PHYSICAL REVIEW E **84**, 046610 (2011)

Sustained strong fluctuations in a nonlinear chain at acoustic vacuum: Beyond equilibrium

Edgar Ávalos
Department of Physics, Chung Yuan Christian University, Chungli, Taiwan 32063, Republic of China

Diankang Sun
New Mexico Resonance, 2301 Yale Boulevard SE, Albuquerque, New Mexico 87106, USA

Robert L. Doney
US Army Research Labs, Aberdeen Proving Grounds, Maryland 21005, USA

Surajit Sen
Department of Physics, State University of New York, Buffalo, New York 14260-1500, USA
(Received 1 May 2011; published 21 October 2011)

Here we consider dynamical problems as in linear response theory but for purely nonlinear systems where acoustic propagation is prohibited by the potential, e.g., the case of an alignment of elastic grains confined between walls. Our simulations suggest that in the absence of acoustic propagation, the system relaxes using only solitary waves and the eventual state does not resemble an equilibrium state. Further, the studies reveal that multiple perturbations could give rise to hot and cold spots in these systems. We first use particle dynamics based simulations to understand how one of the two unequal colliding solitary waves in the chain can gain energy. Specifically, we find that for head-on collisions the smaller wave gains energy, whereas when a more energetic wave overtakes a less energetic wave, the latter gains energy. The balance between the rate at which the solitary waves break down and the rate at which they grow eventually makes it possible for the system to reach a peculiar equilibriumlike phase that is characteristic of these purely nonlinear systems. The study of the features and the robustness of the fluctuations in time has been addressed next. A particular characteristic of this equilibriumlike or quasiequilibrium phase is that *very large* energy fluctuations are possible—and by very large, we mean that the energy can vary between zero and several times the average energy per grain. We argue that the magnitude of the fluctuations depend on the nature of the nonlinearity in the potential energy function and the feature that any energy must eventually travel as a compact solitary wave in these systems where the solitary wave energies may vary widely. In closing we address whether these fluctuations are peculiar to one dimension or can exist in higher dimensions. The study hence raises the following intriguing possibility. Are there physical or biological systems where these kinds of nonlinear forces exist, and if so, can such large fluctuations actually be seen? Implications of the study are briefly discussed.

DOI: 10.1103/PhysRevE.84.046610 PACS number(s): 46.40.Cd, 45.70.−n, 43.25.+y

I. INTRODUCTION

Consider a finite, bounded, interacting, macroscopic, classical many-particle system that has been subjected to a perturbation in the absence of thermal fluctuations, i.e., at "zero temperature." We let the entire energy of the system come from the perturbation. We may not be able to use *linear* response theory here [1–4]. One can still ask how any perturbation disperses into the system due to the presence of the interparticle potential.

Such a problem would need to be addressed using classical dynamics as opposed to nonequilibrium statistical mechanics. Indeed, relaxation in a simple harmonic oscillator chain where any excitation eventually manifests as normal mode oscillations has been extensively studied [5]. Fermi, Pasta, and Ulam (FPU) in collaboration with Tsingou [6] probed the dynamics of such a system in the presence of *linear and nonlinear interactions* using computers in the mid 1950s. Their work led to the realization that unlike what might be naively expected, anharmonicity does not necessarily enhance the propensity to reach an equilibrium state [6,7]. The FPU study also appears to have foreshadowed the entry of solitary waves into much of the statistical and condensed matter physics literature [8–11].

Let us now consider a system without inertial mismatches and with *purely nonlinear* interactions. Linear response theory would suggest that any perturbation would likely be dispersed in time into the system as it returns to an equilibrium phase [1–4]. But, is this hypothesis applicable to purely nonlinear systems? As we shall see, the dynamics of a lossless alignment of elastic grains [12], which has been extensively probed in recent times [13–15], held between perfectly reflecting walls poses a counterexample. Whether the system is initially at zero temperature or is in an excited state, perturbations imparted to such a system do not disperse into an equilibrium state but rather can lead to rich dynamics throughout the system [16]. There is a considerable amount of literature on the dynamics of the FPU system at large times [17]. A key difference between the FPU systems and the ones considered here lies in that our systems do not have a harmonic term in the potential energy function and the potential function is one sided. As will be seen later, the interactions gradually vanish as the grains lose contact. Hence, they exist in "acoustic vacuum" meaning that the grains cannot oscillate while maintaining contact, and therefore the system does not support ordinary sound propagation [13].

In what follows, we describe the system considered and its time evolution following some perturbation. Our studies suggest that the eventual state of such a system can admit very large kinetic energy fluctuations while satisfying the Gaussian distribution of velocities and possible independence from initial conditions. Given that this is a nonlinear system and that there is no dissipation, and that energy fluctuations depend on the nature of the potential, one may naively think that large energy fluctuations are not a surprise. The counter argument could be that there are nonlinear potentials that are used to model solids, liquids, and gases in their equilibrium states. Hence, nonlinearity alone is not the reason. Rather, it is the system and the nature of the nonlinearity that may both be relevant here. To be specific, virial theorem allows one to determine how much of the energy is kinetic and how much is potential. For the potentials considered here, more of the total energy ends up as kinetic energy. Correspondingly, the fluctuations in kinetic energy are larger than the same for linear systems. A second reason why the energy fluctuations in these systems ends up being unusually large is because energy moves as solitary waves in these systems. These waves have a spatial extent that is fixed by the nonlinearity of the potential. Hence the initial energy is not equally distributed among the particles in the system but rather unequally among the number of solitary waves in the system. Here we address how it may be possible to manipulate the fluctuations in these systems. We close with a brief discussion of the possible implications of the work.

II. THE NONLINEAR SYSTEM

The system of interest is an alignment of N identical elastic spheres, each of mass m and radius R, placed in such a way that they are barely in contact with each other. There is no grain-grain interaction when there is no contact [13,15]. We assume that the alignment is held between perfectly reflecting walls. The interaction potential is given as

$$V(x_{i+1}-x_i) = a[2R-(x_{i+1}-x_i)]^n \equiv a\delta_{i,i+1}^n \geqslant 0, \qquad (1)$$

where x_i is the displacement of grain i from the original position (i.e., grains are at a distance of $2R$ between the centers initially), $2R \geqslant x_{i+1}-x_i \geqslant 0$, and $\delta_{i,i+1}$ is the overlap between the grains. Observe that the grains do not interact when the grain-grain contact is broken, and hence for $2R < x_{i+1}-x_i$ or $\delta_{i,i+1} < 0$, $V(x_{i+1}-x_i)=0$. The properties of $V(\delta_{i,i+1})$ are discussed in some detail in Ref. [18].

For spheres, as per Hertz law, $n = 5/2$ [19]. Hence, the interaction potential is fully nonlinear in nature, i.e., there is no $n = 2$ term and hence there is no chance of any oscillatory dynamics of any of the grains. In turn, as alluded to above, this means that there is no acoustic propagation from grain to grain. One can show that for the majority of grain-grain contact potentials $n > 2$ [18]. To illustrate the effects of n in influencing the dynamics of the system, we will use $n = 5/2$ and $n = 2.1$, the latter being closer to the harmonic case while being fully nonlinear. The equation of motion of each grain (except for the two boundary grains) is given by

$$m\frac{d^2x_i}{dt^2} = an\left[\delta_{i,i-1}^{n-1} - \delta_{i,i+1}^{n-1}\right], \quad n \geqslant 2, \qquad (2)$$

where i runs from 1 to N. The calculations are done via a third order Gear algorithm [20] and the outcomes are presented using dimensionless quantities. To make our results dimensionless, we assign 10^{-5} m, 2.36×10^{-5} kg, and 1.0102×10^{-3} s as the units of distance, mass, and time, respectively [21]. In our simulations, the grain diameter was set to 100, the mass was set to unity, and the integration time step dt was set to 10^{-6}. As in many previous studies, we chose $a = 5657$, a value ($=4.14 \times 10^7$ N m$^{-3/2}$) which is in the range of elasticity of silicate materials. In order to compare to previous works, we have typically used $a = 1$ when we probed $n = 2.1$ cases. In the calculations, we saw no change in the total energy after 10^9 time steps.

We now briefly summarize the findings of our earlier work on which the present study will develop. Any initial velocity given to an edge grain develops into a propagating solitary wave in these systems within about ten grain diameters from the edge [22]. The solitary wave velocity depends on the magnitude of the initial perturbation [13,15]. These solitary waves possess an average width $\overline{W}$ that depends on n. For $n = 5/2$, measured at a precision of 10^{-12} in grain diameters, we find $\overline{W} = 7$ [22], whereas for $n = 2.1$ we estimate $\overline{W} = 11$ (this value is precision limited [23]). The finite width of these waves, and the fact that as per virial theorem [24], they carry $\frac{n}{n+2}$ of the total energy as kinetic and the rest as potential, implies that they are "soft" objects.

Given that the grains are held within reflecting boundary walls, any solitary wave must turn around when it hits a wall. It is now well established that such a turnaround is accompanied by squeezing of the solitary wave at the wall with the production of secondary solitary waves. These secondary waves carry much less energy and are formed after the original wave reflects, albeit with less energy than what it hit the wall with. Extrapolating from here, it may seem that solitary waves will continue to break upon any collision. This argument is incorrect because such repeated breakdowns of a nondispersive wave into many smaller ones would mean that eventually the original pulse would be comprised of an infinite number of infinitesimally small pulses. Recent simulations reported in Ref. [25] have suggested that there are solitary wave collisions where one wave grows while the other loses energy as a result of a collision.

This paper is organized as follows. We first report on how solitary waves can grow as a result of collisions in our systems. This is followed by a quick summary of how a single solitary wave decays into an equilibriumlike state and then to our analysis of what happens when multiple perturbations are effected at the same or at different times in these systems. As we shall see, our studies strongly suggest that purely nonlinear, conservative systems, which propagate energy as solitary waves and in acoustic vacuum, are capable of sustaining high levels of energy fluctuations indefinitely. We close with a brief discussion of the implications of this study.

III. RESULTS

A. Solitary wave collisions

We first present the results of our dynamical simulations to suggest that when two solitary waves collide, there are scenarios where one gains energy while the other loses it. To probe this we consider the collision of unequal solitary waves when they are moving in the same and in opposite directions [23]. The studies have been done using chains with 499 or 500 grains, depending on the nature of the collision to be probed. As a result they are data intensive dynamics simulations and care is needed to ensure accuracy of the integration of the equations of motion (our energy conservation is typically accurate to at least a billionth of the total energy). For simplicity, we have chosen scenarios where the waves meet at the center of a grain. Off-center collisions are more challenging to characterize. No matter where the collision, however, the outcomes appear to be identical to the cases discussed here.

In the overtaking collision problem, where a fast moving and hence higher energy solitary wave overtakes a less energetic one at a grain center, we found that the former wave undergoes a slight energy gain by "consuming" a part of the latter, which in turn loses some energy [Figs. 1(a) and 1(c)]. The process is complex because the total combined energy of the *two waves* gets reduced due to the collision process, thereby indicating the formation of secondary solitary waves of low energy content. During this collision, the central grain where the waves meet ends up moving faster than either of the neighboring grains. This new velocity of the central grain appears to determine the magnitude of the resultant solitary waves. The larger postcollision wave forms immediately following the collision process, carrying away as much energy as possible.

In the head-on collision case the outcome is different. Here the more energetic wave loses energy while the lower energy wave gains energy [Figs. 1(b) and 1(d)]. There are secondary solitary waves which form in this case as well. Total energy is conserved in both the cases, which attests to the accuracy of the results.

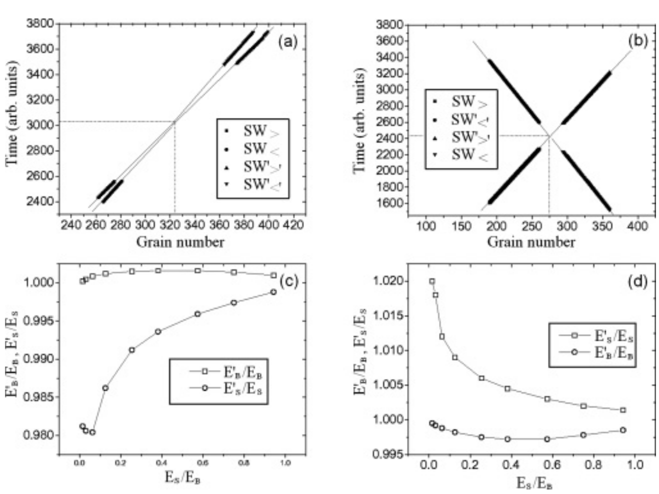

FIG. 1. (a) and (b) show space versus time plots for overtaking and head-on collisions, respectively, of two unequal solitary waves in a chain of 500 grains. The symbols "$SW_{>\text{ or }<}$" and "$SW_{>'\text{ or }<'}$" denote before and after and larger and smaller waves, respectively. (c) and (d) describe the gain or loss of energy suffered due to the collision by the larger and small wave for overtaking and head-on collisions, respectively.

Our studies suggest that the energetics of the collision is typically determined by the dynamics of the central grain where the two unequal solitary waves meet. In both types of collisions probed, the magnitudes of the kinetic energy changes of the more energetic and the smaller solitary waves are maximized when the kinetic energy of the smaller wave is approximately half that of the larger wave. Any kind of analytic work to attain a deeper understanding of these processes requires a more accurate solution of the equations of motion than what is currently available [13,26].

B. δ-Function perturbation(s) at $t = 0$ and quasiequilibrium

When a δ-function perturbation in velocity is given to an edge grain at $t = 0$, we first see the formation of a propagating solitary wave, which then backscatters from the opposite wall and the process of breakdown of the wave begins. After a few round trips of the original pulse, the energy distribution of the system reveals a roughly gray color with significant energy fluctuations in the gray scale plots shown in Fig. 2(a) for $n = 2.1$ and Fig. 2(c) for $n = 2.5$ [see Eqs. (1) and (2)]. The dynamics is characterized by the growth of approximately half of the waves and the decay of the rest during collisions between unequal waves. Longer time simulations do not reveal any significant changes in the energy distribution of the system. We call the gray state the *quasiequilibrium* state (see [15,16] for related discussions). The grains possess a Gaussian distribution of velocities (as expected from the central limit theorem) and the kinetic energy fluctuations seen in the system are "large" [15]. Since the energy propagates as solitary waves, each of width $\overline{W} > 1$, the particles themselves never carry approximately the same amount of kinetic energy when a time average is performed. Hence, the equipartition theorem is not satisfied by these systems [15,16].

As $n \to 2$, $\overline{W} \to \infty$ [13], and the quasiequilibrium phase begins to resemble a state with equipartitioned energy. Comparing the panels in Fig. 2, it is evident that the $n = 2.1$ system achieves a better gray scale than the $n = 2.5$ system. Further, since the typical magnitude of energy fluctuations possesses a system size dependence, as $N \to \infty$, the system should tend to resemble an equilibrium phase.

Let us now ask the following questions: (i) Is it possible to perturb the system in such a way that multiple temporally stable or unstable cold and hot spots may develop in these systems? (ii) How does the system evolve if a second round of perturbation that is identical or similar to the one affected at $t = 0$ is introduced? Since the odd numbered systems are the ones that allow cold spot formation, we henceforth consider only such chains below.

To explore (i), we study a system in which the two edge grains are perturbed in such a way that identical solitary waves end up propagating toward each other [Figs. 2(b) and 2(d)]. When N is odd, the two solitary waves must meet at the center

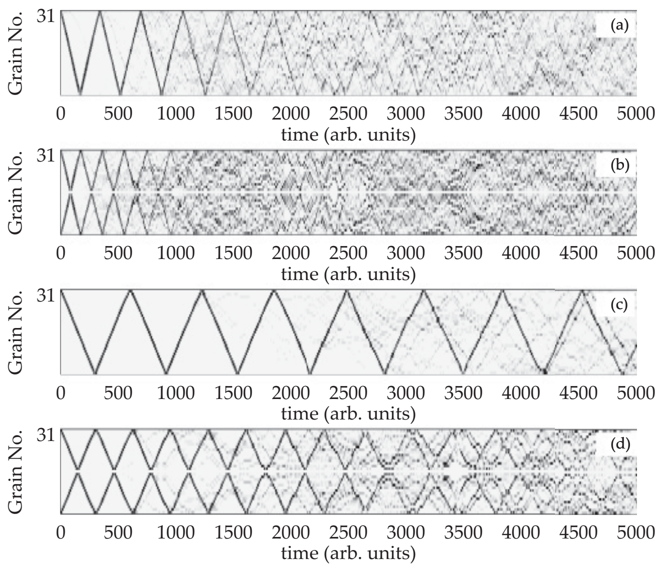

FIG. 2. We show grain position versus time versus energy (in gray scale) in (a) and (c) for 31 grain chains with $n = 2.1$ and 2.5, respectively. In both cases, a velocity imparted to grain 31 at $t = 0$ leads to the formation of a solitary wave. The wave propagates along the dark straight lines with properties that are appropriate for the system. The wave breaks down during wall collisions. However, one of the waves gains energy when later, two unequal waves collide. At large enough times, the system acquires gray scale features to reveal that the energy in the original impulse and in the initial solitary wave that formed has now been distributed throughout the system. As can be seen, the energy distribution is not at all homogeneous. Further, it changes significantly with time. In (b) and (d) we show studies where equal and opposite velocities are imparted at the two end grains of each chain at $t = 0$. This results in two equal and opposite propagating solitary waves colliding in the center of the chain. The central grain does not move and defines a cold spot. Again, for (b) $n = 2.1$ and for (d) $n = 2.5$. The system exhibits large and sustained energy fluctuations.

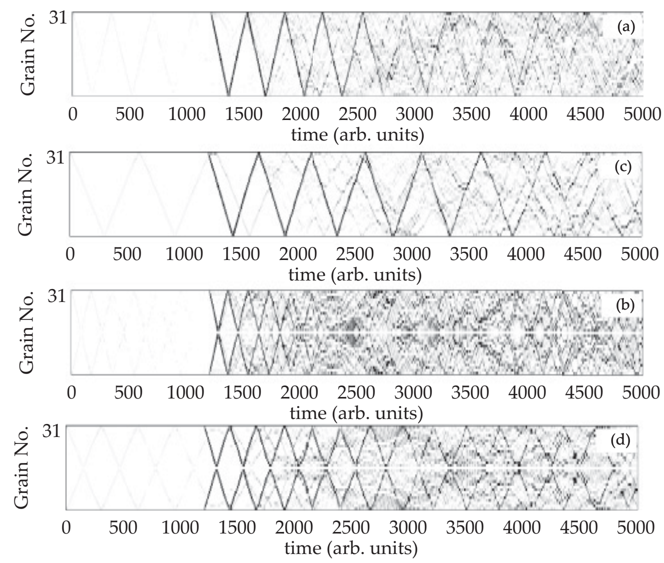

FIG. 3. The study shown here is identical to that shown in Figs. 2(a)–2(d) except that these systems have been perturbed a second time at $t = 1200$. The velocity of perturbation is five times stronger than that in the original perturbation at $t = 0$. The reason for the stronger perturbation is to ensure that the energy fluctuations resulting from such a perturbation can be clearly seen. While the magnitude of the energy fluctuations are larger here, the nature of the fluctuations is very similar to that shown in Figs. 2(a)–2(d).

of the $\frac{N}{2} + 1$ grain. When N is even, they would meet at an interface between grains numbered $\frac{N}{2}$ and $\frac{N}{2} + 1$. In the former case, the center grain would get compressed from both ends and hence would store some potential energy. The center grain of course will be at rest at all times, thereby carrying zero kinetic energy and creating what we will henceforth call a "cold spot." In the latter case, it is not possible to store potential energy at a grain edge and hence the system ends up with slightly more kinetic energy (in this context, see the recent experimental work of Santibanez *et al.* in [27]). In Figs. 2(b) and 2(d) we show the kinetic energy versus grain number versus time results for odd numbered chains when $n = 2.1$ and $n = 2.5$, respectively. Regardless of the magnitude of n and the system size, the odd numbered systems with two opposing perturbations imparted at $t = 0$ reveal the presence of a time invariant cold spot. The results suggest that as $n \to 2$, the system tends to reach the quasiequilibrium state more rapidly. To our knowledge it is not possible to form a sustained cold spot in a system that is in an equilibrium phase and hence in that respect, the quasiequilibrium phase appears to be special. The presence of harmonic forces would allow energy dispersion and hence make the cold spot unsustainable.

C. How robust is the quasiequilibrium phase?

To explore the robustness of the system's state at late enough times, we carried out studies in which a totally new δ-function velocity perturbation was reinitiated at an edge of the system at a later time. The edge used was the same as the one in which the original perturbation was initiated. We typically introduced this new perturbation at $t = 1200$ units. As is evident from Figs. 2(a) and 2(b), the $n = 2.1$ system has reached the quasiequilibrium phase by that time with a well developed temporally fluctuating, grainy, gray scale structure. In contrast, we find that the quasiequilibrium behavior is yet to develop for the $n = 2.5$ case in Figs. 2(c) and 2(d). Naturally, the steeper the potential, the longer the solitary wave lasts and the more slowly the quasiequilibrium phase would emerge. Results from our studies of the system with the second perturbation are shown in Figs. 3(a)–3(d).

The second perturbation is typically stronger than the first one in the studies shown here [Figs. 3(a)–3(d)] [28]. The increase in strength makes it easier to actually "see" the changes that happen due to this perturbation in Figs. 3(a)–3(d). The increase in the energy speeds up the solitary waves and this can be readily seen by the slope changes of the lines in Figs. 3(a)–3(d). It is worth noting that the second perturbation does not have a marked effect on the properties of the fluctuations at large times. In Figs. 3(b) and 3(d), the cold spot is sustained along with the development of short lived hot (shown as darker) regions just as seen in Figs 2(a)–2(d) at late times. In short, the introduction of a second or more perturbation that is identical in position at later times does not change the detailed features of the quasiequilibrium phase.

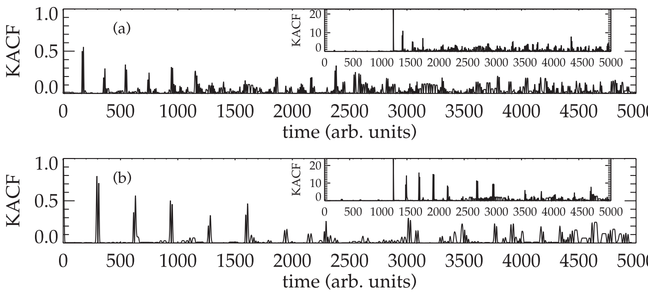

FIG. 4. The normalized kinetic energy autocorrelation function (see text) is shown as function of time for the 31 grain system for cases shown in Figs. 2(b) and 2(d) and Figs. 3(b) and 3(d) (insets).

To check how much of the "memory" of the initial perturbation is retained by the system, we calculated the dynamical kinetic energy auto-correlation function (KACF) defined as $v_1^2(t=0)v_1^2(t)/v_1^2(t=0)^4$ as a function of t when the first and the last grains are simultaneously initiated with equal magnitude and opposite in direction velocity perturbations at $t = 0$. In this case it is not required to perform a time average because we are probing whether the system can ever return to its initial state or not, and hence exploring if KACF can return to unity or if it decays, and if so, how. The results are shown in Figs. 4(a) and 4(b) for the $n = 2.1$ and $n = 2.5$ cases, respectively. The insets in each figure show the same systems when they have been subjected to a second perturbation around $t = 1200$ [hence these pertain to cases shown in Figs. 3(b) and 3(d)]. As is evident by looking at the behavior at large enough times, the peaks in the KACF remain of approximately the same height, thereby suggesting that the system reaches a peculiar, strongly fluctuating, equilibriumlike state. The sharp peaks are indicative of the nature of excitations that are in the system—namely, only the discrete solitary waves.

D. Effects of synchronous and asynchronous perturbations on quasiequilibrium

Figures 5(a)–5(d) show the nature of energy transport in the system when a pair of *solitary wave trains* are initiated at the boundaries using appropriate initial conditions in a chain with odd numbered grains. One way to make solitary wave trains is to make the perturbation last for a few time steps, while another way would be to simultaneously and equally strongly perturb a fixed number of edge grains at some initial time (see [22] for an in-depth discussion on initial conditions). We study two distinct wave train collisions—those between wave trains that are identical and different but initiated simultaneously at $t = 0$ (synchronous), while the other is for wave trains that are initiated with some time delay (asynchronous).

Figures 5(a) and 5(b) show the results for a collision of two identical solitary wave trains, each containing three solitary waves of progressively decreasing energy content. In Fig. 5(a), the waves are released simultaneously, whereas in Fig. 5(b) there is a delay by ten time steps. For synchronous wave train collisions in Fig. 5(a) we find the emergence of a larger cold spot around $t \approx 150$ although this is not completely cold. The short time features of Figs. 5(a) and 5(b) are very similar. However, in Fig. 5(b), the energy propagation behavior turns out to be different than in Fig. 5(a) at late times. Notably, the cold spot gets obliterated in Fig. 5(b). In Fig. 5(c), the signals are released 100 time steps apart. The cold spot moves up in space and becomes visibly asymmetric. Not surprisingly, the long time dynamics is also different compared to Figs. 5(a) and 5(b).

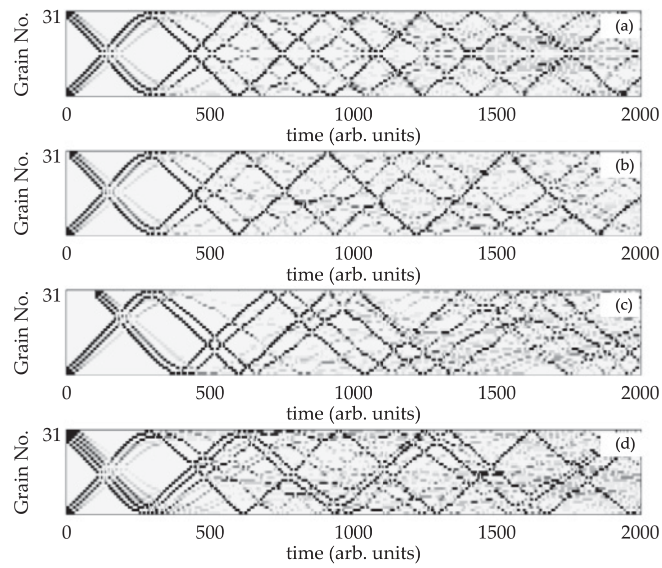

FIG. 5. Collisions between solitary wave trains with three progressively decreasing solitary waves are shown in (a)–(c). In (a) the trains form simultaneously, in (b) the train released from grain 31 is ten steps delayed, and in (c) the train released from grain 31 is 100 steps delayed. The cold spot is not sustainable when symmetry is broken in (b) and (c). (d) A five wave train collides with a three wave train to form longer lived and larger cold regions. Observe that multiple solitary wave collisions at late times can cause hot spots in these systems (see text for more details).

Our studies suggest that for synchronous systems where identical wave trains collide and $n \to 2$ where $\overline{W} \to \infty$ and $N < \infty$, these cold spots would become larger. However, the larger spots are short lived and not sustainable and only a central cold spot remains time invariant. The solitary waves in the train slowly break down forming secondary solitary waves and eventually the system ends up in a quasiequilibrium phase with a central cold spot.

In Fig. 5(d) we show the results for collision when the two wave trains contain different numbers of solitary waves. The cold spots now become more long lived and asymmetric in space and time. There is some initial evidence of several waves colliding at late times to form short lived hot spots (see around $t \approx 1550$–1600). The collision of solitary wave trains hence raises an intriguing possibility, namely, whether these collisions could give rise to the birth of unstable regions with large or small cold or hot spots.

E. Effects of many synchronous perturbations and late time second perturbations

To address this question we consider larger chains with multiple synchronous perturbations. Specifically, we initiate velocities $(v_0, -v_0)$ at grains (1,11), (13,23), (25,35), and (37,47) in a $N = 47$ system and (1,11), (13,23), (25,35),

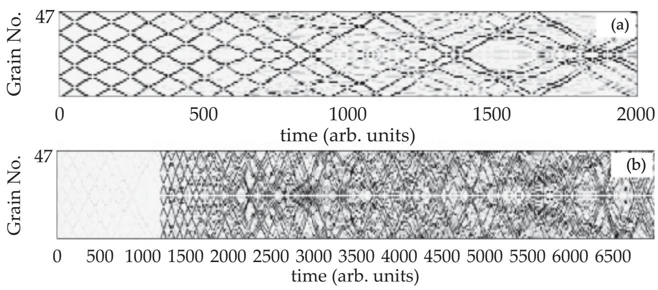

FIG. 6. Here we show a 47 grain system that has been subjected to symmetric perturbation at $t = 0$ at multiple points (see text) in (a) and the same has been repeated at $t = 1200$ in (b). These studies are for $n = 2.1$. (a) reveals that the nature of the system dynamics is similar to and somewhat richer than what is seen for Figs. 3(b) and 3(d).

(37,47), . . . ,(109,119) in a $N = 119$ chain as shown in Fig. 6(a) for $n = 2.1$ and in Figs. 7(a) and 7(b) for $n = 2.5$ and $n = 2.1$, respectively. In Fig. 6(b) we show a study in which the $N = 47$ chain has been subjected to an identical second perturbation which is five times stronger (just to make the effect visible in the plot) at $t = 1200$.

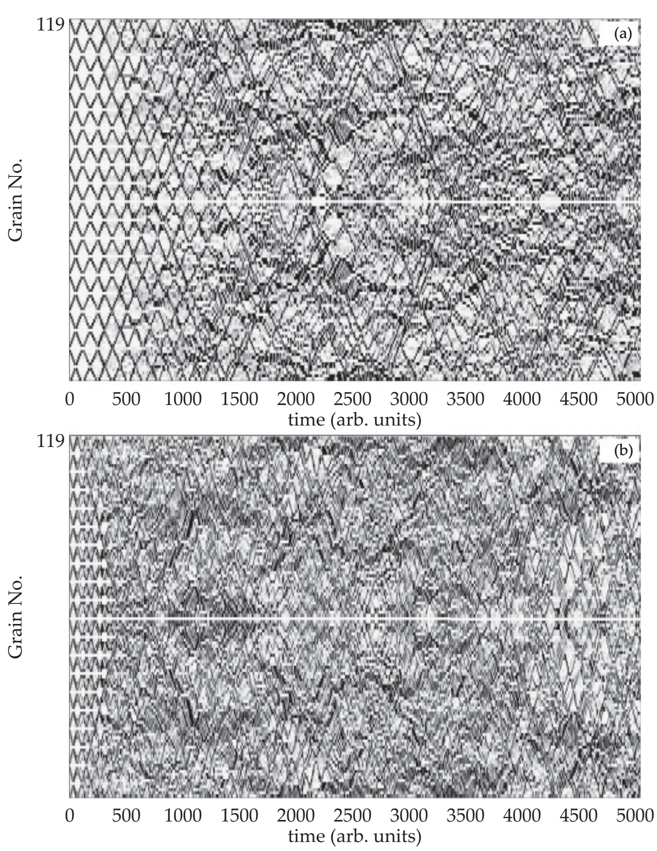

FIG. 7. Here we show the time evolution of a 119 grain chain for $n = 2.5$ in (a) and $n = 2.1$ in (b). Clearly, the presence of multiple perturbations enrich the nature of fluctuations in these systems. Significant kinetic energy is seen near the walls over extended times. Peculiar low and large fluctuation areas that last for a significant number of time steps are also seen (see text).

The dynamics of these systems turn out to be rich and complex and marked by the formation of sizable metastable cold and hot spots in addition to the small central cold spot that arises due to symmetry considerations. In addition, in both the cases, the system shows significant kinetic energy near the boundaries. We hypothesize that the formation of metastable hot spots near the boundaries arise because solitary waves of all energies collide there, and hence they spend more time near the boundaries as they approach and recede from those regions. The many symmetric hot and cold spot regions in the vicinity of the center beyond $t \approx 700$ in Fig. 6(a) are unexpected. No significant change happens to the dynamics when in the $N = 47$ case the system is perturbed a second time. The new perturbation again relaxes to the highly fluctuating state except that the average kinetic energy of the grains ends up increasing appropriately. We calculated the cosine transform of the kinetic energy as a function of time for the case shown in Fig. 6(b). Our results reveal a featureless, Gaussian-like kinetic energy power spectrum centered around zero frequency. The spectrum confirms that the system does not have any characteristic frequencies but rather a smooth distribution of the same that is heavily weighted in the vicinity of the zero frequency mode.

The features seen in Figs. 7(a) and 7(b) are very similar. The differences arise from the fact that solitary waves are wider in Fig. 7(b) where $n = 2.1$. Hence, the energy exchange between the grains or the energy mixing transpires more rapidly. Strong and sustained energy fluctuations are evident in both of these cases. The formation of circular hot spot rings in our space-time versus energy diagrams for t between 1500 and 4000 in Fig. 7(a) and between about 750 and 1500 in Fig. 7(b) are completely unexpected. The large cold spots that form for t between 2500 and 3500 and 4000 and 4500 in Fig. 7(a) and the three similar regions between t of 3500 and 4500 in Fig. 7(b) are intriguing. A second perturbation study of the $N = 119$ system shows that the features seen in Figs. 7(a) and 7(b) are sustained. Hence, those studies are not being shown here.

IV. SUMMARY AND CONCLUSION

We have presented a study of the time evolution of grains in an unloaded granular chain except that we have assumed that the system is lossless. Such a system is an example of a purely nonlinear system in the sense that there is no harmonic term in the interparticle potential in such a system. The choice of considering a system that is lossless is tied to the goal of probing the dynamics of these systems at late times—something that cannot be done for mechanical systems which are inevitably lossy. However, as alluded to in [12], it may be possible to realize these systems as circuits.

The initial part of the study in Sec. III A discussed how one of the two unequal colliding solitary waves in the chain can gain energy. Our simulations suggest that for head-on collisions the smaller wave gains energy. However, our simulations reveal that when a more energetic wave overtakes a less energetic wave, the more energetic wave gains energy [25]. The balance between the rate at which the solitary waves break down and the rate at which they grow eventually makes it possible for the system to reach a peculiar equilibriumlike phase that is characteristic to these purely nonlinear systems [29].

The study of the features and the robustness of the fluctuations in time has been addressed in Secs. III B–III E. A particular characteristic of this equilibriumlike or quasiequilibrium phase is that very large energy fluctuations are possible—and by very large we mean that the energy can vary between zero and several times the average energy per grain. The basic reason for these large fluctuations is that all energy must propagate as solitary waves and the amount of energy carried by a wave can vary. One can ask whether these fluctuations are peculiar to one dimension or whether they can exist in higher dimensions. The study hence raises the following intriguing possibility. Are there physical or biological systems where these kinds of nonlinear forces exist, and if so, can such large fluctuations actually be seen? A related question is how would nature exploit such large energy fluctuations if they are found in some systems? Can such fluctuations naturally arise in biological systems and be responsible for the ability of biological systems to adapt to surrounding changes [30]? These are some of the questions that are currently under investigation.

In closing, it is natural to wonder what happens when in addition to the Hertz term, a quadratic term is introduced in Eq. (1). Such a scenario is realized in loaded granular chains. It turns out that the presence of the quadratic term in the potential function may suppress the strong fluctuations that are characteristic of the equilibrium phase, but this is not always the case. Subtle competitions may arise between the harmonic and the nonlinear forces with unexpected consequences. This topic is currently under investigation and will be the subject of a future paper [31]. The study of purely nonlinear systems hence may involve qualitatively different dynamics than those of strongly nonlinear systems with a quadratic potential.

ACKNOWLEDGMENT

This work has been supported by the US Army Research Office.

[1] See, for example, R. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957); Rep. Prog. Phys. **29**, 255 (1966).

[2] R. Zwanzig, Phys. Rev. **124**, 983 (1961).

[3] H. Mori, Prog. Theor. Phys. **33**, 423 (1965); **34**, 399 (1965).

[4] M. H. Lee, Phys. Rev. B **26**, 2547 (1982); Phys. Rev. Lett. **87**, 250601 (2001).

[5] See, for example, in P. Ullersma, Physica **32**, 27 (1966); **32**, 56 (1966); **32**, 74 (1966); J. T. Hynes, J. Stat. Phys. **11**, 257 (1974); P. E. Phillipson, J. Math. Phys. **15**, 2127 (1974); G. W. Ford, J. T. Lewis, and R. F. O'Connell, J. Stat. Phys. **53**, 439 (1985); J. Florencio and M. H. Lee, Phys. Rev. A **31**, 3231 (1985); D. Vitali and P. Grigolini, *ibid.* **39**, 1486 (1989).

[6] E. Fermi, J. Pasta, and S. Ulam, Los Alamos National Laboratory Report No. LA-1940, 1955 (unpublished).

[7] J. Ford, Phys. Rep. **213**, 271 (1992).

[8] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).

[9] For an extensive discussion on the various forms of solitary waves, see *Solitons in Condensed Matter Physics*, edited by A. R. Bishop and T. Schneider (Springer, Berlin, 1978); M. Remoissenet, *Waves Called Solitons* (Springer, Berlin, 1996).

[10] D. K. Campbell, P. Rosenau, and G. M. Zaslavsky, Chaos **15**, 01510 (2005).

[11] R. Reigada, A. Romero, A. Sarmiento, and K. Lindenberg, J. Chem. Phys. **111**, 1373 (1999); R. Reigada, A. Sarmiento, and K. Lindenberg, Phys. Rev. E **64**, 066608 (2001); K. Ahnert and A. Pikovsky, *ibid.* **79**, 026209 (2009).

[12] We note that there has been a significant amount of work done on the nature of energy loss in granular systems, as, for example, in O. R. Walton and R. L. Braun, J. Rheol. **30**, 949 (1986); N. V. Brilliantov, F. Spahn, J. M. Hertzsch, and T. Pöschel, Phys. Rev. E **53**, 5382 (1996); A. Rosas, A. H. Romero, V. F. Nesterenko, and K. Lindenberg, Phys. Rev. Lett. **98**, 164301 (2007). However, we contend that it is worth studying the lossless behavior of these systems because of the following two reasons: (i) We can look into the long time dynamics of these systems computationally and take advantage of the relatively simple potential energy behavior compared to the FPU system and (ii) because it may be possible to construct electrical circuits that can realize these nonlinear mechanical systems in the lossless limit as shown by several authors, e.g., R. W. Newcomb and N. El-Leithy, Circ. Syst. Signal Proc. **5**, 321 (1986), and references therein.

[13] V. Nesterenko, J. Appl. Mech. Tech. Phys. **5**, 733 (1983); A. N. Lazaridi and V. Nesterenko, *ibid.* **26**, 405 (1985); V. Nesterenko, J. Phys. IV **4**, C8 (1994); A. Chatterjee, Phys. Rev. E **59**, 5912 (1999). For a comprehensive review of the dynamics of granular chains, see V. Nesterenko, *Dynamics of Hetereogeneous Materials*, Chap. 1. (Springer, New York, 2001).

[14] For example, see C. Coste, E. Falcon, and S. Fauve, Phys. Rev. E **56**, 6104 (1997); S. Sen, M. Manciu, and J. D. Wright, *ibid.* **57**, 2386 (1998); E. J. Hinch and S. Saint-Jean, Proc. R. Soc. London, Ser. A **455**, 3201 (1999); E. Hascoët, H. J. Herrmann, and V. Loreto, Phys. Rev. E **59**, 3202 (1999); E. Hascoët and H. J. Herrmann, Eur. J. Phys. B **14**, 183 (2000); S. Sen and M. Manciu, Physica A **268**, 644 (1999); J. Hong and A. Xu, Appl. Phys. Lett. **81**, 4868 (2002); J. Lee, S. Park, and I. Yu, Phys. Rev. E **67**, 066607 (2003); D. T. Wu, Physica A **315**, 194 (2002); A. Rosas and K. Lindenberg, Phys. Rev. E **68**, 041304 (2003); **69**, 037601 (2004); S. Job, F. Melo, A. Sokolow, and S. Sen, Phys. Rev. Lett. **94**, 178002 (2005); R. L. Doney and S. Sen, Phys. Rev. E **72**, 041304 (2005); Phys. Rev. Lett. **97**, 155502 (2006); J. Hong, *ibid.* **94**, 108001 (2005); L. Vergara, *ibid.* **95**, 108002 (2005); V. F. Nesterenko, C. Daraio, E. B. Herbold, and S. Jin, *ibid.* **95**, 158702 (2005); C. Daraio, V. F. Nesterenko, E. B. Herbold, and S. Jin, *ibid.* **96**, 058002 (2006); A. Sokolow, J. Pfannes, R. Doney, M. Nakagawa, J. Agui, and S. Sen, Appl. Phys. Lett. **87**, 154104 (2005); R. Doney, J. Agui, and S. Sen, J. Appl. Phys. **106**, 064905 (2009).

[15] A. recent review of impulse propagation in granular chains appears in S. Sen, J. Hong, J. Bang, E. Avalos, and R. Doney,

Phys. Rep. **462**, 21 (2008). Here, in Fig. 6.3, the presence of large kinetic energy fluctuations arising from single perturbations in granular chains were mentioned. The discussions were based on some earlier works where this proposed equilibriumlike phase was discussed. However, the work presented here predates the development of our understanding of how solitary waves with unequal energies interact with each other and of how multiple solitary waves released in a chain interact.

[16] S. Sen, T. R. Krishna Mohan, and J. M. M. Pfannes, Physica A **342**, 336 (2004); S. Sen, J. M. M. Pfannes, and T. R. Krishna Mohan, J. Korean Phys. Soc. **46**, 571 (2005); T. R. Krishna Mohan and S. Sen, Pramana, J. Phys. **64**, 423 (2005).

[17] See, for example, in R. Livi, M. Pettini, S. Ruffo, M. Sparpaglione, and A. Vulpiani, Phys. Rev. A **28**, 3544 (1983); **31**, 1039 (1985); J.-P. Eckmann and C. E. Wayne, J. Stat. Phys. **50**, 853 (1988); C. Claude, Y. S. Kivshar, O. Kluth, and K. H. Spatschek, Phys. Rev. B **47**, 14228 (1993); H. Krantz, R. Livi, and S. Ruffo, J. Stat. Phys. **76**, 627 (1994); T. Cartegny, T. Dauxois, S. Ruffo, and A. Torcini, Physica D **121**, 109 (1998); V. V. Mirnov, A. J. Lichtenberg, and H. Guclu, *ibid.* **157**, 251 (2001); R. Reigada, A. Sarmiento, and K. Lindenberg, Phys. Rev. E **64**, 066608 (2001); D. K. Campbell, P. Rosenau, and G. Zaslavsky, Chaos **15**, 015101 (2005); M. Sato, B. E. Hubbard, and A. J. Sievers, Rev. Mod. Phys. **78**, 137 (2006); A. F. Vakakis, L. I. Manevitch, Y. V. Mlkhlin, V. N. Pilipchuk, and A. A. Zevin, *Introduction, in Normal Modes and Localization in Nonlinear Systems* (Wiley-VCH Verlag GmbH, Weinheim, 2008); E. Avalos and S. Sen, Phys. Rev. E **79**, 046607 (2009).

[18] J.-Y. Ji and J. Hong, Phys. Lett. A **260**, 60 (1999), see Sec. 2.

[19] H. Hertz, J. Reine Angew. Math. **92**, 156 (1881).

[20] M. P. Allen and D. J. Tildesley, *Computer Simulation of Liquids* (Clarendon, Oxford, 1988).

[21] R. S. Sinkovits and S. Sen, Phys. Rev. Lett. **74**, 2686 (1995); S. Sen and R. S. Sinkovits, Phys. Rev. E **54**, 6857 (1996).

[22] A. Sokolow, E. Bittle, and S. Sen, Europhys. Lett. **77**, 24002 (2007); S. Job, F. Melo, A. Sokolow, and S. Sen, Granular Matter **13**, 10 (2007).

[23] D. Sun, Ph.D thesis, SUNY Buffalo, 2009.

[24] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Addison-Wesley, Reading, 2000).

[25] W. Z. Ying, W. S. Jin, Z. X. Ming, and L. Lei, Chin. Phys. Lett. **24**, 2887 (2007).

[26] S. Sen and M. Manciu, Phys. Rev. E **64**, 056605 (2001).

[27] F. Santibanez, R. Muñoz, A. Caussarieu, S. Job, and F. Melo, Phys. Rev. E **84**, 026604 (2011).

[28] Specifically for these calculations we used $v_1(t = 0) = 0.01$ and $v_1(t = 1200) = 0.05$, where v_1 refers to the velocity of grain 1. When perturbations are initiated at both ends, we have used $v_1 = 0.01$ and $v_N = -0.01$. We have used arbitrary units throughout the work.

[29] A discussion on some features of quasiequilibrium in a mass-spring chain with metastable breathers appears in S. Sen and T. R. Krishna Mohan, Phys. Rev. E **79**, 036603 (2009).

[30] See, for example, R. J. Bagley and J. D. Farmer, in *Artificial Life II*, edited by J. D. Farmer, C. Langton, S. Rasmussen, and C. Taylor (Addison Wesley, Reading, 1991).

[31] Y. Takato and S. Sen (unpublished).