# Turbulent magnetic Prandtl number in kinematic magnetohydrodynamic turbulence: Two-loop approximation

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The turbulent magnetic Prandtl number in the framework of the kinematic magnetohydrodynamic (MHD) turbulence, where the magnetic field behaves as a passive vector field advected by the stochastic Navier-Stokes equation, is calculated by the field theoretic renormalization group technique in the two-loop approximation. It is shown that the two-loop corrections to the turbulent magnetic Prandtl number in the kinematic MHD turbulence are less than 2% of its leading order value (the one-loop value) and, at the same time, the two-loop turbulent magnetic Prandtl number is the same as the two-loop turbulent Prandtl number obtained in the corresponding model of a passively advected scalar field. The dependence of the turbulent magnetic Prandtl number on the spatial dimension $d$ is investigated and the source of the smallness of the two-loop corrections for spatial dimension $d = 3$ is identified and analyzed.

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## I. INTRODUCTION

It is well known that diffusion processes of the magnetic field in a conductive medium, which is described by the magnetohydrodynamics (MHD), are characterized by the dimensionless magnetic Prandtl number $\mathrm{Pr}_m$, the ratio of the coefficient of the kinematic viscosity to the coefficient of the magnetic diffusivity (resistivity). On the other hand, in the case when the conductive fluid is in the state of fully developed turbulence (MHD turbulence), the diffusion processes are rapidly accelerated and this fact is expressed in the appearance of an effective value of the diffusion coefficient, namely, the turbulent magnetic diffusivity. The ratio of the turbulent viscosity to the turbulent magnetic diffusivity (turbulent resistivity) is the so-called turbulent magnetic Prandtl number $\mathrm{Pr}_{m,t}$ [1,2] in analogy with the turbulent Prandtl number of the thermal diffusion [3,4], which obtains its universal value in the limit of fully developed turbulence.

The need for the theoretical investigation and calculation of the possible values of the turbulent magnetic Prandtl number is dictated by its importance in many physical processes and their simulations, such as the turbulent dynamo problem (see, e.g., Refs. [5–10] and references cited therein), astrophysical MHD turbulence phenomena (see, e.g, Refs. [11–14] and references cited therein), and MHD simulations and calculations (see, e.g., Refs. [15,16]). Therefore, the determination of the turbulent magnetic Prandtl number on the fundamental level of a microscopic model of fully developed MHD turbulence is very important.

An effective approach for theoretical investigation of universal properties of models of fully developed turbulence is the renormalization group (RG) method [4,17–19]. The RG technique was used for the calculation of the turbulent magnetic Prandtl number in Ref. [20], where the method of Wilson's recursion relations was applied. Later, in Ref. [21], the turbulent magnetic Prandtl number was calculated in a more general stochastic MHD model by using the field theoretic RG technique, which is based on the standard formalism of the quantum field theory. Within the RG approach, the turbulent magnetic Prandtl number is given in the form of a perturbation series in the corresponding expansion parameter of the model (see the next section). In this respect, the analysis and calculations in Refs. [20,21] have been done in the first-order (one-loop) approximation and it must be said that, up to now, the value of the turbulent magnetic Prandtl number was considered and calculated only in the leading order of the perturbation theory [20–22]. On the other hand, the situation is different in the case of the turbulent Prandtl number in the model of a passive scalar quantity advected by the turbulent environment where, aside from the first-order results in the framework of various RG approaches [23–26], also the next-to-leading (two-loop) approximation result exists [27,28], which was obtained by using the field theoretic RG approach.

In Refs. [27,28], it was shown that the turbulent Prandtl number in the model of a passive scalar field advected by the stochastic Navier-Stokes equation is surprisingly very stable under the perturbation theory (at least up to the second-order approximation in the corresponding perturbation expansion) in the sense that the two-loop corrections to the one-loop value of the turbulent Prandtl number are very small and are less than 2% of its leading one-loop value [28]. It must be also stressed that the obtained result $\mathrm{Pr}_t = 0.7051$ for the two-loop turbulent Prandtl number is in rather good agreement with experimental estimation of possible values of turbulent Prandtl number [3,29,30].

On the other hand, the smallness of the two-loop RG corrections to the turbulent Prandtl number is rather surprising in the situation when the corresponding two-loop corrections to other quantities, which characterize the fully developed turbulence (such as the Kolmogorov constant and the skewness factor), are comparable to, or are even larger than, their one-loop values [31]. In Refs. [27,28], the negligible amount of the two-loop corrections to the turbulent Prandtl number was explained by the absence of the terms proportional to $1/(d-2)$ in the final expressions for them (see Ref. [27] for details). Here, $d$ is the spatial dimension. As we shall see in this paper, the above mentioned cancellation of the terms proportional to $1/(d-2)$ plays an important role here, but the main reason for the almost vanishing of the two-loop corrections to the two-loop Prandtl number is a little bit different.

 

As was already mentioned above, in contrast to the turbulent Prandtl number, the turbulent magnetic Prandtl number has been calculated only to the first order in the framework of the corresponding perturbation theory. In this paper, we would like to start with a systematic investigation of the turbulent magnetic Prandtl number in the second-order approximation by using the field theoretic RG approach. The aim of this paper is to find the two-loop value of the turbulent magnetic Prandtl number in the kinematic MHD turbulence, i.e., in the case when the magnetic field is weak enough and can be considered as a passively advected vector field (i.e., the Lorentz force term in the Navier-Stokes equation is omitted, see the next section), and also to discuss its behavior within the perturbation expansion. As we shall see, although the present model of a passively advected vector field (weak magnetic field) in the framework of the kinematic MHD turbulence is considerably different from the corresponding model of a passive scalar advection studied in Refs. [27,28], nevertheless, it will be shown that the final two-loop turbulent magnetic Prandtl number is the same as the corresponding turbulent Prandtl number of the passive scalar advection. The second aim of this paper is to identify properly the reason for the smallness of the two-loop corrections to the turbulent magnetic Prandtl number in the kinematic MHD turbulence (as well as to the turbulent Prandtl number in the model of passively advected scalar field [27,28]) by using a detailed analysis of their dependence on the spatial dimension $d$. It will be shown that the typical two-loop corrections to the turbulent (magnetic) Prandtl number are from 20% to 30% of its one-loop value for spatial dimensions $d = 4,5,\dots,10$, i.e., they are considerably larger in comparison with 2% corrections for $d = 3$. It is shown that the main reason for so small two-loop corrections for $d = 3$ is related to the almost exact cancellation of the two-loop contributions given, on one hand, by the two-loop Feynman diagrams and, on the other hand, by the expansion to the leading order of scaling functions of the corresponding response functions.

The paper is organized as follows. In Sec. II, the model of the kinematic MHD turbulence is defined and the field theoretic formulation of the model is given. In Sec. III, we perform the ultraviolet (UV) renormalization of the model, the two-loop renormalization constants are calculated, and the stable scaling regime is established. In Sec. IV, the two-loop turbulent magnetic Prandtl number is calculated and its dependence on the value of the spatial dimension is discussed. Obtained results are briefly reviewed and discussed in Sec. V.

## II. FIELD THEORETIC FORMULATION OF THE MODEL

### A. The kinematic MHD turbulence

The advection of a passive solenoidal magnetic field $\mathbf{b} \equiv \mathbf{b}(x)$ $[x \equiv (t,\mathbf{x})]$ in the framework of the kinematic MHD is described by the following system of stochastic equations:

$$\partial_t \mathbf{b} = \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \qquad (1)$$

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial P + \mathbf{f}^{\mathbf{v}}, \qquad (2)$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, $\nu_0$ is viscosity coefficient (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory), $\nu_0 u_0 = c^2/(4\pi\sigma)$ represents the magnetic diffusivity (where we have already extracted dimensionless reciprocal magnetic Prandtl number $u_0$ for convenience), $c$ is the speed of light, $\sigma$ is the conductivity, $P \equiv P(x)$ is the pressure, and $\mathbf{v} \equiv \mathbf{v}(x)$ is a solenoidal (owing to the incompressibility) velocity field. Thus, both $\mathbf{v}$ and $\mathbf{b}$ are divergence-free vector fields: $\partial\cdot\mathbf{v} = \partial\cdot\mathbf{b} = 0$.

The energy pumping given by a transverse Gaussian random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ with zero mean and the correlation function

$$D^b_{ij}(x;0) \equiv \langle f^b_i(x) f^b_j(0)\rangle = \delta(t) C_{ij}(|\mathbf{x}|/L) \qquad (3)$$

represents the source of the fluctuations of the magnetic field $\mathbf{b}$ and maintains the steady state of the system. Here, $L$ is an integral scale related to the corresponding stirring, and $C_{ij}$ is a function finite in the limit $L \to \infty$. In what follows, the detailed form of the function $C_{ij}$ is unimportant; the only condition that must be satisfied is that $C_{ij}$ decreases rapidly for $|\mathbf{x}| \gg L$. If $C_{ij}$ depends on the direction of the vector $\mathbf{x}$ and not only on its modulus $r = |\mathbf{x}|$, then it can be considered as a source of the large-scale anisotropy.

On the other hand, the transverse random force per unit mass $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ in Eq. (2) simulates the energy pumping into the system on large scales. We assume that its statistics is also Gaussian with zero mean and pair correlation function

$$D^v_{ij}(x;0) \equiv \langle f^v_i(x) f^v_j(0)\rangle = \delta(t)\int \frac{d^d\mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (4)$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the ordinary transverse projector, $d$ denotes the spatial dimension of the system, $D_0 \equiv g_0\nu_0^3 > 0$ is the positive amplitude, and the physical value of formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. It plays an analogous role as the parameter $\epsilon = 4 - d$ in the theory of critical behavior, and the introduced parameter $g_0$ plays the role of the coupling constant of the model. In addition, $g_0$ is a formal small parameter of the ordinary perturbation theory and is related to the characteristic ultraviolet (UV) momentum scale $\Lambda$ (or inner length $l \sim \Lambda^{-1}$) by the following relation:

$$g_0 \simeq \Lambda^{2\varepsilon}. \qquad (5)$$

In Eq. (4), the needed infrared regularization is given by a restriction of the integrations from below, namely, $k \geqslant m$, where $m$ corresponds to another integral scale. In what follows, we shall suppose that $L \gg 1/m$.

The correlation function (4) is chosen in the form that, on one hand, is suitable for description of the real infrared energy pumping to the system [for $\varepsilon \to 2$, the function $D_0 k^{4-d-2\varepsilon}$ is proportional to $\delta(\mathbf{k})$ for appropriate choice of the amplitude factor $D_0$, which corresponds to the injection of energy to the system through interaction with the largest turbulent eddies] and, on the other hand, its powerlike form gives possibility to apply the RG technique for analysis of the problem [17–19].

The stochastic model given in Eqs. (1)–(4) represents a simplification of real MHD turbulence problem in the sense that in the real MHD problem, the velocity field $\mathbf{v}$ obeys the stochastic Navier-Stokes equation with the additional Lorentz force term that describes the influence of magnetic field on the velocity field of a conductive fluid. Therefore, the magnetic field **b** in the present model behaves like a passively advected vector field.

### B. Field theoretic formulation of the model

The field theoretic formulation of the present model is based on the well-known theorem [32] that asserts that the stochastic problem (1)–(4) is equivalent to the field theoretic model of the double set of fields $\Phi = \{\mathbf{v},\mathbf{b},\mathbf{v}',\mathbf{b}'\}$ with the following action functional:

$$S(\Phi) = \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \times \left[v_i'(x_1)D^v_{ij}(x_1;x_2)v_j'(x_2) + b_i'(x_1)D^b_{ij}(x_1;x_2)b_j'(x_2)\right] + \int dt\, d^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} + \mathbf{b}'[-\partial_t\mathbf{b} + \nu_0 u_0\triangle\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}]\}, \quad (6)$$

where $x_i = (t_i,\mathbf{x_i})$, $i = 1,2$, $\mathbf{v}'$, and $\mathbf{b}'$, are auxiliary transverse fields that have the same tensor properties as fields $\mathbf{v}(x)$ and $\mathbf{b}(x)$, $D^b_{ij}$, $D^v_{ij}$ are given in Eqs. (3) and (4), respectively, and required summations over dummy indices are assumed.

The pressure term $\partial P$ in Eq. (2) is omitted in action (6) as a result of the fact that the auxiliary vector field $\mathbf{v}'(x)$ is also transverse, i.e., $\partial_i v_i' = 0$, and by using the integration by parts, it is evident that it vanishes, namely,

$$\int dt\, d^d\mathbf{x}\, v_i'\partial_i P = -\int dt\, d^d\mathbf{x}\, P\,\partial_i v_i' = 0.$$

The field theoretic model given by action functional (6) corresponds to a standard Feynman diagrammatic perturbation theory with the following set of bare propagators (in frequency-momentum representation):

$$\langle b_i' b_j\rangle_0 = \langle b_i b_j'\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 u_0 k^2}, \quad (7)$$

$$\langle v_i' v_j\rangle_0 = \langle v_i v_j'\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 k^2}, \quad (8)$$

$$\langle b_i b_j\rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-i\omega + \nu_0 u_0 k^2|^2}, \quad (9)$$

$$\langle v_i v_j\rangle_0 = \frac{g_0\nu_0^3 k^{4-d-2\varepsilon}P_{ij}(\mathbf{k})}{|-i\omega + \nu_0 k^2|^2}, \quad (10)$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of function $C_{ij}(\mathbf{r}/L)$ in Eq. (3). In the Feynman diagrams, these propagators are represented by lines that are shown in Fig. 1 (the end with a slash in the propagators $\langle b_i' b_j\rangle_0$ and $\langle v_i' v_j\rangle_0$ corresponds to the fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively, and the end without a slash corresponds to the fields **b** and **v**, respectively). The triple vertices (or interaction vertices) $b_i'(-v_j\partial_j b_i + b_j\partial_j v_i) = b_i' v_j V_{ijl} b_l$ and $-v_i' v_j \partial_j v_i = v_i' v_j W_{ijl} v_l/2$, where $V_{ijl} = i(k_j\delta_{il} - k_l\delta_{ij})$ and $W_{ijl} = i(k_l\delta_{ij} + k_j\delta_{il})$ (in the momentum-frequency representation) are present in Fig. 2, where momentum **k** is flowing into the vertices via the auxiliary fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively.

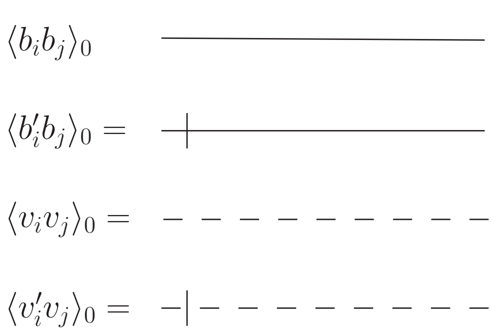


FIG. 1. Graphical representation of the propagators of the model.

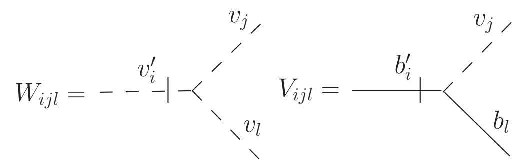


FIG. 2. The interaction vertices of the model.

The advantage of the formulation of the stochastic problem given by Eqs. (1)–(4) through action functional (6) is related to the fact that it allows one to use the well-defined field theoretic means, e.g., the RG technique, to analyze the problem and the statistical averages of random quantities in the stochastic problem are replaced with the corresponding functional averages with weight exp $S(\Phi)$ (see, e.g., Ref. [19] for details).

## III. RENORMALIZATION GROUP ANALYSIS

The RG analysis of a field theoretic model is based on the analysis of UV divergences that, on the other hand, is given by the analysis of the corresponding canonical dimensions. The dynamical model (6) belongs to the class of the so-called two-scaled models [17–19], i.e., to the class of models for which the canonical dimension of some quantity $Q$ is given by two numbers: the momentum dimension $d_Q^k$ and the frequency dimension $d_Q^\omega$. Therefore, the dimensions of all quantities can be found by using the requirement that each term of action functional (6) must be dimensionless separately with respect to the momentum and frequency together with the standard definitions (normalization conditions) $d_k^k = -d_x^k = d_\omega^\omega = -d_t^\omega = 1$. The total canonical dimension $d_Q$ is then defined as $d_Q = d_Q^k + 2d_Q^\omega$ [it is related to the fact that $\partial_t \propto \partial^2$ in the free action (6) with choice of zero canonical dimensions for $\nu_0$ and $u_0$] and it plays the same role in the renormalization theory of our dynamical model as the simple momentum dimension does in static models.

The main conclusion of the corresponding dimensional analysis is the fact that the coupling constant of the model, namely, $g_0$, is dimensionless (i.e., the model is the so-called logarithmic) at $\varepsilon = 0$. Therefore, in the framework of the minimal subtraction (MS) scheme [33], which is used in what follows, all possible UV divergences in the correlation functions of the model have the form of poles in $\varepsilon$. Then, by using the general expression for the total canonical dimension of an arbitrary one-irreducible Green's function $\langle\Phi\cdots\Phi\rangle_{1-ir}$, which plays the role of the formal index of the UV divergence, together with the symmetry properties of the model, one can find that, in the case with $d > 2$, the superficial UV divergences are present only in the one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle b_i' b_j\rangle_{1-ir}$ and, at the same time, action functional (6) has all necessary tensor structures to remove divergences multiplicatively (see, e.g., [19,33]). All divergences can be removed by the counterterms of the forms $\mathbf{v}'\Delta\mathbf{v}$ and $\mathbf{b}'\Delta\mathbf{b}$, which can be explicitly expressed in the multiplicative renormalization of the parameters $g_0, u_0$, and $\nu_0$ in the form

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u, \tag{11}$$

where the dimensionless parameters $g$, $u$, and $\nu$ are the renormalized counterparts of the corresponding bare ones, $\mu$ is the renormalization mass (a scale-setting parameter), an artifact of the dimensional regularization. Quantities $Z_i = Z_i(g,u;d;\varepsilon)$ are the so-called renormalization constants and they contain poles in $\varepsilon$.

On the other hand, the renormalized action functional has the form

$$\begin{aligned} S_R(\Phi) &= \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \\ &\times \left[v_i'(x_1)D_{ij}^v(x_1;x_2)v_j'(x_2) + b_i'(x_1)D_{ij}^b(x_1;x_2)b_j'(x_2)\right] \\ &+ \int dt\, d^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu Z_1\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} \\ &+ \mathbf{b}'[-\partial_t\mathbf{b} + \nu u Z_2\triangle\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}]\}, \end{aligned} \tag{12}$$

where $Z_1$ and $Z_2$ are the renormalization constants, which are related to the renormalization constants defined in Eq. (11) as

$$Z_\nu = Z_1, \quad Z_g = Z_1^{-3}, \quad Z_u = Z_2 Z_1^{-1}. \tag{13}$$

Thus, one is left with two independent renormalization constants $Z_1$ and $Z_2$, and their explicit forms in the MS scheme are

$$Z_1(g;d;\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(1)}(d)}{\varepsilon^j}, \tag{14}$$

$$Z_2(g,u;d;\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(2)}(u,d)}{\varepsilon^j}, \tag{15}$$

where coefficients $z_{nj}^{(1)}$ and $z_{nj}^{(2)}$ are independent of $\varepsilon$ and are determined by the requirement that the one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle b_i' b_j\rangle_{1-ir}$ must be UV finite when written in the renormalized variables, i.e., they have no singularities in the limit $\varepsilon \to 0$. On the other hand, one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle b_i' b_j\rangle_{1-ir}$ are related to the corresponding self-energy operators $\Sigma^{v'v}$ and $\Sigma^{b'b}$, which are expressed via Feynman diagrams, by the Dyson equations. In frequency-momentum representation, they can be written in the form

$$\langle v_i' v_j\rangle_{1-ir} = -[-i\omega + \nu_0 p^2 - \Sigma^{v'v}(\omega,p)]P_{ij}(\mathbf{p}), \tag{16}$$

$$\langle b_i' b_j\rangle_{1-ir} = -[-i\omega + \nu_0 u_0 p^2 - \Sigma^{b'b}(\omega,p)]P_{ij}(\mathbf{p}). \tag{17}$$

Thus, $Z_1$ and $Z_2$ are found from the requirement that the UV divergences are canceled in Eqs. (16) and (17) after the substitution $e_0 = e\mu^{d_e} Z_e$ for $e = \{g,u,\nu\}$. This determines $Z_1$ and $Z_2$ up to a UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme, all the renormalization constants have the following form: 1 + *poles in* $\varepsilon$ and, in the end, one comes to the explicit expressions for coefficients $z_{nj}^{(i)}, i = 1,2$, given in Eqs. (14) and (15) within the corresponding order of the perturbation theory.

The expansion of the renormalization constant $Z_1$ in Eq. (14) is known up to the second order in $g$ (two-loop approximation), i.e., the explicit form of the coefficients $z_{11}^{(1)}$, $z_{21}^{(1)}$, and $z_{22}^{(1)}$ was already calculated. The simplest one-loop result $z_{11}^{(1)}$ was obtained, e.g., in Ref. [34] and it reads as

$$z_{11}^{(1)} = -\frac{S_d}{(2\pi)^d}\frac{(d-1)}{8(d+2)}, \tag{18}$$

where $S_d$ denotes the surface area of the $d$-dimensional unit sphere defined as

$$S_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{19}$$

and $\Gamma(x)$ is Euler's gamma function. The two-loop corrections $z_{21}^{(1)}$ and $z_{22}^{(1)}$ were found by authors of paper [31]. The coefficient $z_{22}^{(1)}$ is simply related to the coefficient $z_{11}$ (see Ref. [31] for details):

$$z_{22}^{(1)} = -\left(z_{11}^{(1)}\right)^2, \tag{20}$$

and the explicit form of the coefficient $z_{21}^{(1)}$ can be found in Ref. [31]. It is a rather huge and complicated expression, therefore, we shall not present it here explicitly.

On the other hand, as for the renormalization constant $Z_2$ in Eq. (15), up to now, it is known only to the first order of the perturbation theory, namely,

$$z_{11}^{(2)} = -\frac{S_d}{(2\pi)^d}\frac{(d-1)}{4du(u+1)}, \tag{21}$$

and it was found, e.g., in Ref. [35]. Therefore, the first necessary step needed for the systematic investigation of the properties of the present model within the second order of the perturbation theory is to find the explicit form of the coefficients $z_{21}^{(2)}$ and $z_{22}^{(2)}$ in Eq. (15). To this end, the corresponding analysis of the structure of the self-energy operator $\Sigma^{b'b}$ given in the Dyson equation (17) must be done.

The self-energy operator $\Sigma^{b'b}$ is given by the sum of singular parts of the corresponding one-irreducible Feynman diagrams. In the two-loop approximation, it can be written as

$$\Sigma^{b'b} = \Gamma^1 + \Gamma^2 = \Gamma^1 + \sum_{l=1}^{8} s_l \Gamma_l^2, \tag{22}$$

where $s_l$, $l = 1,\ldots,8$, are components of the vector

$$\mathbf{s} = (1,1,1,1/2,1,1,1,1), \tag{23}$$

which represents the corresponding symmetry coefficients of the two-loop diagrams that are shown in Fig. 3. The analytic form of the singular part of the one-loop contribution $\Gamma^1$ is given as

$$\Gamma^1 = -\frac{S_d}{(2\pi)^d}\frac{g\nu p^2}{4\varepsilon}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{d-1}{d(u+1)}, \tag{24}$$

which leads to the explicit expression for coefficient $z_{11}^{(2)}$ as it is given in Eq. (21). In Eq. (24), $m$ is an integral scale and it is introduced to provide the needed infrared (IR) regularization (see, e.g., Ref. [31] for details). On the other hand, the two-loop

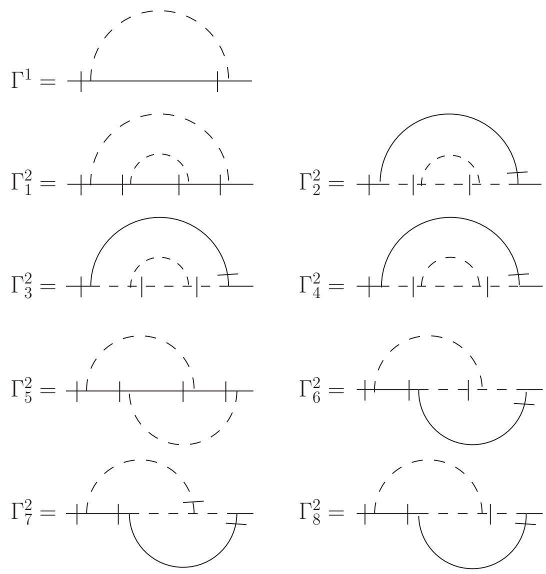


FIG. 3. The one- and two-loop diagrams that contribute to the self-energy operator $\Sigma^{b'b}(\omega,p)$ in Eq. (17).

contributions $\Gamma_l^2$, $l=1,\dots,8$, are given by the calculation of the two-loop diagrams in Fig. 3 and can be written in an integral representation as

$$\Gamma_l^2 = \frac{g^2\nu\, p^2\, S_d}{16(2\pi)^{2d}}\left(\frac{\mu}{m}\right)^{4\varepsilon}\frac{1}{\varepsilon}\left\{\frac{(d-1)^2}{2d(1+u)}\frac{S_d}{\varepsilon}A_l + S_{d-1}\int_0^1 dx\,(1-x^2)^{(d-1)/2}\,B_l\right\}, \quad (25)$$

where $x$ is the cosine of the angle between two independent momenta $\mathbf{k}$ and $\mathbf{q}$ over which the integration is taken in the two-loop case, i.e., $x=\mathbf{k}\cdot\mathbf{q}/(|\mathbf{k}||\mathbf{q}|)$, and the explicit forms of coefficients $A_l$ and $B_l$ for $l=1,\dots,8$ as functions of $d$, $u$, and $x$ are given in the Appendix.

Thus, by using the Dyson equation (17) together with the relation (22) and the explicit expression (25), the coefficients $z_{21}^{(2)}$ and $z_{22}^{(2)}$ in Eq. (15) for the renormalization constant $Z_2$ are given as

$$z_{22}^{(2)} = -\frac{S_d^2}{(2\pi)^{2d}}\frac{(d-1)^2[d(1+u)(2+u)+2(d+2)]}{64d^2(d+2)u(1+u)^3} \quad (26)$$

and

$$z_{21}^{(2)} = \frac{S_d S_{d-1}}{16u(2\pi)^{2d}}\int_0^1 dx\,(1-x^2)^{(d-1)/2}\sum_{l=1}^{8}s_l B_l, \quad (27)$$

where the symmetry coefficients $s_l$, $l=1,\dots,8$, are given in Eq. (23) and functions $B_l$, $l=1\,\dots,8$, are shown in the Appendix.

The fact that fields $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{b}$, and $\mathbf{b}'$ are not renormalized means that, e.g., the renormalized connected correlation functions $W^R=\langle\Phi\cdots\Phi\rangle^R$ are equal to their unrenormalized counterparts $W=\langle\Phi\cdots\Phi\rangle$ and the only difference is in the choice of variables (renormalized or unrenormalized) and in the corresponding perturbation expansion (in $g$ or $g_0$), i.e.,

$$W^R(g,u,\nu,\mu,\dots) = W(g_0,u_0,\nu_0,\dots), \quad (28)$$

where the dots stand for other arguments that are untouched by renormalization, e.g., coordinates and times. Using the fact that unrenormalized correlation functions are independent of the scale-setting parameter $\mu$, one can apply the differential operator $\mu\partial_\mu$ at fixed unrenormalized parameters on both sides of Eq. (28), which leads to the basic differential RG equation

$$[\mu\partial_\mu + \beta_g\partial_g + \beta_u\partial_u - \gamma_\nu\nu\partial_\nu]W^R(g,u,\nu,\mu,\dots) = 0, \quad (29)$$

where the so-called RG functions (the $\beta$ and $\gamma$ functions) are given as

$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon + 3\gamma_1), \quad (30)$$

$$\beta_u \equiv \mu\partial_\mu u = u(\gamma_1-\gamma_2), \quad (31)$$

$$\gamma_i \equiv \mu\partial_\mu \ln Z_i, \quad i=1,2 \quad (32)$$

where relations among renormalization constants (13) were used and $Z_1$ and $Z_2$ are given in Eqs. (14) and (15), respectively.

The IR asymptotic scaling behavior (the scaling behavior deep inside of the inertial interval) of the correlation functions of the model is driven by the IR stable fixed point of the RG equations. The coordinates of a fixed point $(g_*,u_*)$ are determined by the requirement of vanishing of the $\beta$ functions, namely,

$$\beta_g(g_*) = 0, \quad \beta_u(g_*,u_*) = 0. \quad (33)$$

Our interest is concentrated to the nontrivial fixed point with $g_*\neq 0$ and $u_*\neq 0$, and within two-loop approximation, its coordinates are

$$g_* = g_*^{(1)}\varepsilon + g_*^{(2)}\varepsilon^2 + O(\varepsilon^3), \quad (34)$$

$$u_* = u_*^{(1)} + u_*^{(2)}\varepsilon + O(\varepsilon^2), \quad (35)$$

with

$$g_*^{(1)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}, \quad (36)$$

$$g_*^{(2)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}\lambda, \quad (37)$$

$$u_*^{(1)} = \frac{1}{2}\left(-1+\sqrt{\frac{9d+16}{d}}\right), \quad (38)$$

$$u_*^{(2)} = \frac{2(d+2)}{d[1+2u_*^{(1)}]}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right], \quad (39)$$

where $\lambda$ is related to the coefficient $z_{21}^{(1)}$ in Eq. (14) by the following equation:

$$\lambda = \frac{2}{3}\frac{(2\pi)^{2d}}{S_d^2}\left(\frac{8(d+2)}{d-1}\right)^2 z_{21}^{(1)}, \quad (40)$$

and its explicit form can be found in Ref. [31]. (Here, for convenience, we use the same notation as in Refs. [27,31].). On the other hand, the coefficient $\mathcal{B}(u_*^{(1)})$ is given by the coefficient $z_{21}^{(2)}$ in Eq. (27) by substitution $u\to u_*^{(1)}$ as follows:

$$\mathcal{B}(u_*^{(1)}) = \frac{(2\pi)^{2d}}{S_d^2}z_{21}^{(2)}(u_*^{(1)}). \quad (41)$$

The type of a fixed point is determined by the properties of the matrix of the first derivatives

$$\Omega_{ij} = \begin{pmatrix} \partial\beta_g/\partial g & \partial\beta_g/\partial u \\ \partial\beta_u/\partial g & \partial\beta_u/\partial u \end{pmatrix} \quad (42)$$

calculated at the point $(g_*,u_*)$. For IR stable fixed point, the real parts of all its eigenvalues are positive. In our case, the matrix element $\partial\beta_g/\partial u$ vanishes identically ($\beta_g$ does not depend on $u$), therefore, the eigenvalues are given directly by the diagonal elements. It can be shown by numerical analysis of the diagonal elements of the matrix (42) that their real parts are positive for $\varepsilon > 0$, i.e., the fixed point is IR attractive.

It is important to say that the form of $\beta_g$ and $\beta_u$ in Eqs. (30) and (31) does not depend on order of the perturbation expansion, i.e., it is exactly given by the one-loop approximation without higher-loop corrections. This fact leads to the exact values for the anomalous dimensions $\gamma_1$ and $\gamma_2$ at the IR stable fixed point $(g_*,u_*)$, namely,

$$\gamma_1^* = \gamma_2^* = \frac{2\varepsilon}{3}. \quad (43)$$

The existence of the stable IR fixed point means that the correlation functions of the model exhibit scaling behavior with given critical dimensions in the IR range, but we shall not discuss this question here (the corresponding discussion within one-loop approximation can be found in Ref. [35], where also the problem of the anomalous scaling is analyzed).

## IV. THE TURBULENT MAGNETIC PRANDTL NUMBER

In Ref. [27], the second-order approximation RG formula for the turbulent (effective) inverse Prandtl number was derived, which holds inside the inertial interval and does not depend on the renormalization scheme [see Eq. (33) in Ref. [27]]. Using this formula, the second-order corrections to the turbulent Prandtl number have been calculated, and it was shown that the two-loop corrections are very small (they are less than 2% of the leading one-loop result) [27,28].

We shall not repeat all steps in derivation of the formula (33) in Ref. [27], but it can be shown that by using the same arguments as in Ref. [27], the corresponding RG expression can be also derived for the turbulent inverse magnetic Prandtl number $u_{\text{eff}}$ in the present model of the kinematic MHD turbulence. The formula can be also written in the same form as for the turbulent inverse Prandtl number in the model of passively advected scalar field [27], namely,

$$u_{\text{eff}} = u_*^{(1)}\left\{1 + \varepsilon\left[\frac{1+u_*^{(1)}}{1+2u_*^{(1)}}\left(\lambda - \frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right) + \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}[a_v - a_b(u_*^{(1)})]\right]\right\}, \quad (44)$$

where $\lambda$ and $\mathcal{B}(u_*^{(1)})$ are now given in Eqs. (40) and (41), respectively, and quantities $a_v$ and $a_b(u_*^{(1)})$ are given by the corresponding expansions to the leading order in $\varepsilon$ of the scaling functions of response functions $\langle vv'\rangle$ and $\langle bb'\rangle$ of the velocity field and the magnetic field, respectively (see Ref. [27] for details). The explicit form of the coefficient $a_v$ can be found in Ref. [27], therefore, we shall not present it here. On the other hand, the coefficient $a_b(u_*^{(1)})$ can be calculated in the framework of the present model in the same manner as the coefficient $a_\psi$ in Ref. [27]. The corresponding calculation gives

$$a_b(u) = -\frac{S_{d-1}}{2u(2\pi)^d}\int_0^\infty dk\int_{-1}^1 dx\,(1-x^2)^{\frac{d-1}{2}} \times\left[\frac{k}{(1+u)k^2+2ukx+u} - \frac{\theta(k-1)}{k(1+u)}\right], \quad (45)$$

where $\theta(y)$ is a standard Heaviside step function.

In general, the coefficients $a_b$ and $\mathcal{B}$ in Eq. (44) for the turbulent inverse magnetic Prandtl number can be different from the corresponding coefficients $a_\psi$ and $B$ in Eq. (33) in Ref. [27] for the turbulent inverse Prandtl number. It is given by the fact that, although the corresponding quantities are defined by the same set of Feynman diagrams (compare Fig. 3 in this paper to Figs. 2 and 3 in Ref. [27]), the diagrams have different tensor structures, which reflect different internal properties of the advected scalar and vector (magnetic) fields. However, by direct comparison, it is evident that the coefficient $a_b$ in Eq. (45) is the same as the corresponding coefficient $a_\psi$ in Eq. (38) in Ref. [27] and, at the same time, by direct calculations it can be also shown that despite the fact that the models are completely different, the final expression $\mathcal{B}$ in Eq. (44) is the same as the corresponding quantity $B$ in Eq. (33) in Ref. [27]. This is a nontrivial fact, which can be seen only when all two-loop diagrams shown in Fig. 3 are analyzed and calculated.

Thus, by using the following values for needed coefficients for $d = 3$ in Eq. (44), namely,

$$u_*^{(1)} = 1.393, \quad (46)$$

$$\lambda = -1.101, \quad (47)$$

$$a_v = -0.047718/(2\pi^2), \quad (48)$$

$$a_b = -0.04139/(2\pi^2), \quad (49)$$

$$\mathcal{B}(u_*^{(1)}) = -4.432\times10^{-3}, \quad (50)$$

one comes to the following two-loop value of the turbulent magnetic Prandtl number within the kinematic MHD turbulence (for the physical value $\varepsilon = 2$), namely,

$$\text{Pr}_{m,t} = 0.7051, \quad (51)$$

which coincides with the corresponding turbulent Prandtl number $\text{Pr}_t$ of passively advected scalar field [27,28].

The fact that the turbulent magnetic Prandtl number in the present model of a passively advected vector (magnetic) field in the framework of the kinematic MHD turbulence is the same as the turbulent Prandtl number in the corresponding model of a passive scalar advection [27,28] has only one possible explanation. Let us briefly discuss it. The kinematic MHD model differs from the model of passively advected scalar field [27,28] in two details. First of all, it is the internal vector nature of the passively advected magnetic field in the kinematic MHD model and, second, it is a more complicated antisymmetric interaction vertex in the model of kinematic MHD turbulence (see Sec. II). It can be shown by direct calculations that, if the interaction vertex in the vector model is taken in the same form as in the model of passive scalar advection (i.e., when the part $b_i' b_j \partial_j v_i$ of the vertex is omitted), then the results for the two-loop Feynman diagrams given in Fig. 3 are completely different from those obtained in the kinematic MHD model discussed in this paper (thus, they are also different from the corresponding results in the model of passively advected scalar field). It means that the antisymmetric structure of the interaction vertex for the advected vector field in the kinematic MHD model, which is given by the nonlinear terms in the stochastic equation for the magnetic field in Eq. (1) [or by the corresponding trilinear terms, which include fields **b**, **v**, and **b**′ in the action functional (6)], compensates the additional terms related to the vector nature of the advected field in the one- and two-loop Feynman diagrams and, as a result, the final expressions for the diagrams are the same as for the model of passively advected scalar field [27,28]. However, we would like to stress once more that this equivalence between corresponding one- and two-loop Feynman diagrams of the present vector model and the model of a passively advected scalar field discussed in Refs. [27,28] is not evident at first sight, and to see it the corresponding analysis and calculations of the diagrams must be done.

Thus, we can conclude that the diffusion processes of the passive vector field (e.g., a weak magnetic field) advected by the incompressible isotropic turbulent environment driven by the stochastic Navier-Stokes equation in the framework of the kinematic MHD turbulence have the same properties as the corresponding diffusion processes of passive scalar quantities advected by the same stochastic environment, at least, up to the second-order approximation (the two-loop approximation), which is discussed in this paper. However, although we are not able to prove it, nevertheless, it seems that this equivalence between diffusion properties of these two models will be also held in all orders of the perturbation theory as a result of the above discussed mechanism of compensation. Of course, this assertion can not be simply extended and applied to the corresponding models with the presence of an internal asymmetry of the turbulent systems [e.g., the turbulent environment with helicity (spatial parity violation) or anisotropy], where the tensor structure of the passively advected field can play a nontrivial role and can lead to considerable differences in the corresponding diffusion properties of advected fields with different internal tensor structures. However, these questions are out of the scope of this paper and will be studied elsewhere.

In the end, let us analyze the behavior of the two-loop turbulent magnetic Prandtl number within the present model as a function of spatial dimension $d$. It will give us important information about the source of rather miraculous cancellation of large two-loop contributions to the turbulent magnetic Prandtl number (as well as to the turbulent Prandtl number [27,28]), which are generated separately by coefficients $\lambda$ and $\mathcal{B}$, respectively. This cancellation of large contributions leads to the final result for the turbulent (magnetic) Prandtl number, which is less than 2% different from its one-loop value for the physical spatial dimension $d = 3$ (see the corresponding discussion in Refs. [27,28]).

Because the two-loop turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence is the same as the corresponding two-loop turbulent Prandtl number in the model of a passively advected scalar field (this assertion is also true for their $d$ dependence), the analysis and results presented here for the turbulent magnetic Prandtl number as the function of spatial dimension $d$ will also hold for the turbulent Prandtl number of a passively advected scalar field studied in Refs. [27,28]).

For further convenience, it is appropriate to rewrite Eq. (44) into the form that immediately gives us the information about the importance of the contributions of the corresponding terms into the two-loop value of the turbulent magnetic Prandtl number. Thus, one can write

$$u_{\text{eff}} = u_*^{(1)}[1 + \varepsilon(\lambda' - \mathcal{B}' + a_v' - a_b')], \tag{52}$$

where we have used the following notation:

$$\lambda' = \frac{1 + u_*^{(1)}}{1 + 2u_*^{(1)}}\,\lambda, \tag{53}$$

$$\mathcal{B}' = \frac{1 + u_*^{(1)}}{1 + 2u_*^{(1)}}\,\frac{128(d+2)^2}{3(d-1)^2}\,\mathcal{B}(u_*^{(1)}), \tag{54}$$

$$a_v' = \frac{(2\pi)^d}{S_d}\,\frac{8(d+2)}{3(d-1)}\,a_v, \tag{55}$$

$$a_b' = \frac{(2\pi)^d}{S_d}\,\frac{8(d+2)}{3(d-1)}\,a_b. \tag{56}$$

In Table I, first of all, the dependence of the above defined coefficients $a_v'$, $a_b'$, $\lambda'$, and $\mathcal{B}'$ on the value of spatial dimension $d$ is shown. It can be seen that the coefficients $a_v'$ and $a_b'$, which are related to the expansion of the corresponding scaling functions to the leading order in $\varepsilon$ (see Ref. [27] for details), have different behavior as functions of the spatial dimension $d$. The absolute value of the coefficient $a_v'$ is an increasing function of $d$ for relatively small values of the spatial dimension. On the other hand, the absolute value of the coefficient $a_b'$ has the opposite behavior, i.e., it is a decreasing function of the spatial dimension. It is evident that their individual contributions to the two-loop turbulent magnetic Prandtl number are rather large. As it can be seen in Table I, their difference $a_v' - a_b'$, which directly contributes to the two-loop value of the turbulent magnetic Prandtl number in Eq. (52), is also significantly large almost for all small and moderate values of the spatial dimension $d$. The contribution of the difference $a_v' - a_b'$ to the two-loop turbulent magnetic Prandtl number vanishes completely only in two cases, namely, when the spatial dimension of the system has the fractal value near $d = 2.75$ and when $d \to \infty$. The second case is related to the fact that, in the limit $d \to \infty$, both quantities $a_v'$ and $a_b'$ are given by the same asymptotic expression, namely,

$$a_{v,b}' \simeq -\frac{4}{3\sqrt{\pi}}\,\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty dk\int_{-1}^1 dx\,(1-x^2)^{\frac{d-1}{2}} \times\left[\frac{k}{2k^2+2kx+1} - \frac{\theta(k-1)}{2k}\right],\quad d\to\infty. \tag{57}$$

On the other hand, as it can be seen in Table I, the coefficients $\lambda'$ and $\mathcal{B}'$ have the same behavior, namely, they are both increasing functions of the spatial dimension $d$, at least for the small and moderate values of $d$. Again, their individual contributions to the two-loop turbulent magnetic Prandtl number are large. However, their difference $\lambda' - \mathcal{B}'$ as a function of $d$ becomes negligible in comparison to the

TABLE I. The dependence of the coefficients $a_v'$, $a_b'$, $\lambda'$, and $\mathcal{B}'$ given in Eqs. (53)–(56), their differences $a_v' - a_b'$ and $\lambda' - \mathcal{B}'$, and the one-loop value $\mathrm{Pr}_{m,t}^{(1)}$ and the two-loop value $\mathrm{Pr}_{m,t}$ of the turbulent magnetic Prandtl number on spatial dimension $d$. The relative contribution $\epsilon$ of the two-loop corrections with respect to the one-loop result for the turbulent magnetic Prandtl number as function of spatial dimension $d$ is also presented.

| $d$ | 2.1 | 2.25 | 2.5 | 2.75 | 3 | 3.25 | 3.5 | 4 | 5 | 6 | 8 | 10 | 100 | $d \to \infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_v'$ | −0.0421 | −0.1308 | −0.2259 | −0.2828 | −0.3181 | −0.3404 | −0.3546 | −0.3686 | −0.3719 | −0.3646 | −0.3459 | −0.3297 | −0.2434 | ≈−0.23 |
| $a_b'$ | −0.2946 | −0.2905 | −0.2848 | −0.2800 | −0.2759 | −0.2724 | −0.2694 | −0.2645 | −0.2575 | −0.2528 | −0.2471 | −0.2436 | −0.2322 | ≈−0.23 |
| $\lambda'$ | −3.7020 | −1.9304 | −1.1304 | −0.8446 | −0.6958 | −0.6010 | −0.5357 | −0.4503 | −0.3604 | −0.3147 | −0.2710 | −0.2514 | −0.2214 | −2/9 |
| $\mathcal{B}'$ | −4.3108 | −2.1385 | −1.2509 | −0.9227 | −0.7470 | −0.6364 | −0.5601 | −0.4619 | −0.3621 | −0.3140 | −0.2693 | −0.2499 | −0.2215 | −2/9 |
| $a_v' - a_b'$ | 0.2525 | 0.1597 | 0.0589 | −0.0028 | −0.0422 | −0.0680 | −0.0852 | −0.1041 | −0.1144 | −0.1118 | −0.0988 | −0.0861 | −0.0112 | 0 |
| $\lambda' - \mathcal{B}'$ | 0.6088 | 0.2081 | 0.1205 | 0.0781 | 0.0512 | 0.0354 | 0.0244 | 0.0116 | 0.0017 | −0.0007 | −0.0017 | −0.0015 | 0.0001 | 0 |
| $\mathrm{Pr}_{m,t}^{(1)}$ | 0.6501 | 0.6636 | 0.6839 | 0.7019 | 0.7179 | 0.7322 | 0.7452 | 0.7676 | 0.8023 | 0.8280 | 0.8633 | 0.8866 | 0.9869 | 1 |
| $\mathrm{Pr}_{m,t}$ | 0.2388 | 0.3823 | 0.5034 | 0.6101 | 0.7051 | 0.7832 | 0.8482 | 0.9419 | 1.0358 | 1.0683 | 1.0805 | 1.0750 | 1.0093 | 1 |
| $\epsilon \times 10^2 [\%]$ | 63.3 | 42.4 | 26.4 | 13.1 | 1.8 | 7.0 | 13.8 | 22.7 | 29.1 | 29.0 | 25.2 | 21.2 | 2.3 | 0 |

values of the difference $a_v' - a_b'$ starting from the spatial dimension $d = 4$ (see Table I). On the other hand, the contribution of the difference $\lambda' - \mathcal{B}'$ to the two-loop value of the turbulent magnetic Prandtl number is important for $d < 4$ and it rapidly increases in the limit $d \to 2$, where it obtains infinite value. This behavior is related to the existence of the terms proportional to $1/(d-2)$ in the difference $\lambda' - \mathcal{B}'$ (see Ref. [27] for details).

The final two-loop numerical values of the turbulent magnetic Prandtl number $\mathrm{Pr}_{m,t}$ for various spatial dimensions $d$ together with the corresponding one-loop values $\mathrm{Pr}_{m,t}^{(1)}$ are also given in Table I. Besides, in Table I, the relative contribution $\epsilon$ of the two-loop corrections with respect to the one-loop result, which is defined as

$$\epsilon = \left| \frac{\mathrm{Pr}_{m,t} - \mathrm{Pr}_{m,t}^{(1)}}{\mathrm{Pr}_{m,t}^{(1)}} \right|, \tag{58}$$

is presented for various values of the spatial dimension $d$. It is evident that the two-loop corrections to the turbulent magnetic Prandtl number are significant for small and moderate values of the spatial dimension $d$ and, typically, they are of the order of tens of percent of the corresponding one-loop results. The only exceptions are, on one hand, the narrow interval of the spatial dimensions around $d = 3$, where the two-loop corrections to the turbulent magnetic Prandtl number vanish or are very small and, on the other hand, the spatial dimensions very close to $d = 2$, where the two-loop corrections diverge in the limit $d \to 2$ as a result of the presence of the terms proportional to $1/(d-2)$. The vanishing of the two-loop corrections to the turbulent magnetic Prandtl number near $d = 3$ is related to the fact that, here the correction $\lambda' - \mathcal{B}'$, which is given by the corresponding sets of the two-loop Feynman diagrams, and the correction $a_v' - a_b'$, which is related to the expansion of the corresponding scaling functions to the leading order in $\varepsilon$, are comparable as for their absolute values and have opposite signs (see Table I).

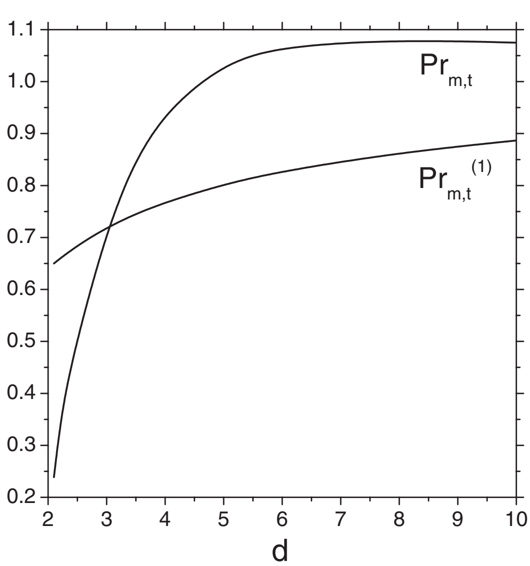


FIG. 4. The dependence of the one-loop $\mathrm{Pr}_{m,t}^{(1)}$ and two-loop $\mathrm{Pr}_{m,t}$ values of the turbulent magnetic Prandtl number on the spatial dimension $d$.

In Fig. 4, the dependence of the two-loop turbulent magnetic Prandtl number on spatial dimension $d$ is shown and compared to the corresponding one-loop behavior. It is evident that the two-loop corrections are small only for spatial dimensions near $d = 3$. As was discussed above, this miraculous vanishing of the importance of the two-loop corrections to the turbulent magnetic Prandtl number for physical value of the spatial dimension $d = 3$ is not given by the smallness of the individual contributions $\lambda' - \mathcal{B}'$ and $a_v' - a_b'$ (they are relatively large and, separately, they give corrections about 10% to the one-loop turbulent magnetic Prandtl number), but rather by their simultaneous cancellation, which leads to the final fact that the two-loop corrections to the turbulent magnetic Prandtl number are less than 2% of the corresponding one-loop result.

We would like to stress once more that all obtained results and discussion also hold for the turbulent Prandtl number of passively advected scalar quantity studied in Refs. [27,28].

## V. CONCLUSION

In this paper, by using the field theoretic RG technique, we have calculated and analyzed the turbulent magnetic Prandtl number of a passively advected magnetic field by the Navier-Stokes turbulent environment in the framework of the kinematic MHD turbulence in the second-order approximation (two-loop approximation) of the corresponding perturbation theory. It is shown that the two-loop turbulent magnetic Prandtl number within the present model is the same as the corresponding two-loop turbulent Prandtl number of a passively advected scalar field by the stochastic Navier-Stokes equation studied in Refs. [27,28]. It is shown that the reason for the equivalence of the diffusion processes in these two different models is given by a mechanism that compensates additional parts in the one- and two-loop Feynman diagrams in the studied vector model by the antisymmetric form of the corresponding vertex. It is supposed that this compensation mechanism will also hold in all orders of the perturbation theory, at least when full symmetry of the developed turbulent environment is assumed. However, this is a nontrivial hypothesis, which can not be seen at first sight and can be proven only by direct calculations at the corresponding level of approximation. In this paper, the equivalence is shown and discussed at two-loop level. We also suppose that the equivalence of the diffusion processes of the advected vector and scalar fields in discussed models was not valid when one supposes the presence of an asymmetry of the turbulent environment (e.g., the presence of anisotropy, spatial parity violation, compressibility, etc.). In these cases, it is rather probable that the internal tensor structure of the advected field will have a nontrivial impact on the properties of the diffusion processes, which will lead to the different behavior of the corresponding turbulent Prandtl numbers. We hope to return to these questions in the near future.

Further, the dependence of the two-loop turbulent magnetic Prandtl number on the spatial dimension $d$ has been calculated and analyzed to find the reason of the smallness of the two-loop corrections to the turbulent magnetic Prandtl number for real spatial dimension $d = 3$. In this case, the two-loop correction to the one-loop value of the turbulent magnetic Prandtl number is less than 2%. It is shown that this result is given by almost exact cancellation of two relatively large contributions given by the corresponding two-loop Feynman diagrams $(\lambda' - \mathcal{B}')$ and by the expansions to the leading order in parameter $\varepsilon$ of the scaling functions of the corresponding response functions $(a'_v - a'_b)$. However, this situation is rather specific and holds only for narrow interval of fractal dimensions around $d = 3$. For spatial dimensions $d \geqslant 4$, the crucial contribution to the two-loop value of the turbulent magnetic Prandtl number is given by the difference $a'_v - a'_b$ and the contribution given by the two-loop Feynman diagrams $\lambda' - \mathcal{B}'$ is negligibly small. As it can be seen in Table I, this fact leads to the two-loop corrections that are larger than 20% of the corresponding one-loop result. However, the contribution of the difference $\lambda' - \mathcal{B}'$ to the two-loop value of the turbulent magnetic Prandtl number is important for $d < 4$ and it rapidly increases in the limit $d \to 2$, where it obtains infinite value, which is related to the existence of the terms proportional to $1/(d-2)$ in the difference $\lambda' - \mathcal{B}'$.

The results obtained in this paper are given for the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence, but all discussed results are also valid for the turbulent Prandtl number of a passive scalar field advected by the Navier-Stokes velocity field as a result of the aforementioned mathematical equivalence of these two models.

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## APPENDIX

The explicit form of the coefficients $A_l$ and $B_l$ for $l = 1,\ldots,8$ in Eq. (25) is the following:

$$A_1 = \frac{1}{d(1+u)^2}, \quad A_2 = \frac{(3+u)}{4(d+2)(1+u)},$$

$$A_3 = \frac{1}{4(d+2)}, \quad A_i = 0, \quad i = 4,\ldots,8$$

$$B_1 = \frac{2ux(d-1)}{d(1+u)^3\sqrt{1+2u+u^2(1-x^2)}}X_1,$$

$$B_2 = \frac{B_{21}X_2 + B_{22}X_3 + B_{23}(X_4+X_5) + B_{24}X_6}{2d},$$

$$B_3 = \frac{B_{31}X_2 - B_{32}X_3}{2dx(1+u)},$$

$$B_4 = \frac{B_{41} + B_{42}X_1 + B_{43}X_3 + B_{44}X_7}{4d},$$

$$B_5 = \frac{2x}{d(1+u)^2\sqrt{1+2u+u^2(1-x^2)}}X_1,$$

$$B_6 = \frac{2x[B_{61}X_2 + B_{62}(X_4+X_5) + B_{63}X_6]}{d(1+u)(1-2u+u^2+4ux^2)},$$

$$B_7 = -[B_{71} + B_{72}X_1 + B_{73}X_3 + B_{74}(X_4+X_5) + B_{75}X_6 + B_{76}X_7]/[4d(1+u)],$$

$$B_8 = \frac{B_{81}X_1 + B_{82}X_2 + B_{83}X_3}{dx(1+u)},$$

where

$$X_1 = \arctan\left(\frac{1+u(1+x)}{\sqrt{1+2u+u^2(1-x^2)}}\right) - \arctan\left(\frac{1+u(1-x)}{\sqrt{1+2u+u^2(1-x^2)}}\right),$$

$$X_2 = \arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right),$$

$$X_3 = \arctan\left(\frac{2+x}{\sqrt{4-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{4-x^2}}\right),$$

$$X_4 = \arctan\left(\frac{2+x}{\sqrt{2(1+u)-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{2(1+u)-x^2}}\right),$$

$$X_5 = \arctan\left(\frac{1+u+x}{\sqrt{2(1+u)-x^2}}\right) - \arctan\left(\frac{1+u-x}{\sqrt{2(1+u)-x^2}}\right),$$

$$X_6 = \ln\left(\frac{2}{1+u}\right),$$

$$X_7 = i\pi + \ln\left(\frac{1-x^2+x\sqrt{x^2-1}}{x^2-1+x\sqrt{x^2-1}}\right),$$

and

$$B_{21} = -2\sqrt{1-x^2}[1+12x^2+u^3(4x^2-1) + u^2(16x^4-1)+u(1-16x^2+48x^4)]/ [(1+u)^2x(1-2u+u^2+4ux^2)],$$

$$B_{22} = \frac{[4-(7+d)x^2+2x^4]}{(1-u)x\sqrt{4-x^2}},$$

$$B_{23} = 2x\{-3+2u^3+2x^2+4u(1-4x^2+2x^4) + u^2(13-14x^2)-d[1+u^2+2u(2x^2-1)]\}/ [(u-1)(1+u)^2\sqrt{2(1+u)-x^2} \times(1-2u+u^2+4ux^2)],$$

$$B_{24} = 2\{1-2x^2-3u^2(1-2x^2)+2u[1+4x^2(1-x^2)] + d[1+u^2-2u(1-2x^2)]\}/ [(1-u)(1+u)^2(1-2u+u^2+4ux^2)],$$

$$B_{31} = \frac{2(1-5x^2+4x^4)}{\sqrt{1-x^2}},$$

$$B_{32} = \frac{4-(d+7)x^2+2x^4}{\sqrt{4-x^2}},$$

$$B_{41} = \frac{(10-4d)x^2-4}{(1+u)x^2(1-x^2)},$$

$$B_{42} = \frac{4\{1+u^2(1-x^2)+u[2+(d-2)x^2]\}}{(u-1)x^3\sqrt{1+2u+u^2(1-x^2)}},$$

$$B_{43} = \frac{8u[4+(d-3)x^2]}{(1-u^2)x^3\sqrt{4-x^2}},$$

$$B_{44} = \frac{3x^2-2+2(d-2)x^4+2u(x^2-1)^2}{(1+u)x^3(x^2-1)^{3/2}},$$

$$B_{61} = -\frac{4\sqrt{1-x^2}}{1+u},$$

$$B_{62} = \frac{3+u-2x^2}{(1+u)\sqrt{2(1+u)-x^2}},$$

$$B_{63} = \frac{2x}{1-u},$$

$$B_{71} = -\frac{2(6+u+u^3+8ux^2+4u^2x^2)}{(1+u)(1-2u+u^2+4ux^2)},$$

$$B_{72} = \frac{4(1-d-du-u^2+u^2x^2)}{(1+u)x\sqrt{1+2u+u^2(1-x^2)}},$$

$$B_{73} = \frac{4(2d-x^2)}{(1-u)x\sqrt{4-x^2}},$$

$$B_{74} = 8x[u^4-2+2x^2-3u^3(4x^2-5)+u(12x^2-11) - d(1+3u)(1-2u+u^2+4ux^2) + u^2(-3-2x^2+8x^4)]/ [(1-u^2)(1-2u+u^2+4ux^2)\sqrt{2(1+u)-x^2}],$$

$$B_{75} = 8\{2x^2-1+u^3(4x^2-3)+u^2(5+6x^2-8x^4) + u(4x^2-1)+d(u-1)[1+u^2+2u(2x^2-1)]\}/ \{(u^2-1)[1+u^2+2u(2x^2-1)]^2\},$$

$$B_{76} = \{2x^2(5-8x^2)-2u^3(5-16x^2+20x^4) - u(1-76x^2+96x^4)-u^5(5-12x^2+8x^4) + u^6(1-x^2)+u^2(5+31x^2-32x^4-32x^6) - 2u^4(-5+16x^2-16x^4+8x^6) + d[-2+u^5+8x^2+2u^4(4x^2-1) - 4u^2(1-8x^4)+2u^3(1+8x^4) + u(5-16x^2+32x^4)]\}/ \{(1+u)x[1+u^2+2u(2x^2-1)]^2\sqrt{x^2-1}\},$$

$$B_{81} = \frac{1-d(1+u)+u^2(x^2-1)}{(1-u)\sqrt{1+2u+u^2(1-x^2)}},$$

$$B_{82} = -\sqrt{1-x^2},$$

$$B_{83} = \frac{2d-x^2}{(1-u)\sqrt{4-x^2}}.$$


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