Drift wave turbulence in the presence of a dust density gradient

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We present turbulent properties of electrostatic drift waves in a nonuniform collisional plasma composed of magnetized electrons and ions in the presence of immobile dust particles. For this purpose, we derive a pair of nonlinear quasi-two-dimensional equations exhibiting the coupling between the generalized ion vorticity and the density fluctuations associated with collisional drift waves. The effect of a dust density gradient on the initial drift instability and fully developed turbulence is examined numerically.

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I. INTRODUCTION

Low-frequency (in comparison with the ion gyrofrequency) electrostatic drift waves in a nonuniform collisionless magnetized electron-ion plasma are supported by inertialess electrons and inertial ions [1]. In the pseudo-three-dimensional (3D) drift wave electric field $\mathbf{E} = -\nabla\phi$, where ϕ is the wave potential, the electrons have a helical trajectory due to their $\mathbf{v}_E = (c/B_0^2)\mathbf{E} \times \mathbf{B}_0$ and $\mathbf{v}_D = -(ck_B T_e/eB_0^2 n_e)\mathbf{B}_0 \times \nabla n_e$ drift motions in a plane perpendicular to the external magnetic field $\mathbf{B}_0 = \hat{\mathbf{z}}B_0$ as well as their rapid motion along the external magnetic field direction where $\hat{\mathbf{z}}$ is the unit vector along the z axis in a Cartesian coordinate system, B_0 is the strength of the magnetic field, c is the speed of light in vacuum, k_B is the Boltzmann constant, e is the magnitude of the electron charge, T_e is the electron temperature, and n_e is the electron number density. Furthermore, the motion of the two-dimensional ions in a plane perpendicular to the axial direction is governed by the $\mathbf{v}_E$ and the polarization $\mathbf{v}_p = -(c/B_0\omega_{ci})d_t\nabla_\perp^2\phi$ drifts where we have denoted $d_t = \partial_t + \mathbf{v}_E \cdot \nabla$, $\omega_{ci} = eB_0/m_i c$ is the ion gyrofrequency, and m_i is the ion mass.

Hence, the electron and ion density fluctuations are different due to the differential electron and ion motions in a magnetized plasma with an equilibrium density gradient. The resulting space charge separation, in turn, causes dispersive drift oscillations to propagate across the homogenous magnetic field and density gradient directions. In a collisional electron-ion magnetoplasma, dispersive drift waves are excited by the combined action of the density inhomogeneity and electron-ion collisions. The latter produce a phase lag between the magnetic field-aligned electron velocity perturbation as well as the drift wave potential and the electron density perturbation. As a result, drift waves grow exponentially, extracting energy from the equilibrium density gradient. It is understood that nonthermal drift waves cause anomalous cross-field diffusion of the plasma particles [1–3]. The plasma particle confinement is significantly improved if there are cylindrically symmetric zonal (sheared) flows [4–6] that are nonlinearly excited by finite amplitude dispersive drift waves in plasmas. Thus, zonal flows act like a transport barrier [7].

Charged dust impurities are common in space and laboratory plasmas [8–10]. The presence of charged dust grains in an electron-ion plasma modifies the equilibrium quasineutrality condition $n_{i0} = n_{e0} + \epsilon Z_d n_{d0}$ where n_{j0} is the unperturbed number density of the particle species j (j equals i for ions, e stands for electrons, and d stands for charged dust grains), Z_d is the number of constant charges residing on dust, and $\epsilon = 1$ (-1) is for negative (positive) dust. Due to the modification of the quasineutrality condition, the difference between the divergence of the electron and the ion fluxes involving the $\mathbf{V}_E$ drift is finite. This leads to a flutelike Shukla-Varma (SV) mode [11] in a nonuniform dusty magnetoplasma. The SV mode governs the dynamics of the ion vorticity, which evolves in the form of a dipolar vortex.

In this paper, we present an investigation of the pseudo-3D dissipative drift wave turbulence in a nonuniform collisional dusty magnetoplasma composed of the electrons, ions, and stationary charged dust grains. By using the two-fluid model and the guiding center drift approximation (viz. $|d/dt| \ll \omega_{ci}$), we derive a pair of nonlinear generalized Hasegawa-Wakatani (HW) equations [12] governing the evolution of the potential and density fluctuations associated with the low-frequency (in comparison with ω_{ci}) long wave length (in comparison with the thermal ion gyroradius) collisional drift waves in our dusty plasma.

Generalized HW type equations in the presence of dust have been derived before and have been analyzed linearly in Refs. [13] and [14]. A linear analysis of dissipative drift modes in the presence of a dust density gradient has been presented in Ref. [15] and has been discussed further in Ref. [16] with respect to nonmodal growth, including dust charging effects.

The work presented in Refs. [13–16] (and others for similar topics) has shown that a dynamically variable dust charge can have a profound effect on linear drift wave stability. In a comprehensive dusty plasma turbulence model, therefore, the charging process should be taken into account. Here, we have refrained from including the charging process in order not to multiply the number of free parameters beyond what is absolutely necessary. While we recognize that this, indeed, would be a worthwhile effort, it is another kind of study itself. An important point why we have so far refrained from performing this study ourselves is the many uncertainties related to the dust grain charging model for a nonuniform collisional magnetoplasma that have been used so far. In particular, the dust grain charging is a dynamical process and requires a complete kinetic theory for fluctuating (in the electrostatic field of pseudo-3D drift waves) electron and ion currents that reach the surface of dust grains in our plasma.

In the following, we study turbulence properties of the modified (by the presence of charged dust particles) drift waves in a nonuniform collisional magnetoplasma with static charged dust impurities. Thus, we are assuming that the timescales for the excitation of drift waves and their evolution are much shorter than the dust plasma and dust gyroperiods and much longer than the electron-ion relaxation period.

Furthermore, the assumption of constant dust charge would remain valid since the dust grain charging frequency is typically much larger than the modified drift wave frequency. Thus, our nonlinear simulation study of the dissipative drift wave turbulence is based on a slightly different set of dusty HW-SV equations compared to Refs. [13–16], excluding dust charge fluctuation effects. Our simulation results reveal that the equilibrium dust density gradient controls the formation of turbulent structures and the associated cross-field transport of the plasma particles in a nonuniform dusty magnetoplasma.

The paper is organized in the following fashion. In Sec. II, we derive the governing nonlinear equations for the low-frequency (in comparison with the ion gyrofrequency) dissipative drift wave turbulence in a nonuniform dusty magnetoplasma composed of magnetized electrons and ions, as well as unmagnetized stationary charged dust grains. We use the two-fluid equations in the guiding center approximation to derive a pair of nonlinear equations that exhibits coupling between the plasma density perturbation and the drift wave potential. Section III presents a local linear dispersion relation and its analysis. Governing equations for nonlinearly interacting finite amplitude drift waves are analyzed numerically in Sec. IV. Vortex development is discussed in Sec. V, the turbulent state is reviewed in Sec. VI, and a summary and discussions are contained in Sec. VII.

II. NONLINEAR DRIFT WAVE EQUATIONS

Let us consider the electrostatic drift waves with the electric field $\mathbf{E} = -\nabla_\perp\phi - \hat{\mathbf{z}}\partial_z\phi$ in a plasma with a uniform magnetic field $\hat{\mathbf{z}}B_0$ and the equilibrium density gradient $\partial_x n_{j0}$. In a low-β ($\beta = 8\pi n_{e0}k_B T_e/B_0^2 \ll 1$) plasma, the electron and ion fluid velocities for $|d/dt| \ll \omega_{ci},\nu_e$ are, respectively,

$$\mathbf{u}_e \approx \frac{c}{B_0}\hat{\mathbf{z}}\times\nabla\phi - \frac{ck_BT_e}{eB_0n_e}\hat{\mathbf{z}}\times\nabla n_e + \hat{\mathbf{z}}u_{ez}, \tag{1}$$

and

$$\mathbf{u}_i \approx \frac{c}{B_0}\hat{\mathbf{z}}\times\nabla\phi - \frac{c}{B_0\omega_{ci}}\left(\frac{d}{dt}+\nu_i\right)\nabla_\perp\phi, \tag{2}$$

with the magnetic field-aligned electron fluid velocity perturbation given by

$$u_{ez} = \frac{e}{m_e\nu_e}\frac{\partial}{\partial z}\left(\phi - \frac{k_BT_e}{e}\frac{n_{e1}}{n_{e0}}\right), \tag{3}$$

where m_e is the electron mass, ν_e ($\ll\omega_{ce} = eB_0/m_ec$) is the electron-ion collision frequency, ν_i is the ion-neutral collision frequency, and $n_{e1} = n_e - n_{e0}$ ($\ll n_{e0}$) is a small perturbation in the electron number density. The two-dimensional (2D) ions are assumed to be cold. The assumption of 2D ions ensures the decoupling of the dust ion-acoustic and drift waves.

Inserting Eqs. (1)–(3) into the electron and ion continuity equations, we obtain the evolution equations for the electron and ion number density perturbations n_{e1} ($\ll n_{e0}$) and n_{i1} ($\ll n_{i0}$), respectively,

$$\frac{dn_{e1}}{dt} - \frac{c}{B_0}\hat{\mathbf{z}}\times\nabla n_{e0}\cdot\nabla\phi + \frac{n_{e0}e}{m_e\nu_e}\frac{\partial^2}{\partial z^2}\left(\phi - \frac{k_BT_e}{e}\frac{n_{e1}}{n_{e0}}\right), \tag{4}$$

and

$$\frac{dn_{i1}}{dt} - \frac{c}{B_0}\hat{\mathbf{z}}\times\nabla n_{i0}\cdot\nabla\phi - \frac{cn_{i0}e}{B_0\omega_{ci}}\left(\frac{d}{dt}+\nu_i\right)\nabla_\perp^2\phi. \tag{5}$$

Now, subtracting Eq. (5) from Eq. (4), we obtain the generalized ion vorticity equation,

$$\left(\frac{d}{dt}+\nu_i\right)\nabla_\perp^2\phi + \omega_{ci}K_d\frac{\partial\phi}{\partial y} + \frac{\Omega_{LH}^2}{\nu_e}\frac{\partial^2}{\partial z^2}\left(\phi - \frac{k_BT_e}{e}\frac{n_{e1}}{n_{e0}}\right), \tag{6}$$

where $K_d = (\epsilon Z_d c/B_0 n_{i0})\partial_x n_{d0}$, $\Omega_{LH} = (\omega_{ce}\omega_{ci}/\alpha)^{1/2}$ is the lower-hybrid resonance frequency in our dusty plasma, the drift scale is $\rho_s = c_s/\omega_{ci}$, and $c_s = (\alpha k_BT_e/m_i)^{1/2}$ is the modified ion-sound speed [17], with $\alpha = n_{i0}/n_{e0} > 1$.

We note that the K_d term arises due to a nonzero value coming from the difference between the divergence of the electron and the ion fluxes involving the $(c/B_0)\mathbf{E}\times\hat{\mathbf{z}}$ drift in our nonuniform dusty plasma. In the absence of the magnetic field-aligned electron motion, Eq. (6) describes the dynamics of the SV mode [11]. In a quasineutral dusty plasma with immobile charged dust grains, Eqs. (4) and (6) are closed with the help of $n_{i1} = n_{e1} \equiv n_1$, which is valid as long as the frequency (wavelength) of the drift waves is much higher than the plasma and dust gyrofrequencies (the electron Debye radius).

It is appropriate to normalize the wave potential ϕ by k_BT_e/e, the density perturbation n_1 by n_{e0}, as well as the time and space variables by the ion gyroperiod ω_{ci}^{-1} and the effective ion sound gyroradius ρ_s, respectively. Then, the governing nonlinear equations for the collisional drift waves in our nonuniform dusty magnetoplasma are

$$\left(\frac{d}{dt}+\gamma\right)\Omega + a\frac{\partial\phi}{\partial y} - g(\phi - n) = 0, \tag{7}$$

$$\frac{dn}{dt} + b\frac{\partial\phi}{\partial y} - \left(\frac{d}{dt}+\gamma\right)\Omega = 0, \tag{8}$$

where $\Omega = \nabla_\perp^2\phi$, $n = n_1/n_{e0}$, $\gamma = \nu_i/\omega_{ci}$, $a = K_d\rho_s$, and $b = K_i\rho_s$, where $K_i = -(1/n_{e0})\partial n_{i0}/\partial x > 0$. We have Fourier decomposed the drift wave potential and electron density perturbations along the z axis but have kept their variations in the x-y plane so that $g = k_z^2\omega_{ce}/\nu_e\alpha$ is the dissipative coupling coefficient. Equations (7) and (8) are the generalization of the HW equations [12]. The latter are recovered in the limits $a = 0$ and $\gamma = 0$.

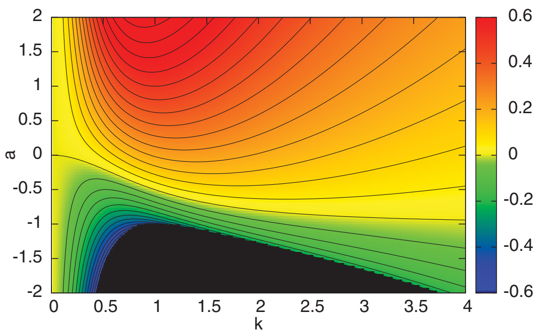

FIG. 1. (Color online) Growth rate $\omega_I(k,a)$ as a function of wave number k and dust density gradient a. Black marks the region of stability.

III. LINEAR DISPERSION RELATION

Linearization of Eqs. (7) and (8) with an ansatz $\phi,n \sim \exp[-i\omega t + i\mathbf{k}\cdot\mathbf{x}]$ obtains the dispersion relation,

$$\omega^2 k^2 + \omega A - \gamma g k^2 + i\omega G - igB, \quad (9)$$

where $A = ak_y$, $B = bk_y$, and $G = gk^2 + \gamma k^2 + g$. Inserting $\omega = \omega_R + i\omega_I$ and solving the imaginary part of the dispersion relation for $\omega_I \ll \omega_R$ results in the real frequency,

$$\omega_R = \frac{gB}{G} = \frac{bk_y}{1 + (1+\gamma/g)k^2}. \quad (10)$$

The growth rate is obtained from the real part as

$$\omega_I = \frac{-1}{Kk}\{1 \mp \sqrt{1 + K^2[(\omega_R^2 + \gamma g)k^2 + A\omega_R]}\}, \quad (11)$$

with $K = 2k/G$. Only the negative root is relevant for unstable solutions ($\omega_I > 0$). Instability, thus, occurs if

$$(\omega_R^2 + \gamma g)k^2 + A\omega_R > 0. \quad (12)$$

This always is achieved if $A\omega_R \sim AB \sim ab \geqslant 0$. The usual resistive drift wave instability (expressed by the first term), thus, is enhanced for co-aligned gradients with $a > 0$ if $b > 0$.

The growth rate is reduced for counteraligned gradients. Drift modes are damped ($\omega_I < 0$) for a large counteraligned dust density gradient when $|a| > [k/(1+k^2)]^2|b|$. For $k = 1$, this corresponds to $a < -b/4$. As drift wave turbulence is most active around $k \approx 1$, it can be expected that (for $b \equiv 1$) the turbulence is damped strongly for $a < -0.25$ and is enhanced approximately proportional to $\sqrt{a}$ for positive a.

In Fig. 1, the growth rate $\omega_I(k,a)$ is plotted as a function of wave number and dust density gradient.

IV. NONLINEAR NUMERICAL SOLUTIONS

For numerical simulations, we write Eqs. (7) and (8) as

$$\partial_t \Omega + [\phi,\Omega] = -a\,\partial_y\phi - \gamma\,\Omega + g\,(\phi - n), \quad (13)$$

$$\partial_t n + [\phi,n] = -(b+a)\,\partial_y\phi + g\,(\phi - n), \quad (14)$$

where the advection nonlinearity $[A,B] = \hat{\mathbf{z}} \times \nabla A \cdot \nabla B$ is expressed in terms of the Poisson brackets. The equations are solved numerically with a third order Karniadakis time stepping scheme in combination with the Arakawa method for the Poisson brackets [18–20]. Hyperviscous operators $\nu^4\nabla^4$ with $\nu^4 = -2\times10^{-4}$ are added to the right-hand side of both Eqs. (13) and (14) for numerical stability, acting on Ω and n, respectively. The Poisson equation is solved spectrally.

The equations are for the turbulence computations discretized on a doubly periodic 512×512 grid with box dimension $L_x = L_y = 64\rho_s$. The initial drift wave development is computed on a 128×256 grid with box dimensions $L_x = 64\rho_s$ and $L_y = 128\rho_s$. Nominal plasma parameters are $g = 0.5$ and $b = 1$.

V. VORTEX DEVELOPMENT

First, we study the initial evolution of a drift wave growing out of a density perturbation in the presence of a dust density gradient a. The ion viscosity is set to $\gamma = 0.1$. A Gaussian density perturbation is initialized with amplitude $n = 1$, which generates a potential perturbation of the same initially Gaussian form. This leads to the formation of an $\mathbf{E}\times\mathbf{B}_0$ drift vortex where the plasma is convected around the perturbation.

For $a = 0$, the usual drift wave dynamics is encountered, which is shown in Fig. 2(a): The singular density structure ($t = 0$, left frame) propagates in the electron diamagnetic drift direction (upward, in the positive y direction) and generates drift wave structures in its wake, which are shown for times $t = 25$ (middle frame) and $t = 50$ (right frame). At a later stage, via secondary instabilities, the drift wave nonlinearly develops into a saturated turbulent state, whose properties are discussed in the next section.

In Fig. 2(b), the evolution for $a = -0.5$ is shown: The vortex is elongated in the y direction, while the structure propagates but is damped. The mode is still damped when the viscosity γ is set to zero and the initial amplitude is set to $n_0 = 10$.

Figure 2(b) shows the case of $a = +0.5$: The structure is much slower when propagating in the y direction but forms a nearly standing drift wave structure, which is elongated in the x direction, and grows fast in amplitude until nonlinearly secondary modes form by the Kelvin-Helmholtz breakup of the streamers.

The linear and nonlinear developments of the drift wave perturbations, thus, strongly depend on the sign of the dust density gradient. Co-alignment of the plasma density gradient with the dust density gradient enhances the mode growth, while counteralignment is strongly damped.

VI. TURBULENT STATE

The computations are started from a random quasiturbulent spectral bath and are run into saturation. The spectral bath is generated by initializing the density field in Fourier space by (within some range) random amplitudes that approximately follow a power law spectrum. In this way, all k modes are excited from the start, and a fully developed turbulent state is reached rather rapidly in the simulation.

Statistical averages are taken in the interval between $200 \leqslant t \leqslant 1000$, which, for all parameters, is well in the saturated regime.

The turbulent activity can be characterized by the potential energy $U = (1/2)\int dV\,n^2$, the kinetic energy $K = (1/2)\int dV(\nabla\phi)^2$, and the generalized enstrophy $W = (1/2)\int dV(n-\Omega)^2$. The mean of these energetic quantities

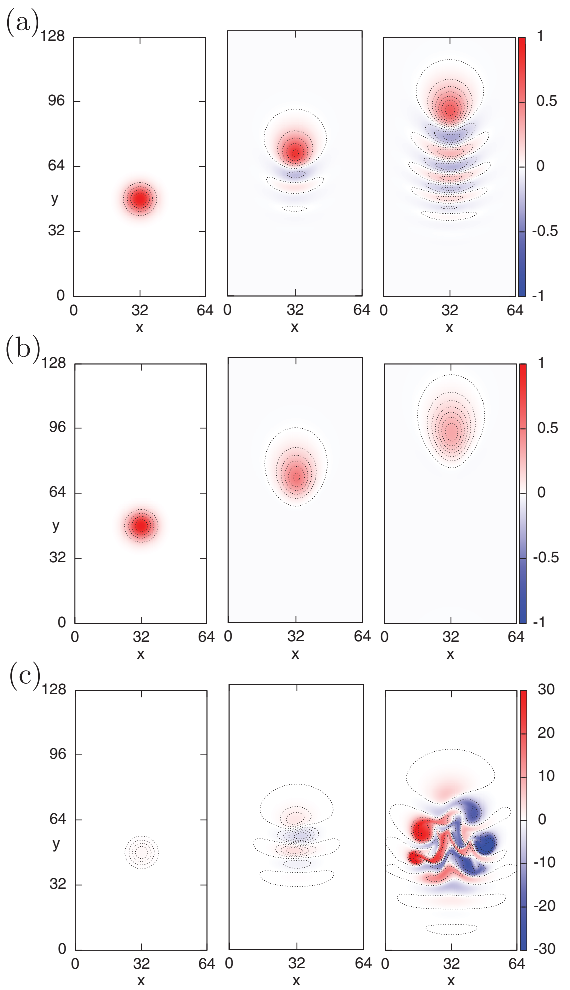

FIG. 2. (Color online) Development of a drift vortex with an initial Gaussian density perturbation $n(x,y)$ in the presence of a dust density gradient a for (left) $t = 0$, (middle) $t = 25$, and (right) $t = 50$: (a) $a = 0$, (b) $a = -0.5$, and (c) $a = +0.5$. Note: The amplitude scale (density perturbations in drift scaled units) for $n(x,y)$ in case (c) is 30 times larger than in (a) and (b).

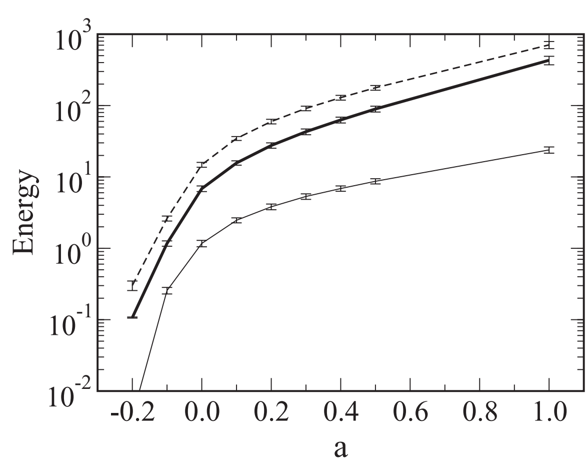

FIG. 3. The potential energy U (bold curve), the kinetic energy K (thin curve), and the generalized enstrophy W (dashed curve) as a function of the dust density gradient a.

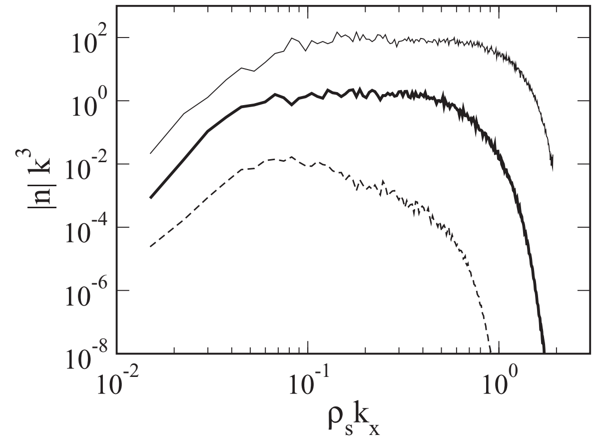

FIG. 4. The wave number spectra of density fluctuations in the x direction multiplied by a factor $(k_x \rho_s)^3$: $a = 0$ (bold line), $a = 1$, $a = -0.2$ (dashed line).

together with the standard deviation of the fluctuations in the saturated state are displayed in Fig. 3 for various values of the dust gradient parameter a.

The turbulent energies closely follow the predictions from the linear analysis: Negative values of a (counteraligned gradients) have a strong damping influence on the turbulent

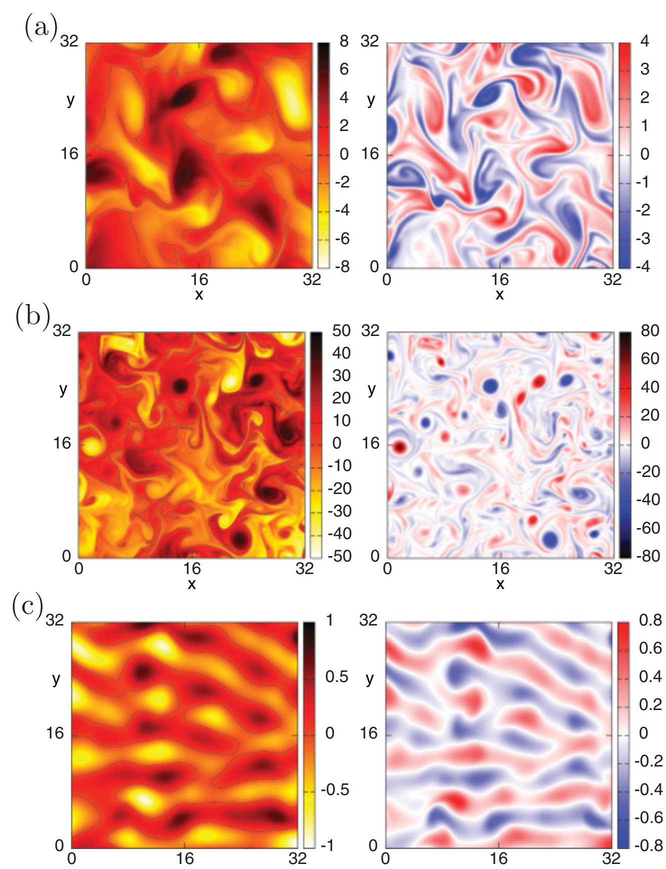

FIG. 5. (Color online) Density (left) $n(x,y)$ and (right) vorticity $\Omega(x,y)$ for (a) $a = 0$, (b) $a = 1$, and (c) $a = -0.2$. Only a quarter of the actual computational domain is shown here.

fluctuations. Co-aligned gradients enhance the drive of the drift wave turbulence.

The wave number spectra $|n(\rho_s k_x)|(\rho_s k_x)^3$ are shown in Fig. 4. The standard HW case ($a = 0$, bold line) is arbitrarily scaled to 1. For a co-aligned dust density gradient ($a = 1$, thin line), modes around $0.5 \leqslant \rho_s k_x \leqslant 1$ on the order of a few drift scales ρ_s are pronounced more strongly in relation. For counteraligned gradients ($a = -0.2$, dashed line), the smaller scales are most strongly damped.

This change in structural scales is also visible in 2D x-y plots of density perturbations (left column) and vorticity (right column) in Fig. 5. The top row shows the standard case for $a = 0$ of the typical drift wave turbulence. In the middle row, a strong co-aligned dust density gradient $a = 1$ leads to a stronger perturbation of structures at smaller scales, visible by a more frayed out density field (contour line drawn for $n = 0$) and more strongly pronounced small-scale vorticity structures. The bottom row shows the case of $a = -0.2$ with counteraligned gradients where quasilinear drift wave structures appear, which extend in the x direction and propagate in the y direction.

VII. SUMMARY AND DISCUSSIONS

Summarizing, we have investigated the properties of nonlinearly interacting finite amplitude electrostatic drift waves in a nonuniform collisional dusty magnetized plasma composed of 3D magnetized electrons, 2D magnetized ions, and immobile charged dust impurities. The dynamics of the dissipative drift wave turbulence in our model of a dusty magnetized plasma is governed by generalized HW equations In the latter, one encounters nonlinear coupling between the quasineutral electron density fluctuation and the ion vorticity that are driven by the combined action of the dust density gradient and a dissipative electron current arising from the magnetic field-aligned electron motion reinforced by the parallel electric force and the parallel variation in the electron pressure perturbation for the constant electron temperature.

Numerical simulations of the governing nonlinear equations reveal features of the fully developed drift wave turbulence. The presence of a co-aligned dust density gradient is found to enhance the drift wave turbulence and the cross-field plasma particle transport. On the other hand, counteraligned gradients lead to a damping of the drift fluctuations. Thus, the dust density inhomogeneity plays a decisive role in the formation of drift wave vortex structures in nonuniform dusty magnetized plasmas.

In conclusion, we stress that the present results should be helpful in understanding the features of the fully developed low-frequency drift wave turbulence that might emerge from the forthcoming laboratory dusty plasma experiments in an external magnetic field and from nonuniform magnetized space dusty plasmas.

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