alpha-\beta+\alpha\beta)t & \left(\alpha > \frac{\beta}{1+\beta}\right) \end{cases} \tag{26}$$

for $t \to \infty$.

Inserting Eqs. (15) and (22) into Eq. (8), we obtain the *matrix product dynamical state*

$$P_t(\emptyset) = C_{z^t}\frac{1-\Lambda}{1-z}, \tag{27}$$

$$P_t(\tau_L\cdots\tau_1) = C_{z^t}\frac{1-\Lambda}{1-z}\Lambda^L\langle W|X_{\tau_L}\cdots X_{\tau_1}|V\rangle. \tag{28}$$

When $\alpha < \frac{\beta}{1+\beta}$ (convergent phase), the matrix product dynamical state converges to the matrix product stationary state [11]

$$\lim_{t\to\infty} P_t(\emptyset) = \lim_{z\to 1}(1-\Lambda) = \frac{\beta-\alpha-\alpha\beta}{\beta(1-\alpha)}, \tag{29}$$

$$\lim_{t\to\infty} P_t(\tau_L\cdots\tau_1) = \lim_{z\to 1}(1-\Lambda)\Lambda^L\langle W|X_{\tau_L}\cdots X_{\tau_1}|V\rangle = \frac{\beta-\alpha-\alpha\beta}{\beta(1-\alpha)}\left[\frac{\alpha}{(1-\alpha)\beta}\right]^L \times \langle W|X_{\tau_L}\cdots X_{\tau_1}|V\rangle. \tag{30}$$

III. WAITING TIME

The waiting time is one of the most important quantities in queueing theory, which corresponds to the number of time steps that a particle needs to leave the system after entering the system.

Before we derive the waiting time distribution we determine the distribution of the number N of particles in the system. In standard queueing theory N is always identical to the length L of the system since the queue has no internal structure. In the EQP we only know that, by definition, N cannot be larger than L. The probability that the number of particles is $N=0$ at time t is, of course, equal to $Q_t(0)$. For $N \in \mathbb{N}$, we find:

$$P_t^{A+B}(N) = P_t^A(N) + P_t^B(N) = \sum_{\substack{\tau_L\cdots\tau_1\in\tilde{S}:\\ \#\{j|\tau_j=1\}=N}} P_t(\tau_L\cdots\tau_1) = C_{z^t}\frac{1-\Lambda}{1-z}\langle W|(\Lambda D+\Lambda^2 DE)^N|V\rangle = C_{z^t}\frac{\Lambda(1-\Lambda)(1+\beta\Lambda)}{1-z}[\Lambda(1-\beta+\beta\Lambda)]^{N-1}, \tag{31}$$

where $P_t^A(N)$ [respectively, $P_t^B(N)$] is the probability of finding N particles in the system and the site 1 being occupied (respectively, empty) at time t. We calculate $P_t^A(N)$ and $P_t^B(N)$ as well:

$$P_t^A(N\in\mathbb{N}) = C_{z^t}\frac{1-\Lambda}{1-z}\langle W|(\Lambda D+\Lambda^2 DE)^{N-1}\Lambda D|V\rangle = C_{z^t}\frac{\Lambda(1-\Lambda)}{1-z}[\Lambda(1-\beta+\beta\Lambda)]^{N-1}, \tag{32}$$

$$P_t^B(N\in\mathbb{N}) = P_t^{A+B}(N) - P_t^A(N) = C_{z^t}\frac{\beta\Lambda^2(1-\Lambda)}{1-z}[\Lambda(1-\beta+\beta\Lambda)]^{N-1}, \tag{33}$$

$$P_t^B(0) = Q_t(0) = C_{z^t}\frac{1-\Lambda}{1-z}. \tag{34}$$

We also set $P_t^A(0)=0$. Indeed Eqs. (31)–(33) agree with the results derived in Ref. [12] in a more complicated way. By using the result (31), the average number of particles at time t is found to be [12]

$$\langle N_t\rangle = \sum_{N\geqslant 1} N P_t^{A+B}(N) = C_{z^t}\frac{\Lambda}{(1-z)(1-\Lambda)(1+\beta\Lambda)} \tag{35}$$

$$\simeq \begin{cases} \frac{\alpha(1-\alpha)}{\beta-\alpha-\alpha\beta} & \left(\alpha<\frac{\beta}{1+\beta}\right) \\ 2\sqrt{\frac{\beta t}{\pi(1+\beta)^3}} & \left(\alpha=\frac{\beta}{1+\beta}\right), \\ \frac{\alpha-\beta+\alpha\beta}{1+\beta}t & \left(\alpha>\frac{\beta}{1+\beta}\right)\end{cases} \tag{36}$$

for $t\to\infty$.

Now we turn to the waiting time, i.e., the time that a new particle stays in the system. For a given number N of particles in the system, the probability that the waiting time is $T\in\mathbb{N}$ is given by

$$(A):\binom{T-N}{N}\beta^{N+1}(1-\beta)^{T-2N}=\mathcal{A}(T,N), \tag{37}$$

$$(B):\binom{T-N-1}{N}\beta^{N+1}(1-\beta)^{T-2N-1}=\mathcal{B}(T,N). \tag{38}$$

Here A (respectively, B) corresponds to the case where the rightmost site is occupied (respectively, empty). Note that $\binom{x}{y}$ denotes the binomial coefficient, which should not be confused with a two-dimensional column vector. The average waiting times *for given* N in the cases A and B are, respectively,

$$\langle T_{N,A}\rangle = \sum_{T\geqslant 2N} T\mathcal{A}(T,N) = \frac{N+1}{\beta}+N-1, \tag{39}$$

$$\langle T_{N,B}\rangle = \sum_{T\geqslant 2N+1} T\mathcal{B}(T,N) = \frac{N+1}{\beta}+N. \tag{40}$$

This result can be interpreted as follows:

(i) For each particle, it takes one time step to move from site 2 to site 1. For $N+1$ particles, it takes, in total, N time steps (respectively, $N+1$ time steps) for the case A (respectively, B).

(ii) For each particle, it takes $\frac{1}{\beta}$ time steps in average to leave the system after arriving at site 1. For $N+1$ particles, it takes, in total, $\frac{N+1}{\beta}$ time steps.

(iii) A new particle entering the system at time t does *not* wait during time t and $t+1$. Thus, we have to subtract 1 from the above.

Let us consider the probability $W_t(T)$ of the waiting time T for a particle entering the system at time t. Using Eqs. (32)–(34), (37), and (38), we find

$$W_t(T) = \sum_{N=0}^{\lfloor T/2\rfloor}\left[\mathcal{A}(T,N)P_t^A(N)+\mathcal{B}(T,N)P_t^B(N)\right] = C_{z^t}\frac{\beta(1-\Lambda)}{1-z}(1-\beta+\beta\Lambda)^{T-1}, \tag{41}$$

where $\lfloor\cdot\rfloor$ denotes the floor function, i.e., $\lfloor T/2\rfloor = T/2$ (if $T\in 2\mathbb{N}$) or $\lfloor T/2\rfloor=(T-1)/2$ (if $T\in 2\mathbb{N}-1$). In the convergent phase, $W_t(T)$ converges to the stationary distribution of the waiting time

$$\lim_{t\to\infty}W_t(T) = \lim_{z\to 1}\beta(1-\Lambda)(1-\beta+\beta\Lambda)^{T-1} = \frac{\beta-\alpha-\alpha\beta}{1-\beta+\alpha\beta}\left(\frac{1-\beta+\alpha\beta}{1-\alpha}\right)^T, \tag{42}$$

which agrees with the result in Ref. [10].

To finish this section, we investigate the average waiting time:

$$\langle T_t\rangle = C_{z^t}\frac{\beta(1-\Lambda)}{1-z}\sum_{T\geqslant 1}T(1-\beta+\beta\Lambda)^{T-1} = C_{z^t}\frac{1}{\beta(1-z)(1-\Lambda)}. \tag{43}$$

The order of the closest singularity $z=1$ to the origin depends on the parameters (α,β) [12]:

$$\lim_{z\to1}\frac{1-z}{\beta(1-z)(1-\Lambda)} = \frac{1-\alpha}{\beta-\alpha-\alpha\beta}\quad\left(\alpha<\frac{\beta}{1+\beta}\right), \tag{44}$$

$$\lim_{z\to1}\frac{(1-z)^{\frac{3}{2}}}{\beta(1-z)(1-\Lambda)} = \sqrt{\frac{1}{\beta(1+\beta)}}\quad\left(\alpha=\frac{\beta}{1+\beta}\right), \tag{45}$$

$$\lim_{z\to1}\frac{(1-z)^2}{\beta(1-z)(1-\Lambda)} = \frac{\alpha-\beta+\alpha\beta}{\beta}\quad\left(\alpha>\frac{\beta}{1+\beta}\right), \tag{46}$$

and, thus, we have

$$\langle T_t\rangle\to\frac{1-\alpha}{\beta-\alpha-\alpha\beta}\quad\left(\alpha<\frac{\beta}{1+\beta}\right), \tag{47}$$

$$\langle T_t\rangle = 2\sqrt{\frac{t}{\pi\beta(1+\beta)}}+o(\sqrt{t})\quad\left(\alpha=\frac{\beta}{1+\beta}\right), \tag{48}$$

$$\langle T_t\rangle = \frac{\alpha-\beta+\alpha\beta}{\beta}t+o(t)\quad\left(\alpha>\frac{\beta}{1+\beta}\right), \tag{49}$$

as $t\to\infty$. We note that one of the central results of queueing theory, Little's theorem [6], is indeed satisfied in the convergent phase ($\alpha<\frac{\beta}{1+\beta}$) [10]:

$$\alpha\lim_{t\to\infty}\langle T_t\rangle = \lim_{t\to\infty}\langle N_t\rangle. \tag{50}$$

We also note that, in the divergent phase and on the critical line ($\alpha\geqslant\frac{\beta}{1+\beta}$) as well as in the convergent phase, there is a physically natural relation between the average waiting time and the average number of particles:

$$J_{1t}\cdot\langle T_t\rangle\simeq\langle N_t\rangle\quad(t\to\infty). \tag{51}$$

Here J_{1t} is the current of particles passing through the exit (outflow), which will be derived in the next section. Note that this relation also holds for the usual queueing process; see the Appendix.

IV. DENSITY AND CURRENT

We consider the probability ρ_{jt} that the site j is occupied at time t, i.e., the density profile. The initial condition implies that $\rho_{jt}=0$ for $j>t$ and $\rho_{tt}=\alpha^t$. The density profile for general j and t can be calculated as

$$\begin{aligned}\rho_{jt} &= \sum_{\tau_k=0,1} P_t(1\tau_{j-1}\cdots\tau_1) + \sum_{\substack{\tau_k=0,1\\ L\geqslant j+1}} P_t(1\tau_{L-1}\cdots\tau_{j+1}1\tau_{j-1}\cdots\tau_1)\\ &= C_{z^t}\frac{1-\Lambda}{1-z}\left[\Lambda^j\langle W|D(D+E)^{j-1}|V\rangle + \sum_{L\geqslant j+1}\Lambda^L\langle W|D(D+E)^{L-j-1}D(D+E)^{j-1}|V\rangle\right]\\ &= C_{z^t}\frac{1-\Lambda}{1-z}\Lambda^j\{\langle W|D(D+E)^{j-1}|V\rangle + \Lambda\langle W|D[1-\Lambda(D+E)]^{-1}D(D+E)^{j-1}|V\rangle\}\\ &= C_{z^t}\frac{\Lambda^j}{(1-z)(1+\beta\Lambda)}.\end{aligned} \tag{52}$$

By definition, the particle current J_{1t} passing through the exit (the right end) during t and $t+1$ is given by

$$J_{1t} = \beta\rho_{1t}. \tag{53}$$

The particle current J_{jt} through the bond between the sites $j(\geqslant 2)$ and $j-1$

$$\begin{aligned}J_{jt} = &\sum_{\tau_k=0,1} P_t(10\tau_{j-2}\cdots\tau_1)\\ &+\sum_{\substack{\tau_k=0,1\\ L\geqslant j+1}} P_t(1\tau_{L-1}\cdots\tau_{j+1}10\tau_{j-2}\cdots\tau_1)\end{aligned} \tag{54}$$

also satisfies the relation

$$J_{jt} = \beta\rho_{jt} \tag{55}$$

since $DE(D+E)^{j-2}|V\rangle = \beta D(D+E)^{j-1}|V\rangle$.

For the generic choice of parameters α and β, the density profile near the right end converges as

$$\rho_{jt} \to \lim_{z\to 1}\frac{\Lambda^j}{1+\beta\Lambda} = \begin{cases}\frac{1}{1+\beta} & \left(\alpha \geqslant \frac{\beta}{1+\beta}\right)\\ (1-\alpha)\left[\frac{\alpha}{(1-\alpha)\beta}\right]^j & \left(\alpha < \frac{\beta}{1+\beta}\right)\end{cases}, \tag{56}$$

for $t\to\infty$. Here we took the limit with the site number j independent of time t. In particular, for $j=1$, we have

$$\lim_{t\to\infty}\rho_{1t} = \begin{cases}\frac{1}{1+\beta} & \left(\alpha \geqslant \frac{\beta}{1+\beta}\right)\\ \frac{\alpha}{\beta} & \left(\alpha < \frac{\beta}{1+\beta}\right)\end{cases}, \tag{57}$$

$$\lim_{t\to\infty}J_{1t} = \begin{cases}\frac{\beta}{1+\beta} & \left(\alpha \geqslant \frac{\beta}{1+\beta}\right)\\ \alpha & \left(\alpha < \frac{\beta}{1+\beta}\right)\end{cases}, \tag{58}$$

confirming the relation (51).

Let us now consider rescaled density profiles in the divergent phase and on the critical line, where the average system length grows of order t and $\sqrt{t}$, respectively [see Eq. (26)]. First, we observe that the density profile ρ_{jt} can be interpreted as the expected number of noninteracting asymmetric random walkers at time t on site j since the expression (52) satisfies the equation

$$\rho_{j,t+1} = \alpha\rho_{j-1,t} + \gamma\rho_{jt} + \delta\rho_{j+1,t} \tag{59}$$

with $\gamma = (1-\alpha)(1-\beta)$ and $\delta = (1-\alpha)\beta$, which is the same as for $Q_t(L)$ [cf. Eq. (20)]. We extend the domain $j\in\mathbb{N}$ to $j\in\mathbb{Z}$ so ρ_{jt} can be regarded as the expected number of walkers with the initial condition that a random walker exists at each site $i\in\mathbb{Z}_{\leqslant 0}$ with probability

$$\rho_{i0} = \frac{1}{1+\beta} + (1-\alpha)\left[\frac{\alpha}{(1-\alpha)\beta}\right]^i - \frac{1-\alpha-\alpha\beta}{1+\beta}\frac{1}{(-\beta)^i}. \tag{60}$$

Each walker at site j hops to its left site $j+1$ with probability α, to the right site $j-1$ with probability δ, or stays at site j with probability γ ($\alpha+\gamma+\delta=1$); see Fig. 3. Let $\epsilon_{jt}^{(i)}$ be the probability that the walker starting from the site i is in site j at time t, which is distributed around $Vt+i$ as [18]

$$\epsilon_{jt}^{(i)} \simeq \frac{1}{\sqrt{2\pi\sigma t}}\exp\left[-\frac{(j-Vt-i)^2}{2\sigma t}\right] \tag{61}$$

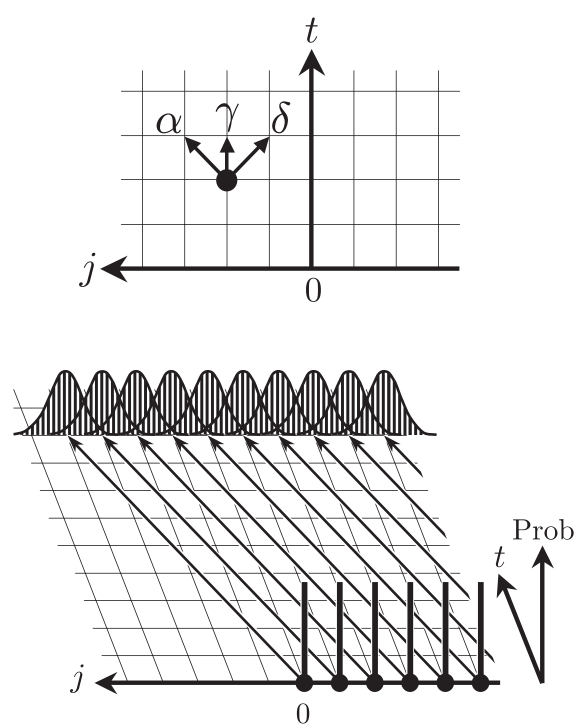

FIG. 3. Schematic picture of the noninteracting-random-walker interpretation.

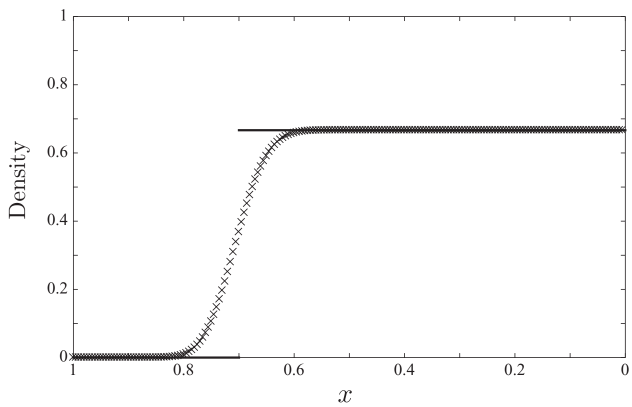

FIG. 4. Rescaled density profile in the divergent phase. The parameters are chosen as $(\alpha,\beta) = (4/5,1/2)$. The markers × correspond to Eq. (52) with $t = 200, x = j/t$, and the line to the asymptotic form (64).

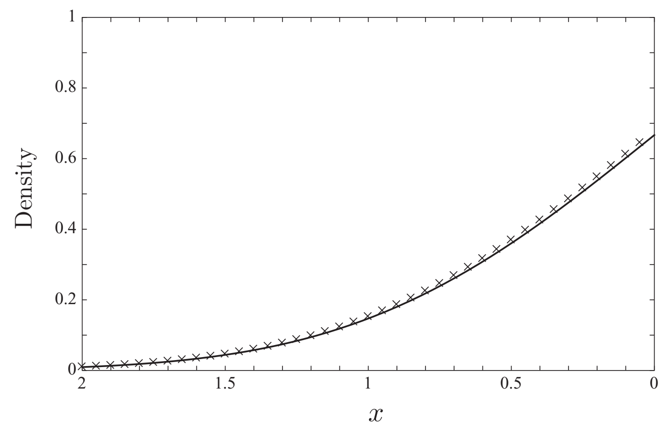

FIG. 5. Rescaled density profile on the critical line. The parameters are chosen as $(\alpha,\beta) = (1/3,1/2)$. The markers × correspond to Eq. (52) with $t = 400, x = j/\sqrt{t}$, and the line to the asymptotic form (66).

for the generic case $0 < \alpha < 1$ and $0 < \beta < 1$. Here $\sigma = \alpha + \delta - (\alpha-\delta)^2$, and $V = \alpha - \delta$ which is equal to the velocity for the system length; see Eq. (26). The density profile is expressed as

$$\rho_{jt} = \sum_{i\leqslant 0} \rho_{i0}\epsilon^{(i)}_{jt}. \tag{62}$$

In the divergent phase $\alpha > \frac{\beta}{1+\beta}$ (with $\alpha < 1$ and $0 < \beta < 1$), noting the initial condition

$$\lim_{i\to-\infty} \rho_{i0} = \frac{1}{1+\beta} \tag{63}$$

and the form (61), we find that the density profile with rescaling of the position $j = xt$ converges as

$$\rho_{xt,t} \to \begin{cases} \frac{1}{1+\beta} & (0 < x < V) \\ 0 & (V < x < 1) \end{cases}. \tag{64}$$

Figure 4 gives an example for the rescaled density profile in the divergent phase.

On the critical line $\alpha = \frac{\beta}{1+\beta}$ $(0 < \beta < 1)$, noting the initial condition

$$\lim_{i\to-\infty} \rho_{i0} = \frac{2}{1+\beta} \tag{65}$$

and the form (61) with $V = 0$, we find that the density profile (52) with the rescaling $x = \frac{j}{\sqrt{t}}$ converges as

$$\rho_{x\sqrt{t},t} \to \frac{1}{1+\beta}\,\mathrm{erfc}\left(\frac{x}{2}\sqrt{\frac{1+\beta}{\beta}}\right). \tag{66}$$

Here erfc is the complementary error function: $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}dy$. Figure 5 gives an example for the rescaled density profile on the critical line.

Now we consider some special cases. When $\alpha = 1$, the position of the leftmost particle is t by definition, and we have $\rho_{tt} = 1$. In this case, $\Lambda = z$ and the density profile becomes simply

$$\rho_{jt} = C_{z^t}\frac{z^j}{(1-z)(1+\beta z)} = \frac{1-(-\beta)^{t-j+1}}{1+\beta}. \tag{67}$$

In particular, the density profile observed by the leftmost particle is independent of time t and exhibits oscillations.

When $\beta = 1$ and $\alpha > \frac{1}{2}$, another oscillation occurs. Since $\gamma = 0$, the walker starting from the site i can exist on the site j at time t only if $j - i - t \in 2\mathbb{Z}$ and is distributed around $Vt + i$ at time t as

$$\epsilon^{(i)}_{jt} \simeq \sqrt{\frac{2}{\pi\sigma t}}\exp\left[-\frac{(j-Vt-i)^2}{2\sigma t}\right]. \tag{68}$$

We also note that the initial condition for the walker on site i converges as

$$\lim_{\substack{i\to-\infty\\ i\in 2\mathbb{Z}+1}} \rho_{i0} = 1-\alpha, \qquad \lim_{\substack{i\to-\infty\\ i\in 2\mathbb{Z}}} \rho_{i0} = \alpha. \tag{69}$$

In the limit $t\to\infty$ with the scaling $j = xt$, we have

$$\rho_{jt} \to \begin{cases} 1-\alpha & (0 < x < 2\alpha-1, j-t \in 2\mathbb{Z}+1) \\ \alpha & (0 < x < 2\alpha-1, j-t\in 2\mathbb{Z}) \\ 0 & (2\alpha-1 < x < 1) \end{cases}. \tag{70}$$

Figure 6 gives an example for the rescaled density profile in this case. Also when $\alpha = \frac{1}{2}$ and $\beta = 1$, the walker starting from the site i can exist on the site j at time t only if $j - i - t \in 2\mathbb{Z}$ and is distributed around $Vt + i$ at time t as Eq. (68). Noting $V = 0$ and the initial condition $\rho_{i0} = 1$, we find the rescaled density profile (66).

For the very special case $\alpha = \beta = 1$, which is completely deterministic, the site j is occupied if $j \leqslant t$ and $t - j$ is even or empty otherwise. The density profile oscillates between 1 and 0.

V. CONCLUDING REMARKS

We have studied the dynamical properties of the EQP with deterministic bulk hopping $p = 1$. We found the exact dynamical state in a matrix product (MP) form with a two-dimensional representation of the matrices and vectors. The MP dynamical state approaches the MP stationary state in the convergent phase $\alpha < \frac{\beta}{1+\beta}$ as $t\to\infty$. We have obtained the time-dependent distributions of the system length L (22), the number of particles N (31) and the waiting time T (41), and

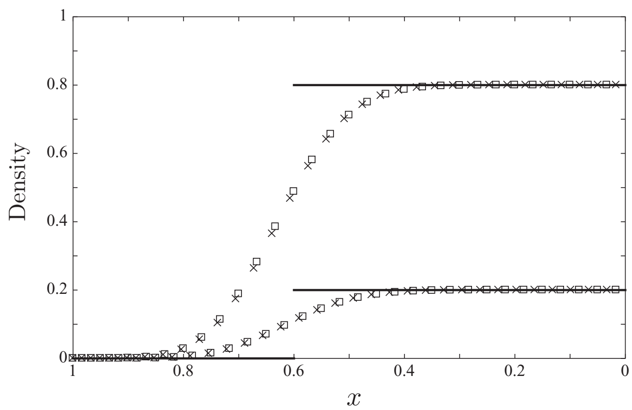

FIG. 6. Rescaled density profile for $(\alpha,\beta) = (4/5,1)$. The markers $\square$ and $\times$ correspond to Eq. (52) with $t = 60$ and 61, respectively, and $x = j/t$. The line corresponds to the asymptotic form (70).

the time-dependent density (52) and current (53) profiles of site j. An interesting point is that essentially they are given in the form

$$C_{z^t}\Psi(z)\Phi(z)^x \quad (x = L,N,T,j) \tag{71}$$

with functions Ψ and Φ including the square root r [cf. Eq. (24)].

We found that the asymptotic density profile in the divergent phase (with the generic choice of parameters) is flat. In contrast, the density profile for $p < 1$ is nontrivial [19]. One of the important tasks for future studies is to determine the form of the density profile and its dependence on the system parameters for the general case. The stationary state for the EQP with probabilistic hopping $p < 1$ has a matrix product form with infinite dimensional matrices [11]. This fact makes us expect that the MP dynamical state can be extended to the $p < 1$ case, which approaches the MP stationary state in the limit $t \to \infty$ as in the following diagram:

$$\begin{array}{ccc}
\text{dynamical state of unknown form} & \xrightarrow{p \to 1} & \text{MP dynamical state with 2D matrices} \\
\Big\downarrow t \to \infty & & \Big\downarrow t \to \infty \\
\text{MP stationary state with } \infty\text{D matrices} & \xrightarrow{p \to 1} & \text{MP stationary state with 2D matrices.}
\end{array}$$

In the probabilistic hopping case $p < 1$, however, the master equation cannot be simplified similarly to Eqs. (5)–(7), and the factorization ansatz (8) is no longer valid [19].

What we have investigated here is a very basic model of queues with excluded-volume effect. Apart from the generalization to $p < 1$ one can consider various other generalizations of the EQP, for example, multilane queues with some types of queue-changing rules.

ACKNOWLEDGMENTS

CA acknowledges support from the JSPS Fellowship Program for Research Abroad. The authors thank Martin R. Evans and Gunter M. Schütz for useful discussions and the Max-Planck-Institut für Physik komplexer Systeme in Dresden, where part of this work was performed, for its hospitality.

APPENDIX: RESULTS ON THE USUAL QUEUEING PROCESS

The usual queueing process is characterized by the number N of particles that is equal to the length of the system since its spatial structure is not taken into account. We denote the probability that the number of particles is N at time t by $P_t(N)$. At each time, a particle enters the system with probability α. When a particle leaves the system at time $t-1$, the next particle can leave the system at time t with probability β. If a new particle enters the system at time t and there is no other particle, this new particle can leave the system simultaneously. The critical line for the usual queueing process is $\alpha = \beta$, and the system is convergent or divergent if $\alpha < \beta$ or $\alpha > \beta$, respectively. The dynamical solution to the master equation

$$P_{t+1}(0) = [(1-\alpha) + \alpha\beta]P_t(0) + (1-\alpha)\beta P_t(1), \tag{A1}$$

$$P_{t+1}(N) = (1-\alpha)\beta P_t(N+1) + [(1-\alpha)(1-\beta) + \alpha\beta] \times P_t(N) + \alpha(1-\beta)P_t(N-1) \quad (N \in \mathbb{N}) \tag{A2}$$

with the initial condition

$$P_0(0) = 1, \quad P_0(N) = 0\, (N \in \mathbb{N}) \tag{A3}$$

is given by

$$P_t(N) = C_{z^t}\frac{1-\Theta}{1-z}\Theta^N, \tag{A4}$$

where

$$\Theta = \frac{1-(1-\alpha-\beta+2\alpha\beta)z - s}{2(1-\alpha)\beta z}, \tag{A5}$$

$$s = \sqrt{[1-(1-\alpha-\beta+2\alpha\beta)z]^2 - 4\alpha\beta(1-\alpha)(1-\beta)z^2}. \tag{A6}$$

When $\alpha < \beta$, the system approaches the stationary state

$$\lim_{t\to\infty} P_t(N) = \lim_{z\to 1}(1-\Theta)\Theta^N = \frac{\beta(1-\alpha)}{\beta-\alpha}\left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^N. \tag{A7}$$

The average number of particles at time t is

$$\langle N_t\rangle = \sum_{N\geqslant 1} N P_t(N) = C_{z^t}\frac{\Theta}{(1-z)(1-\Theta)} \simeq \begin{cases} \frac{\alpha(1-\beta)}{\beta-\alpha} & (\alpha < \beta) \\ 2\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} & (\alpha = \beta) \\ (\alpha-\beta)t & (\alpha > \beta) \end{cases}. \tag{A8}$$

The particle current leaving the system (outflow) J_t at time t is given by

$$J_t = \alpha\beta P_t(0) + \beta\sum_{N\geqslant 1} P_t(N) = C_{z^t}\frac{\beta[\Theta + \alpha(1-\Theta)]}{1-z} \to \begin{cases} \alpha & (\alpha < \beta) \\ \beta & (\alpha \geqslant \beta) \end{cases}, \tag{A9}$$

for $t \to \infty$. Note that, in the EQP case, $\lim_{t\to\infty} J_{1t}$ (58) is not equal to the exit probability β in the divergent phase since the rightmost site can be empty. The probability of the waiting time T for a given number N of particles is $\binom{T}{N}\beta^{N+1}(1-\beta)^{T-N}$ from which we find the probability of the waiting time for a given time t:

$$\sum_{N=0}^{T} P_t(N)\binom{T}{N}\beta^{N+1}(1-\beta)^{T-N} = C_{z^t}\frac{\beta(1-\Theta)}{(1-z)}(1-\beta+\beta\Theta)^T. \tag{A10}$$

The average waiting time is then

$$\langle T_t\rangle = C_{z^t}\frac{\beta(1-\Theta)}{(1-z)}\sum_{T\geqslant 1} T(1-\beta+\beta\Theta)^T = C_{z^t}\frac{1-\beta+\beta\Theta}{\beta(1-z)(1-\Theta)} \simeq \begin{cases} \frac{1-\beta}{\beta-\alpha} & (\alpha<\beta) \\ 2\sqrt{\frac{(1-\alpha)t}{\pi\alpha}} & (\alpha=\beta), \\ \frac{\alpha-\beta}{\beta}t & (\alpha>\beta) \end{cases} \tag{A11}$$

and the relation $J_t\langle T_t\rangle \simeq \langle N_t\rangle$ ($t\to\infty$) holds.

[1] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, (Springer, New York, 1999).
[2] B. Derrida, J. Stat. Mech. (2007) P07023.
[3] G. M. Schütz, *Exactly Solvable Models for Many-Body Systems Far from Equilibrium*, edited by C. Domb and J. L. Lebowitz, Phase Transitions and Critical Phenomena, Vol. 19 (Academic Press, San Diego, CA, 2001).
[4] R. A. Blythe and M. R. Evans, J. Phys. A **40**, R333 (2007).
[5] A. Schadschneider, D. Chowdhury, and K. Nishinari, *Stochastic Transport in Complex Systems: From Molecules to Vehicles* (Elsevier Science, Amsterdam, 2010).
[6] J. Medhi, *Stochastic Models in Queueing Theory* (Academic Press, San Diego, CA, 2003).
[7] T. L. Saaty, *Elements of Queueing Theory With Applications* (Dover, London, 1961).
[8] W. J. Hopp and M. L. Spearman, *Factory Physics* (McGraw-Hill, Boston, 2008).
[9] C. Arita, Phys. Rev. E **80**, 051119 (2009).
[10] D. Yanagisawa, A. Tomoeda, R. Jiang, and K. Nishinari, JSIAM Lett. **2**, 61 (2010).
[11] C. Arita and D. Yanagisawa, J. Stat. Phys. **141**, 829 (2010).
[12] C. Arita and A. Schadschneider, Phys. Rev. E **83**, 051128 (2011).
[13] R. B. Stinchcombe and G. M. Schütz, Europhys. Lett. **29**, 663 (1995).
[14] R. B. Stinchcombe and G. M. Schütz, Phys. Rev. Lett. **75**, 140 (1995).
[15] T. Sasamoto and M. Wadati, J. Phys. Soc. Jpn. **66**, 2618 (1997).
[16] G. M. Schütz, Eur. Phys. J. B **5**, 589 (1998).
[17] C. Arita, A. Ayyer, K. Mallick, and S. Prolhac, J. Phys. A **44**, 335004 (2011).
[18] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[19] C. Arita and A. Schadschneider (unpublished).