Nonlinear wave propagation in a gravitating quantum fluid

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The nonlinear wave propagation in a Bose-Einstein gravitationally condensate gas is investigated using a gravitating quantum fluid model. The small-amplitude dynamics is shown to be governed by a Korteweg–de Vries equation with a nonlocal term. The quantum effect provides the necessary dispersion, and the gravitational effect is responsible for the nonlocal term. This novel equation is solved analytically. The implications of such a soliton-like solution are outlined.

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I. INTRODUCTION

At very low temperatures, all particles (bosons) in a dilute Bose gas confined in a trapping potential condense to the same quantum ground state and form a Bose-Einstein condensate (BEC) [1,2]. The BEC has been observed experimentally in magnetically trapped dilute vapors of alkali metals [3,4]. In most cases, the confining traps are well approximated by harmonic potentials. In general, the phenomena associated with BEC are studied by the Gross-Pitaevskii equation (GPE). The GPE has also been used to develop the quantum hydrodynamical model [5–10] to study the dynamics of quantum degenerate gases.

In the astrophysical context, the concept of BEC is introduced to study the possible existence of Bose stars [11,12] as well as, in the late-time cosmological phase transition theories, to explain the dark matter physics [13,14]. The condensate is described by the nonrelativistic GPE with Newtonian gravitational potential as trapping potential (natural trap) or by the gravitating quantum fluid equations [15,16].

Moreover, O'Dell *et al.* [17,18] have shown that the particular configuration of intense off-resonant laser beams introduces a gravity-like attractive nonlocal interaction between atoms within the laser wavelength, which gives rise to a stable BEC without an additional trap. Actually, any particle that is trapped in a sufficiently deep and wide potential well is settled in quantum bound states. Recently, Nesvizhevsky *et al.* [19] have shown experimentally the gravitational quantum bound states of neutrons, where the gravitational field provides the necessary confining potential well. This is experimental evidence, where gravitational and quantum forces act simultaneously. Thus, all of these studies suggest the possibility of realizing the gravitating BEC in the laboratory.

The nonlinear structures such as envelope solitons (dark and bright), vortices, etc., are investigated in BEC with the harmonic trapping potential modeled by the GPE [20–23]. Also, it has already been shown [24] that a purely attractive gravitational-like potential prevents the collapse of localized waves and gives rise to the formation of localized structures (solitons). Moreover, in the case of repulsive interactions (s-wave scattering length $a_s > 0$: bosons are repulsive) when the internal energy (interaction energy) is very large compared to the kinetic energy of the atoms, BEC is properly described by hydrodynamic equations for superfluid at zero temperature, where the pressure that arises due to short-range interactions between particles is related to density by a barotropic equation of state [1]. Therefore, in this paper we will discuss the weakly nonlinear localized structures in one spatial dimension in a BEC with gravitational attractive nonlocal trapping potential using the reductive perturbation theory (RPT). The medium is modeled as a gravitating quantum neutral fluid. Using the RPT, we have derived a modified form of the Korteweg–de Vries (KdV) equation. The modification occurs due to the gravitational effects. It is interesting to note that unlike the usual gravitating neutral fluid (dispersionless), here the quantum diffraction term provides the necessary dispersion to balance the wave-breaking nonlinearity and form soliton-like structures.

This paper is organized as follows: In Sec. II, the basic equations, linear dispersion relation, and the Jeans criteria for instability are discussed. The nonlinear evolution equation is derived in Sec. III. The gravitational effects on the soliton solution of this nonlinear equation are discussed in Sec. IV. The results are summarized and discussed in Sec. V.

II. BASIC EQUATIONS AND LINEAR MODE

To model the BEC, we assume that at zero temperature, all the bosons have condensed, and consider the mean-field (frictionless) analysis using the GPE in a homogeneous condensate [1]. This equation has the form of the following nonlinear Schrödinger equation with condensate wave function $\psi(\mathbf{r},t)$:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + g\mid\psi(\mathbf{r},t)\mid^2\right]\psi(\mathbf{r},t), \tag{1}$$

where $g(=4\pi a_s\hbar^2/m$ and a_s is the s-wave scattering length. Here $a_s > 0$, as we have considered the repulsive self-interactions), is the coupling constant, m is the boson mass, and $V_{\rm ext}(\mathbf{r})$ is the external (trapping) potential to realize Bose-Einstein condensation [1]. We know that the BEC occurs when de Broglie wavelength $\lambda_{\rm DB} \gg a_s$. However, in the presence of the gravitational trap (no additional trap is required), a stable BEC can be formed only if $a_* \gg \lambda_{\rm DB} \gg a_s$, where $a_* = 4\pi^2\hbar^2/mu$ is the Bohr radius associated with

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gravitational coupling u (characteristic values: $a_* \sim 10$ cm, $\lambda_{DB} \sim 10^{-5} - 10^{-3}$ m, and $a_s \sim 3$ nm) [17,18]. These investigations lead us to assume $V_{ext} = V_G$, the gravitational potential. The above equation is coupled with the Poisson equation for V_G. For a homogeneous medium, there is an ambiguity in defining the equilibrium of this Poisson equation. To define the equilibrium in a gravitating system, the concept of the "Jeans swindle" [25–27] is usually introduced. However, to define this equilibrium here, we consider [28] the following Poisson equation for the gravitational potential V_G:

$$\nabla^2 V_G = 4\pi G(\rho - \rho_0), \tag{2}$$

where $\rho = mn|\psi(\mathbf{r},t)|^2$ is the mass density, ρ_0 is its equilibrium value, and n is the total number of bosons in the condensate.

In the standard approach [1] to BEC, we use the Madelung transformation: $\psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)/m}\exp[iS(\mathbf{r},t)/\hbar]$, where $S(\mathbf{r},t)$ has the dimension of an action. Defining the irrotational flow velocity $\mathbf{v} = \nabla\mathbf{S}/m(\nabla\times\mathbf{v} = 0)$ and then substituting this transformation into Eq. (1), we obtain the gravitating quantum fluid (superfluid) equations [15,16]: The mass density conservation and momentum equation can be written as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{3}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p - \rho\nabla V_G + \frac{\hbar^2\rho}{2m^2}\nabla\left(\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right), \tag{4}$$

where p is the pressure and the term containing $\hbar^2$ is the quantum potential (Bohm potential) [29]. In the hydrodynamical (superfluid) model for BEC, the pressure p is related to the mass density ρ by the following barotropic equation of state [1,2]:

$$p = \frac{1}{2}g\left(\frac{\rho}{m}\right)^2 = \left(\frac{2\pi a_s\hbar^2}{m^3}\right)\rho^2. \tag{5}$$

It should be noted that in a BEC, at zero temperature, the pressure arising in the hydrodynamic equation (4) is totally different from the pressure of a normal fluid at finite temperature. In BEC, the pressure term arises directly from the short-range interactions between the particles (bosons) and is not due to the thermal motion. This is the case, in particular, in which the hydrodynamic equation derived for the BEC can be referred to as the hydrodynamic equations of superfluid [1].

Combining Eqs. (2) and (4), together with Eq. (5), we have

$$\nabla\cdot\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{c_s^2}{\rho_0}\nabla\rho - \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\right] = -4\pi G(\rho - \rho_0), \tag{6}$$

where $c_s(=\sqrt{g\rho_0/m^2} = \sqrt{gn/m})$ is the velocity of sound [1].

The linear dispersion relation is well known, and can be obtained from linearized equations (3) and (6) about equilibrium values $\mathbf{v} = 0$ and $\rho(\mathbf{r},t) = \rho_0$. Using the standard Fourier mode solution for the perturbations, the following dispersion relation can be obtained:

$$\omega^2 = c_s^2k^2 - \omega_J^2 + \frac{\hbar^2k^4}{4m^2} = c_s^2\left(k^2 - k_J^2 + H_Q^2\lambda^2k^4\right), \tag{7}$$

where $k = |\mathbf{k}|$, $\omega_J(=\sqrt{4\pi G\rho_0})$ is the Jeans frequency, $k_J(=\omega_J/c_s)$ is the Jeans wave number, $\lambda(=2\pi/k)$ is the perturbation wavelength, and $H_Q[=\hbar/(2mc_s\lambda)]$ is a dimensionless quantum parameter. This is the gravitational analog of the Bogoliubov energy spectrum of a weakly interacting quantum fluid [30]. The above dispersion relation describes the quantum counterpart of the classical acoustic mode in a gravitating medium and a correction from quantum diffraction effects. Also, this quantum diffraction makes the wave dispersive.

The solution of the above dispersion relation can be written as

$$\omega = \pm ic_s\sqrt{k_J^2 - k^2\left(1 + H_Q^2\lambda^2k^2\right)}. \tag{8}$$

This clearly suggests that the presence of quantum diffraction effects in a self-gravitating quantum fluid contributes to its stability against perturbations in gravitational potential by reducing the instability growth rate as obtained earlier for quantum degenerate gravitating gases [31,32]. In the case of the Thomas-Fermi approximation [1,33], Eq. (8) simply becomes the ordinary Jeans instability condition [25] that exhibits stability (instability) for perturbation with $k > (<)k_J$. Thus, the system is stable only if the Jeans wave number $(k_J) <$ the perturbation wave number (k). In other words, as long as the perturbation wavelength $(\lambda = 2\pi/k)$ is less than the Jeans length $(\lambda_J = 2\pi/k_J)$, the system is stable. In the next section, we investigate the weakly nonlinear localized structures for the perturbations $k_J \ll k$ (stable case).

III. NONLINEAR EVOLUTION EQUATION

To investigate the weakly nonlinear localized structures, we consider one spatial dimension (generalization to more spatial dimension is trivial), namely, the x direction. Throughout this paper we use the following normalizations: $\bar{x} = x/\lambda$ (λ is the wavelength of the perturbations), $\bar{t} = c_st/\lambda$, $\bar{\rho} = \rho/\rho_0$, $\bar{v} = v/c_s$. Hereafter we will use these new variables and remove all the bars for simplicity of notation. From Eqs. (3) and (6), we obtain

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{9}$$

and

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + c_s^2\frac{\partial\rho}{\partial x} - 2H_Q^2\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right)\right] = -\left(\frac{\lambda}{\lambda_J}\right)^2(\rho - 1). \tag{10}$$

For weak perturbations, the reductive perturbation technique [34] is employed and the following stretched coordinate is introduced:

$$\xi = \sqrt{\epsilon}(x - t), \quad \tau = \epsilon^{3/2}t, \tag{11}$$

where ϵ is a small nonzero parameter proportional to the amplitude of the perturbation. The dynamical variables ρ and v are expanded about their equilibrium value in a power series of ϵ in the following way:

$$v = \epsilon v^{(1)} + \epsilon^2 v^{(1)}\cdots; \quad \rho = 1 + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + \cdots. \tag{12}$$

With the new independent coordinates (11) and the perturbation expansion (12), we transform the continuity equation (8) and the modified momentum conservation equation (10) into a set of two equations in the form of a power series in ϵ. The resulting system can be written as

$$\frac{\partial}{\partial\xi}(v^{(1)}-\rho^{(1)})+\epsilon\left[\frac{\partial\rho^{(1)}}{\partial\tau}+\frac{\partial}{\partial\xi}(v^{(2)}+\rho^{(1)}v^{(1)}-\rho^{(2)})\right] = O(\epsilon^2), \tag{13}$$

$$\frac{\partial^2}{\partial\xi^2}(\rho^{(1)}-v^{(1)})+\epsilon\frac{\partial}{\partial\xi}\left(\frac{\partial v^{(1)}}{\partial\tau}+\frac{\partial}{\partial\xi}(\rho^{(2)}-v^{(2)})+v^{(1)}\frac{\partial v^{(1)}}{\partial\xi} -\frac{H_Q^2}{2}\frac{\partial^3\rho^{(1)}}{\partial\xi^3}\right)+\epsilon\left[\epsilon^{-2}\left(\frac{\lambda}{\lambda_J}\right)^2\right]\rho^{(1)} = O(\epsilon^2). \tag{14}$$

This perturbation expansion (14) shows that to include the gravitational effect, and also for the perturbation consistent with that of (11) and (12), we must have the following scaling:

$$\frac{\lambda}{\lambda_J}\sim O(\epsilon)\Rightarrow\left(\frac{\lambda}{\lambda_J}\right)^2\sim O(\epsilon^2). \tag{15}$$

Note that we consider the stable case where $k_J \ll k$. Therefore, the above scaling is also consistent with our assumption that perturbation wavelength $\lambda \ll$ Jeans length λ_J.

Equations (13) and (14) are to be satisfied to all orders in ϵ. The zeroth-order terms subject to the boundary conditions $\rho^{(1)},v^{(1)} \to 0$ as $\xi \to -\infty$ give the following relations:

$$\rho^{(1)} = v^{(1)}. \tag{16}$$

Finally, the first-order terms in ϵ together with relation (16) yields the following modified (by the gravitational effect) form of the KdV equation:

$$\frac{\partial}{\partial\xi}\left[\frac{\partial\rho^{(1)}}{\partial\tau}+\left(\frac{3}{2}\right)\rho^{(1)}\frac{\partial\rho^{(1)}}{\partial\xi}-\left(\frac{H_Q^2}{4}\right)\frac{\partial^3\rho^{(1)}}{\partial\xi^3}\right]+\gamma\rho^{(1)}=0, \tag{17}$$

where

$$\gamma=\frac{1}{2}\left(\frac{\lambda}{\lambda_J}\right)^2. \tag{18}$$

The above equation (17) can be written in the following form after the integration with respect to ξ in the interval $(-\infty,\xi]$ subject to the boundary condition that all the perturbed variables and their derivatives vanish at $\xi = -\infty$:

$$\frac{\partial\rho^{(1)}}{\partial\tau}+\left(\frac{3}{2}\right)\rho^{(1)}\frac{\partial\rho^{(1)}}{\partial\xi}-\left(\frac{H_Q^2}{4}\right)\frac{\partial^3\rho^{(1)}}{\partial\xi^3} + \gamma\int_{-\infty}^{\xi}\rho^{(1)}d\xi = 0. \tag{19}$$

From the above Eq. (17) or (19), we see that only the quantum diffraction (term $\propto H_Q$) is responsible for the dispersion of the nonlinear wave. This can also be seen from the linear dispersion relation (7). Thus, in a Bose-Einstein gravitational condensate gas, the quantum diffraction could balance the wave-breaking nonlinearity to form a stable nonlinear structure. In the next section, we derive the soliton solution of the above modified form of the KdV equation and also find the effects of gravity on it.

IV. SOLITARY WAVE SOLUTION

In the absence of gravitational effects, i.e., for $\gamma = 0$, from Eqs. (17) or (19) we recover the KdV equation with negative dispersion for the weakly nonlinear wave. One can obtain the usual KdV equation with positive dispersion by replacing $\rho^{(1)} \to -\rho^{(1)}$ and $\xi \to -\xi$ [35]. The sign of dispersion only determines the direction of propagation of the wave; therefore the KdV equation can be equally applied to a medium with negative dispersion [35]. The difference between the KdV equations with positive and negative dispersion is in the direction of propagation only [35,36]. The usual KdV equation represents a completely integrable Hamiltonian system that has an infinite set of conservation laws [35,37]. Let us consider the energy conservation law: Multiplying the KdV equation [Eq. (19) with $\gamma = 0$] by $\rho^{(1)}(\xi,\tau)$ and then integrating the resulting equation with respect to ξ within the interval $(-\infty,\infty)$ subject to the solitary wave boundary conditions $\rho^{(1)}(\xi,\tau),\partial_\xi\rho^{(1)}(\xi,\tau)$ and $\partial_\xi^2\rho^{(1)}(\xi,\tau)$ (∂_ξ partial derivative with respect to ξ) all $\to 0$ as $|\xi| \to \infty$, the following energy equation is obtained:

$$\frac{\partial W}{\partial\tau}=0,\quad W=\frac{1}{2}\int_{-\infty}^{\infty}{\rho^{(1)}}^2(\xi,\tau)d\xi, \tag{20}$$

where W is the soliton energy. This shows that in the absence of $\gamma(\gamma = 0)$, the soliton energy W is conserved and thus possesses the single soliton solution,

$$\rho^{(1)}(\xi,\tau) = -N\mathrm{sech}^2\left(\frac{\xi+U\tau}{\Delta}\right), \tag{21}$$

where U is the soliton velocity, N is the dimensionless soliton amplitude, and Δ is the dimensionless spatial width of the soliton. The relations between the different soliton parameters N, U, and Δ are as follows:

$$U=\frac{N}{2},\quad \Delta=\frac{H_Q}{\sqrt{N}}. \tag{22}$$

This shows that as the soliton amplitude increases, the velocity also increases, and consequently spatial width decreases so that $N\Delta^2 = H_Q^2$. However, in the presence of gravity ($\gamma \neq 0$), Eq. (19) does not possess a completely integrable Hamiltonian. In other words, the energy of the system is not conserved, and in the presence of γ, the above energy equation (20) becomes

$$\frac{\partial W}{\partial\tau}=-\gamma\int_{-\infty}^{\infty}\rho^{(1)}(\xi,\tau)\left[\int_{-\infty}^{\xi}\rho^{(1)}(\xi',\tau)d\xi'\right]d\xi. \tag{23}$$

Now, in presence of gravity, following the perturbation procedures of Refs. [38–40] with $\gamma(\sim O(\epsilon^2)) \ll 1$ as a perturbed parameter, a slow time-dependent form of the solution given in Eq. (21) is considered:

$$\rho^{(1)}(\xi,\tau) = N(\tau)\mathrm{sech}^2\left(\frac{\xi + U(\tau)\tau}{\Delta(\tau)}\right). \quad (24)$$

Finally, substitution of (24) in (23) yields the following expressions of soliton amplitude and width of the soliton:

$$N(\tau) = N(0)\left(1 - \frac{\tau}{\tau_0}\right)^2, \qquad \Delta(\tau) = \frac{\Delta(0)}{(1 - \tau/\tau_0)}. \quad (25)$$

The soliton energy is given by

$$W(\tau) = \frac{4}{3}N(0)^2\Delta(0)\left(1 - \frac{\tau}{\tau_0}\right)^3, \quad (26)$$

where

$$\tau_0^{-1} = \gamma\Delta(0); \qquad \Delta(0) = \frac{H_Q}{\sqrt{N(0)}}, \quad (27)$$

where $N(0) = N(\tau = 0) > 0$ is the initial amplitude and $\Delta(0)$ is the initial spatial width of the soliton. It should be noted that for a soliton solution to exist one must have $\Delta > 0$ and soliton energy $W(\tau) > 0$; therefore, the above solutions Eqs. (25) and (26) are physically valid only for $\tau \lesssim \tau_0$. Otherwise both Δ and W become negative and no soliton solution exists.

Thus, the above solution reveals that the gravitational effect causes the soliton amplitude $N(\tau)$, soliton energy $W(\tau)$, and soliton velocity $U(\tau)$ to decay with time (τ) according to Eqs. (25) and (26), whereas the soliton width increases with time according to (25). But the product of the amplitude and the square of the width remains constant $[N(\tau)\Delta(\tau)^2 = H_Q^2]$. All of the above physical phenomena are shown graphically in Fig. 1. The amplitude decay for the soliton is fairly heavy; formally speaking, the soliton amplitude completely vanishes at a finite time,

$$\tau_c = \frac{1}{\gamma\Delta(0)} = \left(\frac{\lambda_J}{\lambda}\right)^2\left(\frac{\sqrt{N(0)}}{H_Q}\right). \quad (28)$$

In reality, Eqs. (25) and (26) are no longer valid when the soliton amplitude approaches zero and its energy becomes extremely low. However, the value of τ_c provides a good estimation of a characteristic lifetime of a soliton in a Bose-Einstein gravitational condensate gas. Thus, the solitonic structures exist for $0 \leqslant \tau < \tau_c$.

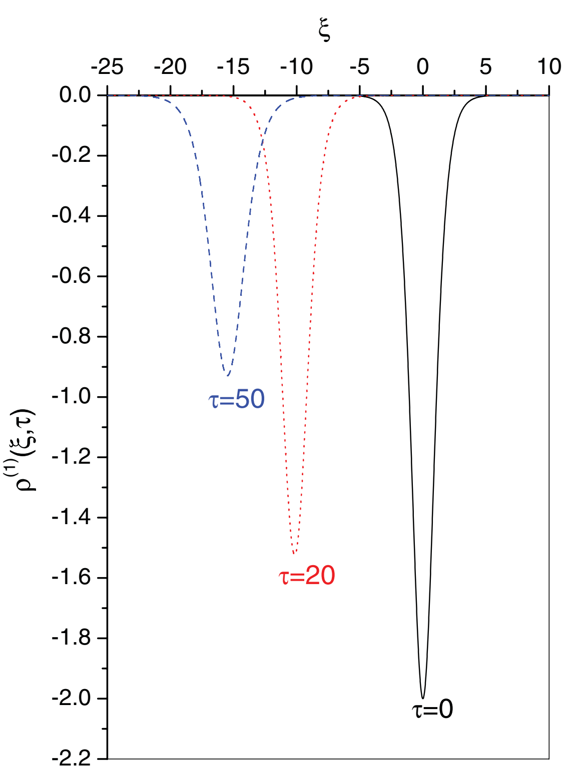

FIG. 1. (Color online) Solitary wave structures in a self-gravitating quantum fluid at different time.

V. SUMMARY AND DISCUSSION

In this paper, we analyze the weakly nonlinear localized structures in a homogeneous BEC in a gravitational trap. For this, we consider a Bose-Einstein gravitational condensate gas described by the gravitating quantum fluid equations. Quantum diffraction due to tunneling effects gives rise to dispersion for which the nonlinear wave is governed by a Korteweg–de Vries equation with a nonlocal term that arises due to gravity for perturbation wavelength much smaller than the Jeans length ($\lambda \ll \lambda_J$). This nonlinear equation is solved analytically with the help of the energy conservation principle. The analytical solution shows that the gravitational effect causes the amplitude, energy, and velocity of the soliton to decay with time. But the product of the soliton amplitude and square of the width remains constant. The physics underlying this decay can be explained in the following way: In a gravitating system, there is a competition between the pressure force pointing outward and gravity pulling inward. As long as Jeans wave number $k_J < k$ (i.e., $\lambda < \lambda_J$), the system is stable and perturbation behaves similarly to a sound wave with decreasing phase velocity. This can easily be seen from the linear dispersion relation

$$\omega^2 = k^2c_s^2\left(1 - \frac{k_J^2}{k^2}\right),$$

with $\hbar \to 0$. This relation is an indication of decrease of wave energy due to this gravitational effect. This decrease is manifested in our weakly nonlinear calculation with an effective term proportional to $\gamma(\equiv k_J^2/k^2)$. Jeans wave number k_J (or Jeans length λ_J) is an indicator of which of these factors is more important for a given mode. In the region $k < k_J(\lambda_J < \lambda)$, perturbations grow exponentially with time and Jeans instability occurs [25–27].

The analytical solution shows that a rarefactive soliton exists in this system. The lifetime τ_c of a soliton is also derived. This lifetime is proportional to the square of the ratio of the Jeans length to perturbation wavelength, whereas it is inversely proportional to the quantum diffraction effect.

Recently, numerical simulation has predicted the formation of solitonic structures in cold dark matter [41]. Thus our findings may be relevant to study the density localization as well as energy localization in cold dark matter, which is thought to be an Bose-Einstein gravitational condensate gas. In this connection, we must mention that we have not yet encountered experimental observations of BEC in a gravitational trap. It would be very interesting to look at this situation in a laboratory. However, we hope that in the future this type of a nonlinear phenomena associated with BEC could be observed in experiment.

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