Noise-controlled dynamics through the averaging principle for stochastic slow-fast systems

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The effect of noise on nonlinear systems is analyzed, considering the case of slow-fast systems. It is known that small noise perturbations can induce a deterministic limit cycle in excitable systems when a specific scaling between the noise strength and the time-scale separation is achieved, a mechanism called self-induced stochastic resonance (SISR). The present study is focused on the impact of order 1 noise using the stochastic averaging principle. We introduce an elementary system of two coupled FitzHugh-Nagumo equations, which display the following nontrivial noise-induced behavior: (i) in the noise-free case, or for very small noise, the system fluctuates around its resting state; (ii) for small noise, oscillations appear due to SISR; (iii) for intermediate noise, the system fluctuates again around its resting state; (iv) for larger noise, new oscillations are observed and their explanation requires the application of the stochastic averaging principle. It is suggested that in the perspective of biological systems, time-scale separation may act as a "noise averager," enabling a noise-controlled dynamical behavior through the averaging principle.

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I. INTRODUCTION

Understanding the impact of noise on nonlinear dynamical systems has been an active field of research for many years and has a wide range of applications, from physics to biology to earth science and economy. Although the effect of noise is frequently reduced to adding variability in response of the system, without any effect on the qualitative dynamics, it has been shown in several situations how the interplay between noise and the nonlinearities of the system may induce counter-intuitive behavior [1,2]. In such situations, noise may play a constructive role, and the nonlinearities enables a useful transformation of the energy contained in the fluctuations into an ordered dynamic, which may be of interest, for instance, in living systems. In this paper, we describe a new mechanism of noise-controlled dynamics in noisy nonlinear systems with multiple time scales.

In this context, previous studies about the constructive role of noise have focused on a class of phenomena called "stochastic resonance" (SR): an appropriate amount of noise applied on some nonlinear systems may enhance their ability to respond coherently to periodic forcing or their performance in a signal detection task [3–5]. In absence of periodic forcing, coherence resonance (CR, or autonomous SR) phenomena [6–10] appear when an appropriate noise perturbation enhances the coherence of a system close to a bifurcation point. In the context of excitable systems below threshold, a phenomenon called self-induced SR (SISR) [11–13] is responsible for noise-induced oscillations. In those systems, generally with several time scales, an appropriate asymptotic scaling relationship between the time-scales separation and the vanishing noise strength leads to coherent oscillations in a regime where the noise-free dynamics is subthreshold. The idea of SISR is that rare events leading to an excursion can become frequent and regular when the slow-fast time-scales ratio scales exponentially with respect to the noise strength.

In the present article, we describe a new phenomenon arising in the same context of noisy slow-fast systems. Unlike SISR, we do not consider here the small noise limit. We rather show how the strength of the noise applied to the system can become a control parameter of the resulting dynamics. This phenomenon is studied on a prototypical slow-fast system, namely the FitzHugh-Nagumo (FHN) model [14], which has been introduced as a minimal nonlinear model describing neuronal excitability. We consider two FHN units, dynamically coupled through a synaptic variable. Previous works [15–17] have focused on understanding the potential impact of noise on coupled excitable units, exploring various regimes of stochastic resonance, and noise-induced oscillations.

We apply mathematical results from stochastic averaging theory to provide an explanation of the observed phenomenon. The fast noisy variable can be approximated by its stationary distribution and an effective description of the slow dynamics features the noise strength as a bifurcation parameter. The principle described here is very general and may have many other applications, as in Ref. [32], where the impact of channel noise on discharge patterns in stochastic conductance-based neuronal models is studied.

The paper is organized as follows. After a description of the model (Sec. II), we present numerical simulations and theoretical explanations of the phenomenon (Sec. III) and conclude by a discussion on several generalizations and an interpretation in terms of neuronal computation (Sec. IV).

II. MODELS

Introduced in Ref. [14], the FitzHugh-Nagumo model is a two-dimensional system, similar to the VanderPol oscillator model, that was constructed as a reduced neuron model where the nonlinearities have been simplified to isolate the mathematical properties of excitation (and propagation in its spatially extended version, see Discussion). The system of equation is given by

$$\epsilon \frac{dx}{dt} = x - x^3/3 - y + I, \quad \frac{dy}{dt} = ax - by.$$

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According to the values of the parameters, this dynamical system can be mainly in three different regimes, namely excitable (one single stable fixed point), multistable (two or three stable fixed points), and oscillatory (one stable limit cycle). We refer the interested reader to Refs. [18,19] for more details about the properties of this model.

A. Stochastic FitzHugh-Nagumo model

To account for the effect of external fluctuations, we consider a stochastic white-noise perturbation on the fast variable, leading to the following Stochastic Differential Equation (SDE):

$$\epsilon dx_t = \left(x_t - \tfrac{1}{3}x_t^3 - y_t + I\right)dt + \sigma dW_t, \tag{1}$$

$$dy_t = (ax_t - by_t)\,dt, \tag{2}$$

with $a,I \in \mathbf{R}$, $b > 0$, $\sigma > 0$, and W_t a standard Brownian motion. This white noise perturbation can be justified as a diffusion approximation of a sum of many independent synaptic events coming from the rest of the network [20]. We do not study here the impact of noise on the slow variable coming from ion channel stochasticity; that is, we assume that this source of noise is negligible compared to external noise.

This model has been studied with various methods and within different parameter regimes. In particular, we consider here only the case where noise is added on the fast variable x, but the impact of a noisy perturbation on the slow variable y has also been studied, for example, in Ref. [21], using geometric singular perturbation methods for stochastic slow-fast systems, or in Ref. [22], using time-scales expansion of the corresponding Fokker-Planck equation. Inspired from large deviations theory, several mathematical results have been obtained concerning the phenomenon of SISR [11–13,23]. We also mention several works on stochastic bifurcations for stochastic FHN models using the moment equations method [24,25].

B. Two coupled systems

We introduce a coupling between two FHN units, system 1 and system 2, based on a simplified model of synaptic transmission. Although processes involved at the synapse are very complex, we can consider as a first approximation [26] that system 2's input $p_{1\to 2}$ evoked by system 1's activity is given by the convolution of membrane potential $x^{(1)}$ with some kernel $K^{1\to 2}$:

$$p^{1\to 2}(t) = \int_0^t K^{1\to 2}(t-s)x_s^{(1)}ds. \tag{3}$$

Experimental results of Ref. [26] and analysis of kinetic models of receptor binding [27] indicate that the choice of an exponential kernel is a reasonable approximation. Thus, we assume in the following that

$$K^{1\to 2}(t,s) = e^{-(t-s)/\tau}. \tag{4}$$

It appears that with this choice, the postsynaptic potential $p_{1\to 2}(t)$ can be seen as a dynamical variable, solution of a linear differential equation driven by $x_t^{(1)}$, with a time constant $\tau > 0$:

$$\frac{dp^{1\to 2}}{dt} = -\tau p^{1\to 2} + x_t^{(1)}. \tag{5}$$

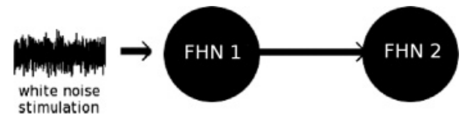

FIG. 1. Schematic representation of the two coupled systems. First FHN system (FHN 1) is driven by a white-noise stimulation and its output drives the second FHN system (FHN 2).

We further assume that the synaptic time-constant τ is of order 1. This assumption corresponds to the case of fast glutamate synapses of type AMPA-kainate, for which the typical decay time-scale τ is of order 1–5 ms (cf. Ref. [27] and Ref. [28], Chap. 7), which is the same order as the time-scale of the recovery variable y (typical time of an action potential plus recovery is 2–5 ms). With this simplified synaptic model, we construct the following system as a model of two coupled neurons. As pictured in Fig. 1, system 1 receives a noisy stimulation and targets to system 2 as follows:

$$\epsilon dx_t^{(1)} = \left[G\left(x_t^{(1)},y_t^{(1)}\right) + I_1\right]dt + \sigma dW_t, \tag{6}$$

$$\frac{dy_t^{(1)}}{dt} = ax_t^{(1)} - by_t^{(1)}, \tag{7}$$

$$\frac{dp^{1\to 2}}{dt} = -\tau p^{1\to 2} + x_t^{(1)}, \tag{8}$$

$$\epsilon\frac{dx_t^{(2)}}{dt} = G\left[x_t^{(2)},y_t^{(2)}\right] + p^{1\to 2} + I_2, \tag{9}$$

$$\frac{dy_t^{(2)}}{dt} = ax_t^{(2)} - by_t^{(2)}, \tag{10}$$

where $G(x,y) = x - x^3/3 - y$.

III. RESULTS

A. Simulations

We present in this section stochastic simulations of both the single stochastic FitzHugh-Nagumo model and the two coupled systems. We fix all the parameters ($a = b = 1$, $I_1 = -0.5$, $I_2 = -0.3$, $\tau = 1$, $\epsilon = 0.0001$) and investigate the impact of the noise strength σ. The value of ϵ is set to 10^{-4}, enhancing the time-scale separation in the system. We have chosen this value since it enables a clearer observation of the impact of σ and corresponds to the asymptotic regime described by the stochastic averaging principle (cf. Sec. III B). We will also display some results for higher values of ϵ to explore the robustness of the results obtained for $\epsilon = 10^{-4}$.

Note that with this choice of parameters ($a = b = 1$), variables $y^{(1)}$ and $p^{1\to 2}$ can be identified since they satisfy the same differential equation.

Stochastic simulations reveal four regimes that we describe below. Notice that the dynamics observed in those regimes are purely noise-induced, in the sense that the deterministic system ($\sigma = 0$) has only a single stable fixed point.

a. Regime 1: very small noise does not induce oscillations. With our choice of parameters, the noise-free system has a single stable fixed point. Under a very small random perturbation, we observe numerically that the system fluctuates for a long time around this resting state, as displayed in Figs. 2 and 3, case $\sigma = 0.01$.

b. Regime 2: small noise induces coherent oscillations in both systems through SISR. When the noise strength σ is increased, a jump from the stable fixed point to the other branch of the cubic x nullcline becomes more probable. In fact, there is a competition between two time-scales that will determine whether this jumping probability is of order 1 or not:

(i) The expected time for such a jump to occur is of order e^{V_y/σ^2} for small σ, with V_y a positive number depending on the ordinate y at the jumping point.

(ii) Due to the time-scale separation, a time-interval of order 1 in the slow time scale is actually seen as a time-interval of order ϵ^{-1} by the fast variable.

As a consequence, we expect that if $\epsilon^{-1} \approx e^{V_y/\sigma^2}$, meaning that $\sigma^2 \ln(\epsilon) \to -\alpha < 0$ when both $\epsilon \to 0$ and $\sigma \to 0$, then the probability of a jump at a value $y(\alpha)$ such that $V_{y_\alpha} = \alpha$ becomes very close to one. We observe numerically this phenomenon in Figs. 2 and 3, with $\sigma = 0.03$ (transition to SISR), and $\sigma = 0.05, 0.1$. We refer the interested reader to Refs. [11–13,23] for more details.

In Fig. 4 (right panel) and Fig. 5, the interspike intervals' mean and coefficient of variation (CV) are displayed as a function of σ. The CV is computed as the ratio between the standard deviation and the mean of the interspike intervals. These figures show the emergence of an oscillatory behavior. Furthermore, an intermediate value of the noise strength σ minimizes the coefficient of variation, a phenomenon referred to as stochastic resonance.

c. Regime 3: medium noise destroys coherent oscillations. Increasing the noise strength destroys the particular scaling between σ and ϵ that is crucial for SISR. We observe numerically in Figs. 2 and 3 for $\sigma = 0.95, 1, 2$ that the fast variable $x_t^{(1)}$ jumps constantly between the two attracting branches of the x nullcline, and that meanwhile both $y_t^{(1)}$ and $p^{1\to 2}$ fluctuate around new equilibrium values, which are not equilibrium points for the deterministic system. A key remark here is that these emergent equilibria for the slow variable and for the synaptic variable depends on the noise strength σ. We will see in Sec. IV how the mathematical theory of stochastic averaging accounts for this surprising phenomenon.

d. Regime 4: large noise induces a bifurcation leading to oscillations only in system 2, through stochastic averaging in system 1. In Fig. 2, the time course of system 2's slow variable $y^{(2)}$ is displayed (gray), showing the emergence of oscillations

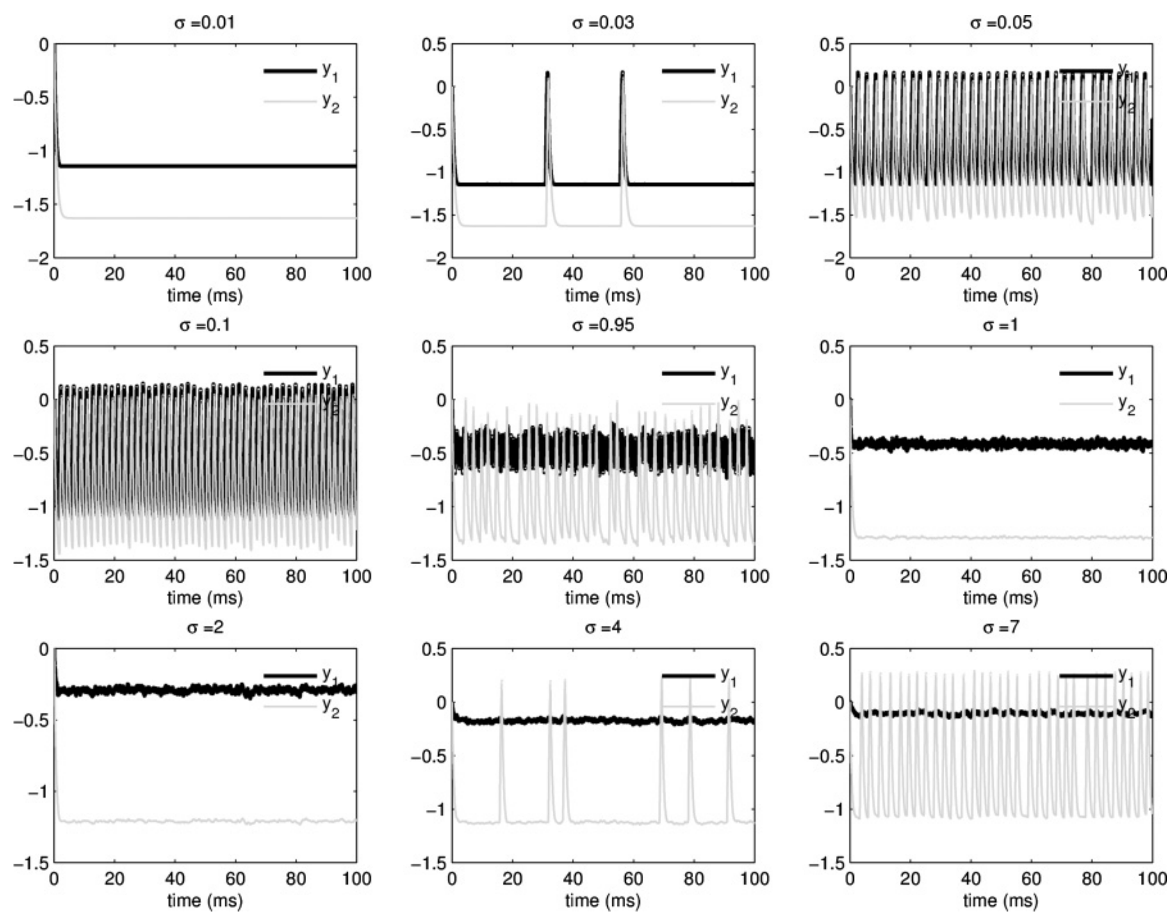

FIG. 2. Numerical simulations of two coupled stochastic FizHugh-Nagumo systems with the same parameters: $a = b = 1$, $\tau = 1$, $I_1 = -0.5$, $I_2 = -0.3$, $\epsilon = 0.0001$. Time courses of variables $y^{(1)}(t)$ (black) and $y^{(2)}(t)$ (gray) are displayed for different values of the noise strength $\sigma \in [0.01, 7]$ (increasing from left to right and from top to bottom).

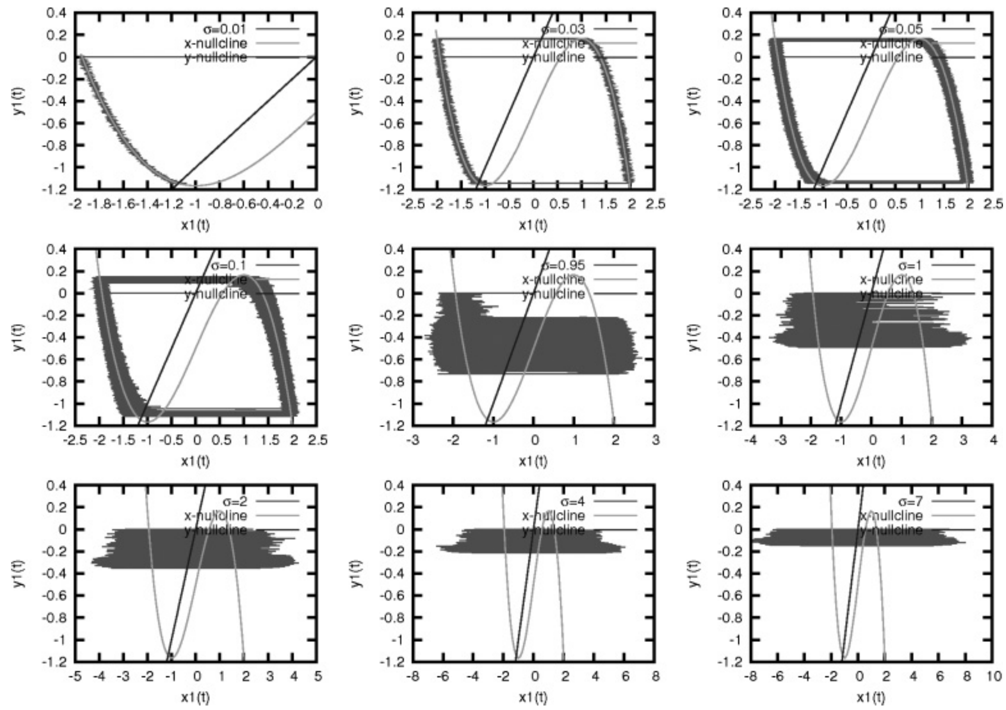

FIG. 3. Numerical simulations of a single stochastic FizHugh-Nagumo model (unit 1) with parameters $a = b = 1$, $\tau = 1$, $I = -0.5$, $\epsilon = 0.0001$. With this choice of parameters, the noise-free system ($\sigma = 0$) is in an excitable regime. Variables $x(t)$ and $y(t)$ are displayed in the phase plane for different values of the noise strength $\sigma \in [0.01,7]$ (increasing from left to right and from top to bottom), together with the nullclines of the deterministic system.

when a large noise is applied to system 1 (cf. $\sigma = 7$). In Fig. 4 (right panel), the interspike intervals' mean and coefficient of variation are displayed as a function of σ, showing the emergence of an oscillatory behavior characterized by the joint decrease in the mean and coefficient of variation of interspike intervals. One recovers the same phenomenology for higher values of ϵ as displayed in Fig. 5. However, transitions between different regimes become much smoother as time-scales are less separated.

In this large noise regime, the fast variable $x^{(1)}$ of system 1 fluctuates very rapidly between the two branches of the x nullcline, and following the averaging principle, the slow variable becomes concentrated around an equilibrium value, which depends on the noise strength σ, and the same is true for the synaptic variable $p^{1\to 2}$. Thus, input to system 2 is approximately a constant that is fixed by the value of σ, and changing σ may thus lead to bifurcations in system 2. In our case, a Hopf-bifurcation in system 2, with $p^{1\to 2}$ as a dynamical parameter modulated by σ, explains the emergence of periodic solutions in system 2 for large values of σ.

B. Theory

The aim of this section is to provide a theoretical explanation of regimes 3 and 4. Indeed, the transition to and from regime 2 has already been intensively investigated. After giving a brief presentation of the main mathematical result of stochastic averaging for slow-fast diffusion processes, we discuss their application in the perspective of understanding the noise-induced bifurcation, thus explaining regime 4.

1. Stochastic averaging principle

The first results on stochastic averaging go back to Ref. [29], where the following set up is studied. Consider a stochastic differential equation in $\mathbf{R}^{n+m}$:

$$dx_t^\epsilon = \frac{1}{\epsilon} g\left(x_t^\epsilon, y_t^\epsilon\right)dt + \frac{1}{\sqrt{\epsilon}}\sigma\left(x_t^\epsilon, y_t^\epsilon\right)dW_t, \tag{11}$$

$$dy_t^\epsilon = f\left(x_t^\epsilon, y_t^\epsilon\right)dt, \tag{12}$$

with initial conditions $x^\epsilon(0) = x_0$, $y^\epsilon(0) = y_0$, and where $y^\epsilon \in \mathbf{R}^n$ is called the slow variable, $x^\epsilon \in \mathbf{R}^m$ is the fast variable, with $f : \mathbf{R}^{m+n} \to \mathbf{R}^n$ and $g : \mathbf{R}^{n+m} \to \mathbf{R}^m, \sigma : \mathbf{R}^{n+m} \to \mathcal{M}_m$ smooth functions ensuring existence and uniqueness for the solution (x^ϵ, y^ϵ), and W a m-dimensional standard Brownian motion.

In order to approximate the behavior of (x^ϵ, y^ϵ) for small ϵ, the idea is to average out the equation for the slow variable with respect to the stationary distribution of the fast one. More precisely, one first assumes, for each $y \in \mathbf{R}^n$ fixed, the *frozen* fast SDE:

$$dx_t = g(x_t, y)dt + \sigma(x_t, y)dW_t \tag{13}$$

admits a unique invariant measure, denoted $\rho^y(dx)$. Such an assumption is generally related to the uniform nondegeneracy of the matrix σ.

Then, one defines the averaged drift vector field $\bar{f}$:

$$\bar{f}(y) := \int_{\mathbf{R}^m} f(x,y)\rho^y(dx) \tag{14}$$

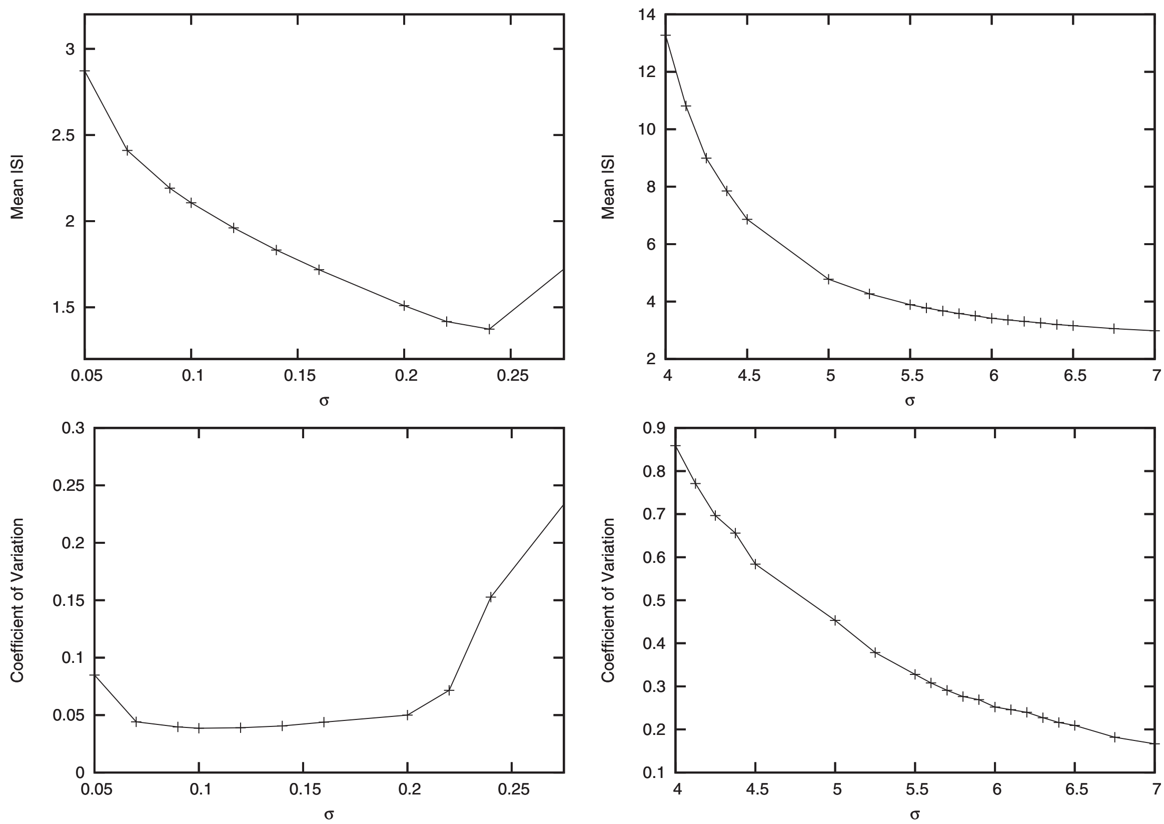

FIG. 4. Numerical estimation of the interspike intervals' (ISI) mean and coefficient of variation for unit 2, under various noise strength σ. The two left panels correspond to regime 2, namely noise-induced oscillations by SISR. The CV is minimal for an intermediate value of σ. The two right panels correspond to regime 4, namely noise-induced oscillations by stochastic averaging, further explained in Sec. III B. Strikingly enough, in this regime the CV is a *decreasing* function of the noise strength σ. Parameters $a = b = \tau = 1$, $I = -0.5$, $\epsilon = 0.0001$.

and $\bar{y}$ the solution of the following ordinary differential equation (ODE):

$$\frac{d\bar{y}}{dt} = \bar{f}(\bar{y}) \tag{15}$$

with initial condition $\bar{y}(0) = y_0$.

Under some mild technical assumptions (see Ref. [29]), mostly strong dissipativity conditions ensuring a uniformly exponential mixing ergodicity property for the frozen fast system, the main mathematical result is then [29]

$$\lim_{\epsilon\to 0} \mathbf{P}\Big[\sup_{t\in[0,T]} \left\|y_t^\epsilon - \bar{y}_t\right\|^2 > \delta\Big] = 0, \tag{16}$$

for any $\delta > 0$ and $T > 0$.

This result ensures that, during finite time intervals $[0,T]$, the probability of seeing x^ϵ outside a δ tube around $\bar{x}$

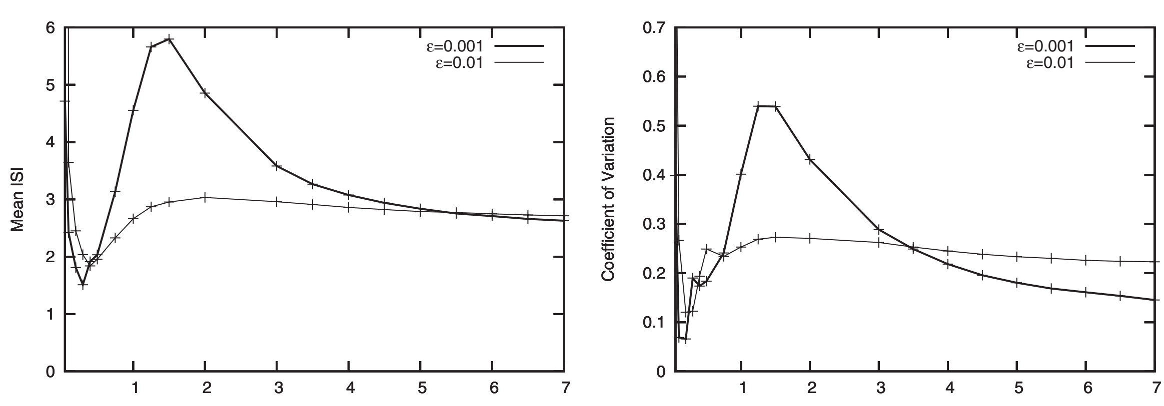

FIG. 5. Impact of the time-scale separation parameter ϵ: numerical estimation of the interspike intervals' (ISI) mean and coefficient of variation for unit 2, under various noise strength σ, and for two values of $\epsilon = 0.01, 0.001$. Transition between different regimes is smoother when relaxing the time-scales separation. Still, the various regimes identified in the case of a stronger time-scales separation are reflected faithfully in the strongly nonmonotonic evolution of the spiking statistics of unit 2 as a function of σ. Parameters $a = b = \tau = 1$, $I = -0.5$.

(deterministic) can be made arbitrary small when decreasing ϵ. Many other results have been developed since, extending the set-up to the case where the slow variable has a diffusion component or to infinite-dimensional settings, for instance, and also refining the convergence study, providing results concerning the limit of $\epsilon^{-1/2}(y^{\epsilon} - \bar{y})$ or establishing large deviation principles. We refer the interested reader to Ref. [30] for an introductory account of the theory and its applications and to Ref. [31] for recent mathematical results on this topic.

For our purpose, the key observation here is that the averaged vector field $\bar{f}$ actually depends on the diffusion coefficient σ of the fast variable, through the quasistationary distribution ρ^{y}. Note that this property has been also exploited in Ref. [32] to explain stochastic bifurcations in the Hodgkin-Huxley model with stochastic ion channels. We will exploit this observation in the following paragraph to analyze a stochastic bifurcation phenomenon in our example (regime 4).

2. *Noise-induced bifurcations for the coupled stochastic FHN system*

We are now in position to apply the above theory to the stochastic FHN system. The first point is to compute the invariant measure for the following one-dimensional *frozen* diffusion, for each $y \in \mathbf{R}$:

$$dx_t = \left(x_t - \tfrac{1}{3}x_t^3 - y + I\right)dt + \sigma dW_t. \tag{17}$$

Introducing the potential,

$$U_y(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + xy - Ix, \tag{18}$$

we derive the invariant density $\rho_y(x)$ for the fast variable x:

$$\rho_y(x) = \frac{1}{Z_y}\exp\left[-\frac{U_y(x)}{2\sigma^2}\right], \tag{19}$$

where $Z_y = \int_{x\in\mathbf{R}} \exp\left(-\frac{U_y(x)}{2\sigma^2}\right)dx$ is an integrating constant.

As a consequence, we obtain the averaged vector field $\bar{F}_\sigma$ for the recovery variable y, which depends on σ:

$$\bar{F}_\sigma(y) = \int_{x\in\mathbf{R}} (ax - by)\rho_y(x)dx, \tag{20}$$

$$= a\xi_\sigma(y) - by, \tag{21}$$

where

$$\xi_\sigma(y) = Z_y^{-1}\int_{x\in\mathbf{R}} x\exp\left(-\frac{U_y(x)}{2\sigma^2}\right)dx. \tag{22}$$

The slow variable y can be approximated for small ϵ by the solution $\bar{y}$ of the deterministic ODE:

$$\frac{d}{dt}\bar{y} = \bar{F}_\sigma(\bar{y}). \tag{23}$$

Similarly, in the coupled system of Sec. II B, the synaptic variable $p^{1\to 2}$ can be approximated by the solution of

$$\frac{d}{dt}\bar{p}^{1\to 2} = -\tau\bar{p}^{1\to 2} + \xi_\sigma(\bar{p}^{1\to 2}). \tag{24}$$

The equilibrium points $\bar{p}^*$ solution of

$$-\tau\bar{p}^* + \xi_\sigma(\bar{p}^*) = 0 \tag{25}$$

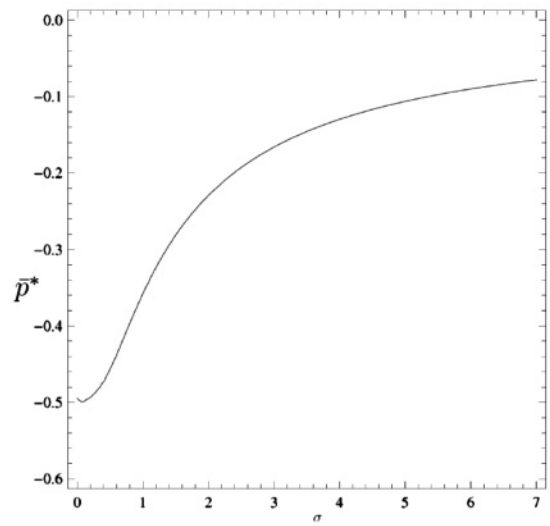

FIG. 6. Numerical computation of the solution $\bar{p}^*$ of Eq. (25) as a function of noise parameter σ. Parameter values: $a = b = 1$, $\tau = 1$, $I = -0.5$.

are thus functions of the noise strength σ applied on the fast variable. Even if an analytical computation of $\bar{p}^*$ is difficult, one can compute numerically the solution of Eq. (25), as shown in Fig. 6, for different values of σ. This figure compares very well with plots of Fig. 2, for $\sigma = 1,2,4,7$ displaying the value of the slow variable $y^{(1)}$, which is equal to $p^{1\to 2}$ with our choice of parameters ($a = b = \tau = 1$); one observes fluctuations around an seemingly equilibrium value, which is precisely $\bar{p}^*$ as displayed in Fig. 6.

It is now possible to understand what happens in regime 4: the synaptic variable of system 1 stays close to $\bar{p}^*$ (for small enough ϵ, during finite time intervals), which can be seen as a bifurcation parameter for system 2. When $\bar{p}^*$ crosses a Hopf bifurcation value for system 2, then it starts to oscillate as shown in Fig. 2, case $\sigma = 7$. As the value of $\bar{p}^*$ is determined by the noise strength applied to system 1, the coupled system is thus in some sense controlled by the noise strength σ.

a. Remark: As seen in Fig. 3, subplots $\sigma = 0.03, 0.05, 0.1$, it seems to be necessary to have σ high enough to apply efficiently the averaging principle. Let us explain in more details this observation, which is due to the nonzero value of ϵ. As the invariant distribution for the fast process is *bimodal* (double-well potential), we may distinguish two time-scales for the convergence toward stationarity. First, one of the two peaks grows exponentially fast and then on a much longer time scale, typically $\exp(1/\sigma^2)$ for small σ, the rare escapes from one well to the other occur and the second peak starts to grow. As a consequence, for the averaging principle to work efficiently, one has to require that the fast variable has enough time to reach its stationary bimodal distribution, which requires that ϵ is sufficiently small compared to the noise strength, in the sense that $\epsilon^{-1} \gg \exp(1/\sigma^2)$.

IV. DISCUSSION

The purpose of the present paper was to describe and analyze a new phenomenon of noise-controlled dynamics in slow-fast systems through the stochastic averaging principle.

We have considered a pair of unilaterally coupled FHN systems, where the first one receives a white-noise input and targets the second system. We have observed by numerical simulations that changing the noise strength as a parameter can induce oscillations by two very different mechanisms. The first one is an example of well-known SISR and arises in a proper scaling between small noise and time-scale separation. The second one is a novel mechanism that relies on the stochastic averaging principle: in the non-small noise regime, the behavior of the slow variable can be approximated by an averaged system, using the idea that the fast variable statistical fluctuations are close to stationarity (however, depending on the slow variable). In our setting, the averaged slow variable then converges to a steady state whose value depends on the statistical properties of the noise, here its variance. When considering the coupled system, the averaged slow variable of the first system then becomes a bifurcation parameter for the second system and can induce possibly any type of dynamical behavior, here oscillation through a Hopf bifurcation. However, our analysis relies on the assumption of a strong time-scale separation, and the transition between the different regimes may be less clear if the time-scale separation is less strong. Yet, our opinion is that the theoretical asymptotic analysis in the limit $\epsilon \to 0$ can be used as a tool to unravel a singular stochastic bifurcation structure (at the singular point $\epsilon = 0$), whose effects remain visible for nonzero values of ϵ.

The idea of stochastic averaging principle is a very general and powerful mechanism for studying and constructing noise-controlled dynamical systems. We point out that it is possible to extend the present setting in several directions. First, we have used here a white-noise stimulation and shown how its variance can become a control parameter, but it is possible to construct a system where the control parameter would take into account several statistical properties of the noisy input, such as its higher-order moments or its temporal correlations. Another very interesting direction would be to consider the same kind of ideas but for spatially extended systems, in which noise might control front or pulse propagation.

With this perspective of noise-controlled dynamics through the stochastic averaging principle, slow-fast systems can be seen as building blocks for "translating" the statistical properties of the noise into steady control parameters, which might be of great interest for biological systems. Among many possible explanations of the ubiquity of slow-fast dynamics in the nervous system, the present analysis suggests that it might be related to the stochastic averaging principle. First, when stochastically perturbed on the fast variables, this class of system enables us to reduce the variability (convergence to a deterministic limit when $\epsilon \to 0$). Second, it may transform the statistics of the fluctuations into a precise dynamical behavior, shedding potentially a new light on the role of noise in neuronal dynamics. Our findings suggest that these properties of slow-fast systems might be advantageous in an intrinsically noisy environment, such as the nervous system, but this idea clearly requires further theoretical and experimental investigations.

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