Effects of friction on forced two-dimensional Navier-Stokes turbulence

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Large-scale dissipation mechanisms have been routinely employed in numerical simulations of two-dimensional turbulence to absorb energy at large scales, presumably mimicking the quasisteady picture of Kraichnan in an unbounded fluid. Here, "side effects" of such a mechanism—mechanical friction—on the small-scale dynamics of forced two-dimensional Navier-Stokes turbulence are elaborated by both theoretical and numerical analysis. Given a positive friction coefficient α, viscous dissipation of enstrophy has been known to vanish in the inviscid limit $\nu \to 0$. This effectively renders the scale-neutral friction the only mechanism responsible for enstrophy dissipation in that limit. The resulting dynamical picture is that the classical enstrophy inertial range becomes a dissipation range in which the dissipation of enstrophy by friction mainly occurs. For each $\alpha > 0$, there exists a critical viscosity ν_c, which depends on physical parameters, separating the regimes of predominant viscous and frictional dissipation of enstrophy. It is found that $\nu_c = [\eta'^{1/3}/(Ck_f^2)]\exp[-\eta'^{1/3}/(C\alpha)]$, where η' is half the enstrophy injection rate, k_f is the forcing wave number, and C is a nondimensional constant (the Kraichnan-Batchelor constant). The present results have important theoretical and practical implications. Apparently, mechanical friction is a poor choice in numerical attempts to address fundamental issues concerning the direct enstrophy transfer in two-dimensional Navier-Stokes turbulence. Furthermore, as relatively strong friction naturally occurs on the surfaces and at lateral boundaries of experimental fluids as well as at the interfaces of shallow layers in geophysical fluid models, the frictional effects discussed in this study are crucial in understanding the dynamics of these systems.

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I. INTRODUCTION

The study of two-dimensional (2D) turbulence dates back to early in the 1950s when pioneers in the field discovered for the first time fundamental differences from its three-dimensional (3D) counterpart [1,2]. In particular, it was found that the simultaneous conservation of energy and enstrophy (half mean-square vorticity) by the advection term in 2D turbulence gave rise to the transfer of energy to large scales (inverse transfer) and enstrophy to small scales (direct transfer), although the far reaching implication of this preferential dual transfer was not fully realized until much later by Kraichnan [3,4]. For unbounded turbulence driven by a spectrally localized source, Kraichnan envisaged the existence of a dual cascade and quasisteady dynamics in the inviscid limit. Kolmogorov's phenomenology was then applied to derive the energy spectra $E(k) = C'\epsilon^{2/3}k^{-5/3}$ for the quasisteady energy inertial range and $E(k) = C\eta^{2/3}k^{-3}$ for the steady enstrophy inertial range. Here C and C' are constant and ϵ and η are, respectively, the energy and enstrophy injection rates. For unforced turbulence, Batchelor [5] argued that the dynamics of the 2D enstrophy and 3D energy are analogous and applied Kolmogorov's theory to obtain the above scaling in the enstrophy inertial range. For this case, η is the enstrophy dissipation rate, the parallel of Kolmogorov's energy dissipation rate. Further results of Ref. [5] include decay laws, which are independent of viscosity.

Since Kraichnan's and Batchelor's seminal works, numerical simulations of 2D turbulence have been actively carried out to verify their predictions, and the results constitute a huge literature. Early simulations with low resolutions invariably employed hyperviscosity to reduce dissipation in the enstrophy inertial range. In addition, various large-scale dissipation mechanisms were used to absorb energy at large scales, presumably mimicking the quasisteady picture of Kraichnan in an unbounded fluid. With modern computers, the simulation of Kraichnan's quasisteady turbulence is within reach [6,7]. Nonetheless, the use of hyperviscosity with or without a large-scale dissipation term has remained a routine practice, particularly for simulations at moderate resolutions [8,9]. Commonly used large-scale dissipation mechanisms are represented by negative integer powers of the Laplacian (hereafter referred to as inverse viscosity, also called hypoviscosity by a number of authors) and mechanical friction (also known as Ekman drag in the geophysical context). The former primarily operates at large scales, and its effects on small scales are not well understood. The latter is scale neutral, removing enstrophy (and energy) at all scales. This has serious "side effects" on the small-scale dynamics. Most importantly, the vorticity remains bounded in the inviscid limit. Such behavior is in sharp contrast to the Kraichnan picture, in which the enstrophy grows without bound as the enstrophy inertial range becomes increasingly wider for smaller viscosity. An undesirable consequence is that viscous dissipation of enstrophy vanishes in the inviscid limit [10] (see also Ref. [11]). It follows that for steady or quasisteady dynamics at sufficiently small viscosity, frictional dissipation of enstrophy outweighs its viscous counterpart. The classical enstrophy inertial range then becomes a (frictional) dissipation range, possibly without dramatic changes in its appearance.

The above results have important theoretical and practical implications. On the one hand, mechanical friction should not be employed in numerical simulations aiming to address fundamental issues concerning the enstrophy flux and enstrophy inertial range of 2D Navier-Stokes turbulence as this scale-neutral dissipation mechanism apparently renders dynamical behavior inconsistent with the Kraichnan picture. On the other hand, relatively strong friction naturally occurs on the surfaces of thin films, at lateral boundaries of confined fluids, and at the interfaces of shallow layers in geophysical fluid models. Therefore, the frictional effects presently discussed is crucial in understanding the dynamics of these systems, particularly their departure from the classical picture. Note, however, that the present findings may not be relevant to 2D Navier-Stokes fluids having no-slip boundaries, although this type of boundaries could impose some friction on the dynamics, particularly at large scales. The reason is that such boundaries can also act as vorticity sources (see Ref. [12] and references therein), apparently playing a role opposite to that of friction.

This study elaborates on the effects of mechanical friction on forced 2D Navier-Stokes turbulence discussed in the preceding paragraphs. This is accomplished by both theoretical and numerical analysis. The plan of the paper is as follows. Section II provides some mathematical background, with an emphasis on fundamental consequences of global regularity of 2D turbulence. Section III features a simple proof of vanishing viscous dissipation of enstrophy in the inviscid limit. In addition, for weak friction, the classical spectrum is used to deduce the critical viscosity separating the regimes of predominant frictional and viscous dissipation of enstrophy. Section IV reports results from numerical simulations corroborating the theoretical findings of Sec. III. Concluding remarks are given in the final section.

II. BACKGROUND

In the velocity-vorticity formulation, the forced 2D Navier-Stokes equations are

$$\begin{aligned} \omega_t + \boldsymbol{u}\cdot\nabla\omega &= \nu\Delta\omega + f, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned} \qquad (1)$$

where $\omega = \boldsymbol{n}\cdot\nabla\times\boldsymbol{u}$ is the vorticity with $\boldsymbol{n}$ being the normal to the fluid domain, ν is the viscosity, and f represents a forcing. We consider a doubly periodic domain with all fields concerned having zero spatial average. For the purpose of this section, f is assumed to be time independent and bounded. In the inviscid and unforced dynamics, the vorticity is materially conserved. This conservation law implies that smooth solutions of Eq. (1) remain smooth for all $t < \infty$ [13,14]. This result has a number of important implications, which do not appear to have been fully exploited in the turbulence literature. This section discusses a few implications that are relevant to the subsequent sections.

Because 2D turbulence with smooth initial conditions remains smooth for all finite times, the mean-square vorticity gradient $\langle|\nabla\omega|^2\rangle$ remains finite for $t < \infty$. In fact, $\langle|\nabla\omega|^2\rangle$ does not grow more rapidly than exponentially in time, a manifestation of the effective linearity of the small-scale dynamics [15,16]. This means that for $t < \infty$, the enstrophy dissipation rate $\nu\langle|\nabla\omega|^2\rangle$ vanishes as $\nu \to 0$. Furthermore, the approach $\nu\langle|\nabla\omega|^2\rangle \to 0$ is linear in ν. This is true for both forced and unforced turbulence and is due solely to the finite-time regularity of solutions. There remains the issue of infinite-time singularities in the inviscid limit and the corresponding problem of enstrophy dissipation. It turns out that for power-law scaling of the enstrophy inertial range, $\nu\langle|\nabla\omega|^2\rangle$ does vanish for unforced turbulence. This fact is briefly recalled and discussed in what follows.

For unforced turbulence, the growth of $\langle|\nabla\omega|^2\rangle$ as $\nu \to 0$ is marginally less rapid than the decrease of viscosity. As a result, $\nu\langle|\nabla\omega|^2\rangle$ vanishes uniformly in time in the inviscid limit. In order to establish this fact, Tran and Dritschel [17] considered the global maximum of the enstrophy dissipation rate, say η_T^ν, which is achieved at $t = T(\nu)$, and showed that $\lim_{\nu\to 0}\eta_T^\nu = 0$. Note that $T(\nu)$ diverges in this limit. The convergence of η_T^ν and divergence of $T(\nu)$ are so slow (logarithmic in ν) that they have been numerically detected only recently [18,19]. The vanishing dissipation of enstrophy (and of other norms of the vorticity, see below) in the inviscid limit means that there may be no viscosity-independent decay laws and that a number of existing theoretical and numerical results may need to be reexamined or reinterpreted. These include the well-known decay law $\langle\omega^2\rangle \sim t^{-2}$ of Batchelor [5] and the modified version $\langle\omega^2\rangle \sim t^{-\gamma}$ [20–23] for various values of γ within the interval $(0,1)$.

For forced turbulence, the situation is quite different. Given a persistent enstrophy injection rate η (and energy injection rate ϵ) normally at intermediate wave numbers, the balance $\nu\langle|\nabla\omega|^2\rangle = \eta$ (in an average sense) is inevitable, no matter how small ν. In the inviscid limit, the time taken to achieve this balance diverges as can be seen from the arguments in the preceding paragraph. In this limit, the vorticity has been found to grow without bound, regardless of the nature of the dynamics [11]. These key features were fully envisaged by Kraichnan [3,4] as is evident from his detailed theory, where the enstrophy inertial range has unbounded enstrophy for both the original and modified spectra (by a logarithmic factor). The important point in this theory is that the divergence of enstrophy is toward small scales only. This is absolutely necessary not only for the building up of the predicted k^{-3} inertial range (during the transient stage of undissipated cascade) but also for the maintenance of the quasisteady dynamics (during the permanent stage of fully dissipated direct cascade). Without such a divergence, no enstrophy dissipation at small scales is possible [11] because in the inviscid limit the production of ever-smaller scales by advection (effectively a linear process [15,16]) is unable to withstand viscous effects. The bottom line is that the enstrophy divergence should not be tampered with in attempts to address issues related to the enstrophy dissipation, which has a delicate dependence on viscosity. This principle has not been strictly observed in a number of past and recent numerical studies (see for example Ref. [7]).

III. VANISHING VISCOUS ENSTROPHY DISSIPATION

Consider the addition of a friction term $-\alpha\omega$, where $\alpha > 0$, to Eq. (1):

$$\begin{aligned} \omega_t + \boldsymbol{u}\cdot\nabla\omega &= \nu\Delta\omega - \alpha\omega + f, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned} \qquad (2)$$

It is shown presently that Eq. (2) admits solutions satisfying $\nu\langle|\nabla\omega|^2\rangle \to 0$ as $\nu \to 0$, that is, vanishing enstrophy dissipation by viscous effects. For power-law spectra, this result holds pointwise in time and can be shown by an elementary method. For a similar result by sophisticated techniques (valid for general spectra but in a time-average sense), the reader is referred to a recent paper by Constantin and Ramos [10].

By multiplying the vorticity equation of Eq. (2) by ω, one obtains, after some straightforward manipulation of the viscosity term,

$$\frac{D}{Dt}\frac{\omega^2}{2} = \nu\Delta\frac{\omega^2}{2} - \nu|\nabla\omega|^2 - \alpha\omega^2 + \omega f, \tag{3}$$

where D/Dt denotes the material derivative. At local maxima of ω^2, the first term on the right-hand side of (3) is nonpositive. Hence the growth rate of the vorticity supremum $\|\omega\|_\infty$ is bounded by

$$\frac{d}{dt}\,\|\omega\|_\infty \leqslant -\alpha\,\|\omega\|_\infty + F, \tag{4}$$

where F represents an upper bound for $|f|$. It follows that

$$\|\omega(t)\|_\infty \leqslant \|\omega_0\|_\infty\, e^{-\alpha t} + \frac{F}{\alpha}(1 - e^{-\alpha t}), \tag{5}$$

where ω_0 is the vorticity field at $t = 0$. So the vorticity is bounded independently of viscosity. This is not known to be the case when $\alpha\omega$ is replaced by an inverse viscosity term (truly operating at large scales) $\nu_\gamma(-\Delta)^{-\gamma}\omega$, where $\nu_\gamma > 0$ and $\gamma > 0$.

Alternatively, the evolution of the L^p norms $\langle|\omega|^p\rangle^{1/p}$ (for $p > 1$) is governed by

$$\begin{aligned}\frac{d}{dt}\langle|\omega|^p\rangle^{1/p} &= -\nu(p-1)\langle|\omega|^p\rangle^{1/p-1}\langle|\omega|^{p-2}|\nabla\omega|^2\rangle \\ &\quad - \alpha\langle|\omega|^p\rangle^{1/p} + \langle|\omega|^p\rangle^{1/p-1}\langle|\omega|^{p-2}\omega f\rangle \\ &\leqslant -\alpha\langle|\omega|^p\rangle^{1/p} + \langle|\omega|^p\rangle^{1/p-1}\langle|\omega|^{p-1}|f|\rangle \\ &\leqslant -\alpha\langle|\omega|^p\rangle^{1/p} + \langle|f|^p\rangle^{1/p},\end{aligned} \tag{6}$$

where the term due to viscosity has been omitted in the second step and Hölder's inequality with the pair of conjugate exponents p and $p/(p-1)$ has been used in the final step. It follows that

$$\langle|\omega|^p\rangle^{1/p} \leqslant \langle|\omega_0|^p\rangle^{1/p} e^{-\alpha t} + \frac{F_p}{\alpha}(1 - e^{-\alpha t}). \tag{7}$$

Here F_p is an upper bound for $\langle|f|^p\rangle^{1/p}$, or just $\langle|f|^p\rangle^{1/p}$ itself if f is time independent. Equation (7) implies that all vorticity norms can be bounded independently of viscosity. Therefore, the enstrophy $\langle\omega^2\rangle/2$ and the vorticity supremum $\|\omega\|_\infty$ (obtained from $\langle|\omega|^p\rangle^{1/p}$ in the limit $p \to \infty$) are bounded.

We now derive the main result of this section. The evolution of the mean square vorticity gradient $\langle|\nabla\omega|^2\rangle$ (twice the palinstrophy) is governed by [11]

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle|\nabla\omega|^2\rangle \\ &= \langle\Delta\omega\, \boldsymbol{u}\cdot\nabla\omega\rangle - \alpha\langle|\nabla\omega|^2\rangle - \nu\langle(\Delta\omega)^2\rangle + \sigma \\ &\leqslant \|\omega\|_\infty\,\langle\omega^2\rangle^{1/2}\langle(\Delta\omega)^2\rangle^{1/2} - \alpha\langle|\nabla\omega|^2\rangle - \nu\langle(\Delta\omega)^2\rangle + \sigma \\ &\leqslant \frac{\langle(\Delta\omega)^2\rangle}{\langle|\nabla\omega|^2\rangle}\left[\|\omega\|_\infty\,\langle\omega^2\rangle^{1/2}\frac{\langle|\nabla\omega|^2\rangle}{\langle(\Delta\omega)^2\rangle^{1/2}} + \frac{\langle|\nabla\omega|^2\rangle}{\langle(\Delta\omega)^2\rangle}\sigma \right. \\ &\quad \left. - \alpha\frac{\langle|\nabla\omega|^2\rangle^2}{\langle(\Delta\omega)^2\rangle} - \nu\langle|\nabla\omega|^2\rangle\right],\end{aligned} \tag{8}$$

where $\sigma = -\langle f\Delta\omega\rangle$ is bounded for a broad class of forces, particularly those acting at intermediate scales only. For power-law scaling of the direct transfer range, a bounded vorticity allows for energy spectra steeper than k^{-3} or at best as shallow as k^{-3} with a limited extent, followed by a steeper tail. For such spectra, the ratio $\langle|\nabla\omega|^2\rangle/\langle(\Delta\omega)^2\rangle^{1/2}$ vanishes as $\langle|\nabla\omega|^2\rangle \to \infty$ [11,17]. Therefore, each of the first three terms within the brackets on the right-hand side of Eq. (8) vanishes in that limit. Hence all local maxima of the viscous dissipation term $\nu\langle|\nabla\omega|^2\rangle$ (achieved when $d\langle|\nabla\omega|^2\rangle/dt = 0$) vanish in the limit $\langle|\nabla\omega|^2\rangle \to \infty$. It follows that $\nu\langle|\nabla\omega|^2\rangle$ vanishes uniformly in time as $\nu \to 0$. Note that this result implies that the term due to viscosity in Eq. (6) also vanishes because $\langle|\omega|^{p-2}|\nabla\omega|^2\rangle \leqslant \|\omega\|_\infty^{p-2}\,\langle|\nabla\omega|^2\rangle$. Hence, viscous dissipation of $\langle|\omega|^p\rangle^{1/p}$ vanishes. The same is true for the unforced case.

An immediate implication of the above result is that for steady dynamics in the inviscid limit, the enstrophy injection η is totally dissipated by the scale-neutral mechanical friction. The enstrophy range is no longer inertial but rather becomes dissipative. This unusual and undesirable dissipation range presumably scales as k^{-3} [24] because the scale-neutral friction is not known to have spectrally steepening effects (for small α). For such a range the dissipation of enstrophy occurs uniformly among its wave number octaves. Hence, one may expect a diminishing enstrophy flux through this range rather than the k-independent flux of Kraichnan.

Given a small α, two distinct dynamical regimes can be expected to exist. One corresponds to small viscosity and is characterized by the predominance of frictional enstrophy dissipation. The other corresponds to moderate viscosity and is characterized by the predominance of viscous enstrophy dissipation. There exists a critical viscosity, say ν_c, separating these two regimes, that is, $\nu_c\langle|\nabla\omega|^2\rangle = \alpha\langle\omega^2\rangle$. Thus, we have

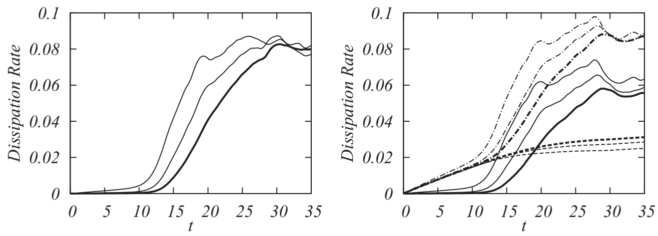
$$\nu_c\langle|\nabla\omega|^2\rangle = \alpha\langle\omega^2\rangle = \frac{\eta}{2}. \tag{9}$$

Past simulations of 2D turbulence using friction belonged to the regime of predominant viscous dissipation. For example, Boffetta and Musacchio [7] have found $\nu\langle|\nabla\omega|^2\rangle \approx 0.9\eta$ (see their Table 1), which they considered strong evidence for the Kraichnan picture of direct enstrophy cascade.

For power-law spectra, the critical viscosity ν_c can be readily determined from Eq. (9). Assume that in the limit $\alpha \to 0$ (inevitably $\nu_c \to 0$), the turbulence approaches the Kraichnan-Batchelor spectrum $E(k) = C\eta'^{2/3}k^{-3}$, where $\eta' = \eta/2$. The spectra of $\langle|\nabla\omega|^2\rangle$ and $\langle\omega^2\rangle$ are then given by $2C\eta'^{2/3}k^1$ and $2C\eta'^{2/3}k^{-1}$, respectively. By using these spectra, we solve Eq. (9) for ν_c and obtain

$$\nu_c = \frac{\eta'^{1/3}}{Ck_f^2}\exp\left(\frac{-\eta'^{1/3}}{C\alpha}\right), \tag{10}$$

where k_f is the forcing wave number, which has been used as an approximation for the low wave-number end of the enstrophy inertial range. The exponential decay of ν_c in Eq. (10) means that for small α, high resolutions are necessary to probe into the regime of predominant frictional dissipation of enstrophy. In the next section we numerically determine ν_c for a moderate range of α and the result is consistent with Eq. (10).



(a) $\alpha = 0$ (b) $\alpha = 0.008$

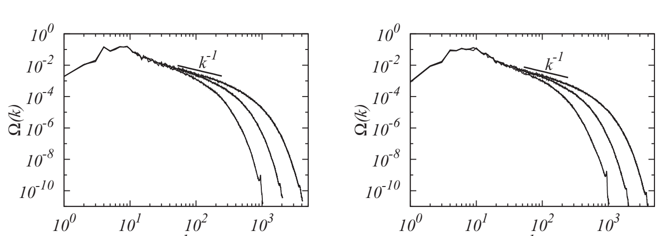
FIG. 1. Viscous (solid), frictional (dashed), and total (dash-dotted) dissipation rates vs time for $\nu = 1.6 \times 10^{-5}$, 4×10^{-6}, and 1×10^{-6} (bold). (a) and (b) correspond to $\alpha = 0$ and $\alpha = 0.008$, respectively. In (a) smaller ν corresponds to less dissipation in the early stage. In (b) smaller ν corresponds to less viscous but more frictional dissipation throughout.

IV. NUMERICAL RESULTS

We now present the results from numerical simulations that support the theoretical predictions discussed in the preceding section. Equations (2) were simulated using a conventional pseudospectral method in a doubly periodic square of side 2π with resolutions up to 8192×8192 grid points. The time stepping was a fourth-order Runge-Kutta scheme, with the frictional and viscous dissipation terms incorporated exactly through an integrating factor. The forcing used was similar to that in Ref. [25], being nonzero only for 16 selected wave vectors having magnitudes lying in the interval $K = (10,11)$:

$$\widehat{f}(\boldsymbol{k},t) = \frac{\eta}{16\,\widehat{\omega}^*(\boldsymbol{k},t)}. \tag{11}$$

In Eq. (11), $\eta = 0.1$ is the constant enstrophy injection rate and the asterisk denotes the complex conjugate. The 16 forced modes were initialized to small nonzero values, with all other Fourier modes initialized to zero. The choice of the forcing region K allows for a direct transfer range of over two decades for the highest resolution simulations and an inverse transfer range of one decade. The latter is sufficiently wide to ensure negligible contamination of the inverse energy transfer by the spectral boundary up to $t = 35$, when the turbulence is clearly on course to become quasisteady. However, it is not possible to get closer to the quasisteady stage by carrying out the simulations significantly beyond $t = 35$, without risking a serious contamination of the inverse energy transfer.



(a) $\alpha = 0$ (b) $\alpha = 0.008$

FIG. 2. Enstrophy spectra at $t = 30$ for the same series of simulations as in Fig. 1. Shallower spectra correspond to smaller viscosity.

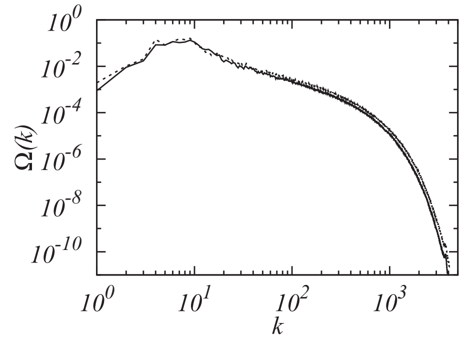
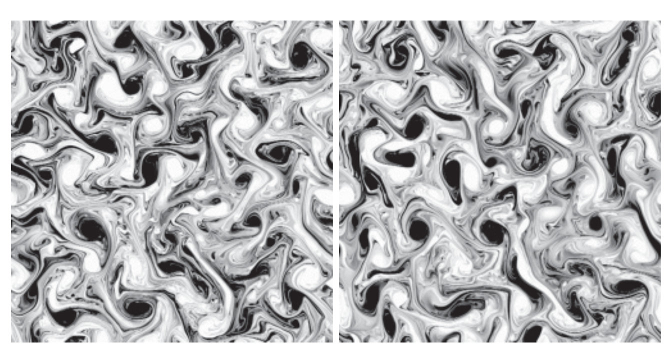


FIG. 3. Enstrophy spectra for the highest resolution runs with $\alpha = 0$ (dotted line) and $\alpha = 0.008$ (solid line).

The dependence of viscous enstrophy dissipation on viscosity in damped (by friction) turbulence is illustrated by three simulations with $\alpha = 0.008$ and $\nu = 1.6 \times 10^{-5}$, 4×10^{-6}, 1×10^{-6}. These simulations correspond to resolutions 2048×2048, 4096×4096, and 8192×8192, respectively. The coefficients α and ν were chosen in such a way that frictional dissipation remained weaker than viscous dissipation for all three simulations and that viscosity alone could adequately resolve the truncation scales. Note that in accordance with the approximately linear scaling of the number of degrees of freedom with the Reynolds number [26], ν was decreased by a factor of 4 when the resolution was doubled. For a comparison with undamped (i.e., $\alpha = 0$) turbulence, three parallel simulations with $\alpha = 0$ at the above viscosity values were also carried out. Figure 1 shows the dissipation rates vs time up to $t = 35$, at which time the turbulence presumably approaches the quasisteady stage. The cases $\alpha = 0$ and $\alpha = 0.008$ are shown in Figs. 1(a) and 1(b), respectively. In the latter, the frictional, viscous, and total dissipation rates vs time are depicted by the dashed, solid, and dashed-dotted lines, respectively. The viscous enstrophy dissipation rate clearly decreases as the viscosity is decreased. This is accompanied by a corresponding increase in the frictional dissipation rate. Note that the damped case appears to approach equilibrium more rapidly than the undamped case.

The enstrophy spectra $\Omega(k) = k^2 E(k)$ at $t = 30$ for the above simulations are shown in Fig. 2, with the undamped and damped cases in Figs. 2(a) and 2(b), respectively. In both cases the spectra seem to be approaching the Kraichnan-Batchelor



(a) $\alpha = 0$ (b) $\alpha = 0.008$

FIG. 4. Vorticity fields corresponding to the spectra of Fig. 3.

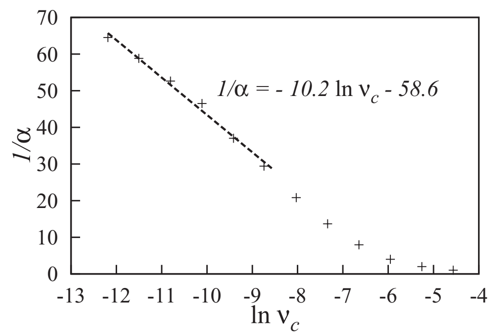


FIG. 5. A plot of $1/\alpha$ vs $\ln \nu_c$ for simulations up to resolution 4096×4096. The dashed line (whose equation is inserted) through the data points corresponding to the six smallest values of α is the line of best fit for these points.

k^{-1} spectrum. This strongly supports the earlier suggestion that the scale-neutral Ekman drag does not affect the form of the spectrum of the enstrophy inertial range in a significant manner. In fact, the two panels are virtually indistinguishable. For a more quantitative comparison, Fig. 3 replots the two highest resolution spectra, one from each of the panels of Fig. 2. The vorticity fields corresponding to these two spectra are given in Fig. 4. It is remarkable that even when frictional dissipation of enstrophy becomes sufficiently strong (more than half as strong as viscous dissipation), there are hardly any noticeable differences between undamped and damped turbulence in both wave number space and physical space.

Finally, we have attempted to numerically determine the dependence of ν_c on α and compare the result with Eq. (10). A separate set of simulations up to resolution 4096×4096 was carried out for this purpose. For each of a dozen values of α within the range $[1/64,1]$, ν_c was determined by varying ν until the equality $\alpha\langle\omega^2\rangle = \nu\langle|\nabla\omega|^2\rangle$ approximately held during a short time period, when the turbulence almost became quasisteady. Figure 5 shows the plot of $1/\alpha$ vs $\ln \nu_c$, which, according to Eq. (10), is expected to render a straight line with slope $-C/{\eta'}^{1/3}$ and intercept $(C/{\eta'}^{1/3})\ln[{\eta'}^{1/3}/(Ck_f^2)]$. For the data points corresponding to the six smallest values of α, the line of best fit (the dashed line in Fig. 5) is given by

$$\frac{1}{\alpha} = -10.2 \ln \nu_c - 58.6. \qquad (12)$$

Solving for C using the slope and intercept of this line, we obtain $C = 3.8$ and $C = 3.2$, respectively. The discrepancy of these two solutions for C is understandable and can be attributed to the fact that the range of α (and ν_c) under consideration is not small enough for the spectra to closely approximate the classical one.

V. CONCLUDING REMARKS

We have examined both theoretically and numerically the effects of mechanical friction on the enstrophy dynamics of forced 2D Navier-Stokes turbulence. On the theoretical side we have shown by an elementary method that friction gives rise to vanishing viscous enstrophy dissipation in the inviscid limit. Similar to freely decaying (undamped) turbulence in the inviscid limit, where viscous dissipation of enstrophy vanishes uniformly in time [17,19], the present result is valid uniformly in time. This uniformity in time is important and worth emphasizing as it appears to have been misunderstood (in the freely decaying case) by some authors (see the remark on page 352 of Ref. [8]). The implication of the present findings is that given a fixed friction coefficient, frictional dissipation of enstrophy becomes predominant for sufficiently small viscosity. This inevitably results in the classical enstrophy inertial range becoming a dissipation range in which the dissipation of enstrophy by friction mainly occurs. This range can at best support a diminishing enstrophy flux rather than the k-independent flux of Kraichnan. For the classical spectrum, which is assumed to be valid in the limit of weak friction, we have derived an expression for the critical viscosity, which separates the regimes of predominant viscous and frictional dissipation of enstrophy. This critical viscosity decreases exponentially with the friction coefficient. On the numerical side, we have carried out a number of numerical integrations of the forced 2D Navier-Stokes equations with a friction term to confirm the theoretical results. Given all else fixed, including the friction coefficient, viscous dissipation of enstrophy has been observed to decrease as the viscosity is decreased. This decrease appears to be slow, probably logarithmically in viscosity as in the case of freely decaying turbulence [17,19]. The numerical results for the critical viscosity are in qualitative agreement with the theoretical finding. We have observed no significant differences between undamped and damped turbulence near the critical viscosity. In particular, the energy spectra of the enstrophy inertial range in the two cases are virtually indistinguishable, both are close to the classical k^{-3} spectrum.

Enstrophy divergence in the inviscid limit is an indispensable feature of the Kraichnan theory. The reason is that the predicted limiting inertial range simply has infinite enstrophy. Another reason, which is less obvious, is that the production of ever-smaller scales (effectively a linear process [15,16]) would not be able to withstand viscous effects in the absence of a diverging vorticity [11]. In other words, only in the presence of a diverging vorticity could the enstrophy be transferred to and eventually dissipated at ever-smaller scales, thereby maintaining the picture of quasisteady dynamics envisaged by Kraichnan. Therefore, in choosing large-scale dissipation operators for numerical reasons or in designing thin film or shallow fluid layer experiments (for comprehensive reviews of this research topic see Refs. [27–29] and references therein), one should be mindful of this constraint. Mechanical friction has been seen here to render "anti-Kraichnan" behavior in the inviscid limit. (For further but largely unrelated effects of friction and other large-scale dissipation mechanisms see Ref. [30,31].) Another important effect of friction not discussed in this study is that it "stabilizes" the virtually inviscid forcing region, which, in Kraichnan's picture, is kept bounded by the dual cascade alone. The implication is that the problem of universality or nonuniversality of the Kraichnan-Batchelor constant may not be reliably resolved by simulations with a friction term.

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