

# Quantum Langevin equation of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables

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We obtain the quantum Langevin equation (QLE) of a charged quantum particle moving in a harmonic potential in the presence of a uniform external magnetic field and linearly coupled to a quantum heat bath through momentum variables. The bath is modeled as a collection of independent quantum harmonic oscillators. The QLE involves a random force which does not depend on the magnetic field, and a quantum-generalized classical Lorentz force. These features are also present in the QLE for the case of particle-bath coupling through coordinate variables. However, significant differences are also observed. For example, the mean force in the QLE is characterized by a memory function that depends explicitly on the magnetic field. The random force has a modified form with correlation and commutator different from those in the case of coordinate-coordinate coupling. Moreover, the coupling constants, in addition to appearing in the random force and in the mean force, also renormalize the inertial term and the harmonic potential term in the QLE.

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## I. INTRODUCTION

The issue of the magnetic response of a charged quantum particle moving in a potential arises in many problems of theoretical and experimental relevance, e.g., Landau diamagnetism [1,2], quantum Hall effect [3,4], two-dimensional electronic systems [5], and others. The additional effect of quantum dissipation due to interaction with the external environment may be studied systematically by employing the system-plus-reservoir model, i.e., the Caldeira-Leggett model [6] (also known as the independent-oscillator model [7–12]). In this scheme, the environment is modelled as a quantum mechanical heat bath or reservoir comprising an infinite number of independent quantum harmonic oscillators with continuously distributed frequencies. One assumes a specific coupling of the dynamical variables of the oscillators to those of the particle.

In the case of bilinear coupling between the particle coordinate and the coordinate of each bath oscillator, a reduced description of the particle motion is given by the quantum Langevin equation (QLE) satisfied by the particle coordinate operator. In this equation, coupling to the bath is described by (i) an operator-valued random force, and (ii) a mean force characterized by a memory function [11,13]. These forces do not depend on the magnetic field whose only appearance in the QLE is through a quantum generalization of the classical Lorentz force.

In this work, we consider the complementary possibility of coupling of a quantum system to a quantum mechanical heat bath through the momentum variables. Although such a scenario has been considered previously by many authors [6,14–19], here we study the additional feature of the presence of an external magnetic field. To this end, we consider a gauge-invariant system-plus-reservoir model. The system comprises a charged quantum particle moving in a harmonic potential in the presence of a magnetic field. The particle is linearly coupled via the momentum variables to a quantum heat bath consisting of independent quantum harmonic oscillators.

Here, we derive a QLE for the particle coordinate operator for the case of an external magnetic field which is uniform

in space. The QLE is obtained by utilizing the well-known Heisenberg equation of motion for evolution of quantum operators and by effectively integrating out the bath degrees of freedom from the equations of motion. We show that similar to the case of coordinate-coordinate coupling, the QLE involves (i) a quantum-generalized Lorentz force term, and (ii) a random force which does not depend on the magnetic field. This latter force, nevertheless, has a modified form, with symmetric correlation and unequal time commutator different from the corresponding results in the case of coordinate-coordinate coupling. Other differences include (i) the memory function characterizing the mean force in the QLE has an explicit dependence on the magnetic field, and (ii) the inertial term and the harmonic potential term in the QLE get renormalized by the coupling constants.

The paper is organized as follows. In the next section, we introduce the system of study, and show that the system is invariant under a gauge transformation. In Sec. III, we derive the Heisenberg equations of motion for the particle and the bath oscillators. In Sec. IV, we derive the QLE for the charged particle for the case of a magnetic field which is uniform in space. Finally, we draw our conclusions.

## II. SYSTEM OF STUDY

Consider a charged particle moving in a harmonic potential in the presence of an external magnetic field. The particle is linearly coupled through the momentum variables to a large number  $N$  of independent quantum harmonic oscillators constituting a heat bath. The Hamiltonian of the system is given by

$$H = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + \frac{1}{2} m \omega_0^2 \mathbf{r}^2 + \sum_{j=1}^N \left[ \frac{1}{2m_j} \left( \mathbf{p}_j - g_j \mathbf{p} + \frac{g_j e}{c} \mathbf{A} \right)^2 + \frac{1}{2} m_j \omega_j^2 \mathbf{q}_j^2 \right], \quad (1)$$

where  $e, m, \mathbf{p}, \mathbf{r}$  are, respectively, the charge, the mass, the momentum operator, and the coordinate operator of the particle, while  $\omega_0$  is the frequency characterizing its motion in the harmonic potential. The  $j$ th heat-bath oscillator has mass  $m_j$ , frequency  $\omega_j$ , coordinate operator  $\mathbf{q}_j$ , and momentum operator  $\mathbf{p}_j$ . The dimensionless parameter  $g_j$  describes the coupling between the particle and the  $j$ th oscillator. The speed of light in vacuum is denoted by  $c$ . The vector potential  $\mathbf{A} = \mathbf{A}(\mathbf{r})$  is related to the external magnetic field  $\mathbf{B}(\mathbf{r})$  through

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}). \quad (2)$$

The relevant commutation relations for the various coordinate and momentum operators are

$$[r_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}, [q_{j\alpha}, p_{k\beta}] = i\hbar\delta_{jk}\delta_{\alpha\beta}, \quad (3)$$

while all other commutators vanish. In the above equation,  $\delta_{jk}$  denotes the Kronecker  $\delta$  function. Here, and in the following, Greek indices ( $\alpha, \beta, \dots$ ) refer to the three spatial directions, while Roman indices ( $i, j, k, \dots$ ) represent the heat-bath oscillators.

We now show that our system of study is gauge invariant. Consider the gauge transformation

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla f(\mathbf{r}), \quad (4)$$

where  $f(\mathbf{r})$  is an arbitrary function of coordinate  $\mathbf{r}$ . The transformed Hamiltonian  $H' = H'(\mathbf{A}')$  is given by the right hand side of Eq. (1) with  $\mathbf{A}$  replaced by  $\mathbf{A}'$ .

Now, our system will be gauge invariant if simultaneous with the transformation (4), one can make a unitary transformation of the state vectors of the system,

$$|\psi(t)\rangle \rightarrow |\psi'(t)\rangle = U|\psi(t)\rangle; U^\dagger = U^{-1}, \quad (5)$$

such that all physical observables remain invariant under the joint transformation. This requires that one should have  $H'(\mathbf{A}') = UH(\mathbf{A})U^\dagger$ , where  $H(\mathbf{A}) \equiv H$ . In our case, finding such a unitary transformation is easily achieved with the choice

$$U = \exp\left(\frac{ie}{\hbar c}f(\mathbf{r})\right). \quad (6)$$

Using the Hadamard formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \dots, \quad (7)$$

and the commutation relations (3), one can check that  $H'(\mathbf{A}') = UH(\mathbf{A})U^\dagger$ , as required.

### III. HEISENBERG EQUATIONS OF MOTION

In this section, we derive the Heisenberg equations of motion for the charged particle and the heat-bath oscillators.

#### 1. Charged particle

For the charged particle, the Heisenberg equations of motion are

$$\begin{aligned} \mathbf{v} \equiv \dot{\mathbf{r}} &= \frac{1}{i\hbar}[\mathbf{r}, H] \\ &= \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) - \sum_{j=1}^N \frac{g_j}{m_j}\left(\mathbf{p}_j - g_j\mathbf{p} + \frac{g_j e}{c}\mathbf{A}\right), \end{aligned} \quad (8)$$

and

$$\begin{aligned} \dot{p}_\alpha &= \frac{1}{i\hbar}[p_\alpha, H] \\ &= \frac{e}{2c}[(\partial_\alpha A_\beta)v_\beta + v_\beta(\partial_\alpha A_\beta)] - m\omega_0^2 r_\alpha. \end{aligned} \quad (9)$$

Equation (8) gives

$$\dot{\mathbf{p}} = m_r \ddot{\mathbf{r}} + \frac{e}{c} \dot{\mathbf{A}} + \sum_{j=1}^N \frac{g_j m_r}{m_j} \dot{\mathbf{p}}_j, \quad (10)$$

where  $m_r$  is the ‘‘renormalized mass,’’ defined as

$$m_r \equiv m / \left[ 1 + \sum_{j=1}^N \frac{g_j^2 m}{m_j} \right]. \quad (11)$$

Next, note that

$$(\mathbf{v} \times \mathbf{B})_\alpha = v_\beta \partial_\alpha A_\beta - v_\beta \partial_\beta A_\alpha, \quad (12)$$

and that

$$\begin{aligned} (\partial_\alpha A_\beta)v_\beta &= v_\beta(\partial_\alpha A_\beta) + [\partial_\alpha A_\beta, v_\beta] \\ &= v_\beta(\partial_\alpha A_\beta) + \frac{i\hbar}{m_r} \partial_\alpha \partial_\beta A_\beta. \end{aligned} \quad (13)$$

Using Eqs. (12) and (13) in Eq. (9), we get

$$\dot{p}_\alpha = \frac{e}{c}(\mathbf{v} \times \mathbf{B})_\alpha + \frac{e}{c}v_\beta \partial_\beta A_\alpha + \frac{i\hbar e}{2m_r c} \partial_\alpha \partial_\beta A_\beta - m\omega_0^2 r_\alpha, \quad (14)$$

that is,

$$\dot{\mathbf{p}} = \frac{e}{c}(\mathbf{v} \times \mathbf{B}) + \frac{e}{c}(\mathbf{v} \cdot \nabla)\mathbf{A} + \frac{i\hbar e}{2m_r c} \nabla(\nabla \cdot \mathbf{A}) - m\omega_0^2 \mathbf{r}. \quad (15)$$

Now, we have

$$\begin{aligned} \dot{A}_\alpha &= \frac{1}{i\hbar}[A_\alpha, H] \\ &= v_\beta(\partial_\beta A_\alpha) + \frac{i\hbar}{2m_r} \partial_\beta \partial_\beta A_\alpha, \end{aligned} \quad (16)$$

so that

$$\dot{\mathbf{A}}(\mathbf{r}) = (\mathbf{v} \cdot \nabla)\mathbf{A} + \frac{i\hbar}{2m_r} \nabla^2 \mathbf{A}, \quad (17)$$

which, on substituting in Eq. (10), gives

$$\dot{\mathbf{p}} = m_r \ddot{\mathbf{r}} + \frac{e}{c}(\mathbf{v} \cdot \nabla)\mathbf{A} + \frac{i\hbar e}{2m_r c} \nabla^2 \mathbf{A} + \sum_{j=1}^N \frac{g_j m_r}{m_j} \dot{\mathbf{p}}_j. \quad (18)$$

On equating Eq. (15) with Eq. (18), we get

$$\begin{aligned} m_r \ddot{\mathbf{r}} &= -m\omega_0^2 \mathbf{r} + \frac{e}{c}(\mathbf{v} \times \mathbf{B}) \\ &\quad + \frac{i\hbar e}{2m_r c}(\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}) - \sum_{j=1}^N \frac{g_j m_r}{m_j} \dot{\mathbf{p}}_j. \end{aligned} \quad (19)$$

On noting that

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \quad (20)$$

where  $\mathbf{j}$  is the current producing the external magnetic field, and also the fact that in practice this current source lies outside the region where the charged particle moves, we have

$$m_r \ddot{\mathbf{r}} = -m\omega_0^2 \mathbf{r} + \frac{e}{c}(\mathbf{v} \times \mathbf{B}) - \sum_{j=1}^N \frac{g_j m_r}{m_j} \dot{\mathbf{p}}_j. \quad (21)$$

In the next subsection, we show that  $\dot{\mathbf{p}}_j = -m_j \omega_j^2 \mathbf{q}_j$ . Using this in the last equation, we get

$$m_r \ddot{\mathbf{r}} = -m\omega_0^2 \mathbf{r} + \frac{e}{c}(\mathbf{v} \times \mathbf{B}) + \sum_{j=1}^N g_j m_r \omega_j^2 \mathbf{q}_j. \quad (22)$$

## 2. Heat-bath oscillators

For the heat-bath oscillators, the equations of motion are

$$\begin{aligned} \dot{\mathbf{q}}_j &= \frac{1}{i\hbar} [\mathbf{q}_j, H] \\ &= \frac{1}{m_j} \left( \mathbf{p}_j - g_j \mathbf{p} + \frac{g_j e}{c} \mathbf{A} \right), \end{aligned} \quad (23)$$

and

$$\begin{aligned} \dot{\mathbf{p}}_j &= \frac{1}{i\hbar} [\mathbf{p}_j, H] \\ &= -m_j \omega_j^2 \mathbf{q}_j. \end{aligned} \quad (24)$$

Combining Eqs. (23) and (24), we get

$$m_j \ddot{\mathbf{q}}_j = -m_j \omega_j^2 \mathbf{q}_j - g_j \dot{\mathbf{p}} + \frac{g_j e}{c} \dot{\mathbf{A}}, \quad (25)$$

which, on using Eqs. (15) and (17), gives

$$\begin{aligned} m_j \ddot{\mathbf{q}}_j &= -m_j \omega_j^2 \mathbf{q}_j + g_j m \omega_0^2 \mathbf{r} - \frac{g_j e}{c} (\mathbf{v} \times \mathbf{B}) \\ &\quad + \frac{i\hbar g_j e}{2m_r c} (\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A})). \end{aligned} \quad (26)$$

Using Eq. (20) and the reasoning given in the sentence following it, we finally have

$$m_j \ddot{\mathbf{q}}_j = -m_j \omega_j^2 \mathbf{q}_j + g_j m \omega_0^2 \mathbf{r} - \frac{g_j e}{c} (\mathbf{v} \times \mathbf{B}). \quad (27)$$

## IV. UNIFORM B: THE QUANTUM LANGEVIN EQUATION

In this section, we derive a QLE for the charged particle interacting with the heat-bath oscillators as modelled by Eq. (1), where we now consider a magnetic field uniform in space. One of the early appearances of a QLE in the case of coordinate-coordinate coupling between the particle and the heat-bath oscillators in the absence of magnetic field was in Ref. [7]. In our case, we follow the program adopted in [13] for the derivation of the QLE. The essential steps are as follows.

*Step 1.* Obtain the Heisenberg equations of motion for the system of the charged particle coupled to the heat bath. Solve these equations for the bath variables, and substitute the solution into the equations for the charged particle to obtain a reduced description of the particle motion. The solution will contain explicit expressions for the dynamical variables at time  $t$  in terms of their initial values.

*Step 2.* Make specific assumptions about the initial state of the system, e.g., assume that the heat bath was at thermal

equilibrium at the initial instant with the bath variables distributed according to a canonical distribution.

*Step 3.* Show that the coordinate operator for the charged particle then represents a stochastic process in time, and satisfies a QLE. The statistical properties of the stochastic process arise from the initial canonical distribution of the heat bath.

Step 1 has been partially carried out in Sec. III. We now carry out the remaining part, and solve the equations of motion for the bath variables by considering the magnetic field  $\mathbf{B}$  to be uniform in space, with components  $B_x, B_y, B_z$ , and magnitude  $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$ . In this case, Eq. (27) has the retarded solution

$$\begin{aligned} \mathbf{q}_j(t) &= \mathbf{q}_j^h(t) + \frac{g_j m \omega_0^2}{m_j \omega_j^2} \mathbf{r}(t) - \frac{g_j m \omega_0^2}{m_j \omega_j^2} \mathbf{r}(0) \cos(\omega_j t) \\ &\quad - \frac{g_j m \omega_0^2}{m_j \omega_j^2} \int_0^t dt' \dot{\mathbf{r}}(t') \cos[\omega_j(t-t')] \\ &\quad - \frac{g_j m \omega_c}{m_j \omega_j B} \Gamma \int_0^t dt' \dot{\mathbf{r}}(t') \sin[\omega_j(t-t')], \end{aligned} \quad (28)$$

where

$$\mathbf{q}_j^h(t) \equiv \mathbf{q}_j(0) \cos(\omega_j t) + \frac{\mathbf{p}_j(0)}{m_j \omega_j} \sin(\omega_j t) \quad (29)$$

is the contribution from the initial condition,

$$\omega_c \equiv \frac{eB}{mc} \quad (30)$$

is the Larmor frequency of precessional motion of the charged particle in the magnetic field, and

$$\Gamma \equiv \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \quad (31)$$

Substituting Eq. (28) into Eq. (22), we get

$$\begin{aligned} m_r \ddot{\mathbf{r}} &+ \int_0^t dt' \dot{\mathbf{r}}(t') \mu(t-t') + m_r \omega_0^2 \mathbf{r} + \mu_d(t) \mathbf{r}(0) \\ &- \frac{e}{c} (\mathbf{v} \times \mathbf{B}) = \mathbf{F}(t), \end{aligned} \quad (32)$$

where

$$\mathbf{F}(t) \equiv \sum_{j=1}^N g_j m_r \omega_j^2 \mathbf{q}_j^h(t) \Theta(t), \quad (33)$$

$$\mu(t-t') \equiv \mu_d(t-t') + \Gamma \mu_{od}(t-t'), \quad (34)$$

where  $\mu_d$ , the diagonal part of function  $\mu$ , and  $\mu_{od}$ , the off-diagonal part, are given by

$$\mu_d(t-t') \equiv \sum_{j=1}^N \frac{g_j^2 m m_r \omega_0^2}{m_j} \cos[\omega_j(t-t')] \Theta(t-t'), \quad (35)$$

$$\mu_{od}(t-t') \equiv \sum_{j=1}^N \frac{g_j^2 m m_r \omega_j \omega_c}{m_j B} \sin[\omega_j(t-t')] \Theta(t-t'). \quad (36)$$

This completes step 1 of the program.

To implement step 2, we now assume that at distant past,  $t = -\infty$ , there was no magnetic field, the charged particle

was held fixed at  $\mathbf{r}(0)$ , while the heat-bath oscillators were kept in weak contact with another heat bath at temperature  $T$  so that it comes to thermal equilibrium. Therefore, at time  $t = 0$ , the heat-bath oscillators are in canonical equilibrium at temperature  $T$  with respect to the free oscillator Hamiltonian

$$H_B = \sum_{j=1}^N \left[ \frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 \mathbf{q}_j^2 \right]. \quad (37)$$

Subsequently, at a time  $t \gtrsim 0$ , the particle is released and the magnetic field is turned on, so that further evolution of the system is governed by Hamiltonian (1). Note that this physical picture is consistent with choosing the retarded solution (28). The state of the system at  $t = 0$ , corresponding to a correlation-free preparation, is given by the total density matrix operator

$$\rho_T(0) = \rho_P(0) \otimes \rho_B, \quad (38)$$

where the initial density matrix operator  $\rho_P(0)$  of the charged particle is given by

$$\rho_P(0) = \delta(\mathbf{r} - \mathbf{r}(0))\delta(\mathbf{p}), \quad (39)$$

while that of the heat bath, which is in canonical equilibrium, is given by

$$\rho_B = \frac{e^{-H_B/k_B T}}{Z_B}; \quad Z_B = \text{Tr}_B(e^{-H_B/k_B T}). \quad (40)$$

Here,  $k_B$  is the Boltzmann constant. The normalization factor is denoted by  $Z_B$ , while  $\text{Tr}_B$  represents partial trace operation with respect to the bath variables.

The statistical average of a heat-bath operator  $O$  with respect to the initial state (38) is given by

$$\langle O \rangle \equiv \text{Tr}_B(O e^{-H_B/k_B T}) / \text{Tr}_B(e^{-H_B/k_B T}). \quad (41)$$

Using known properties of quantum harmonic oscillators, it is straightforward to show that

$$\begin{aligned} \langle q_{j\alpha}(0) \rangle &= 0, \\ \langle p_{j\alpha}(0) \rangle &= 0, \\ \langle q_{j\alpha}(0) q_{k\beta}(0) \rangle &= \frac{\hbar}{2m_j \omega_j} \coth\left(\frac{\hbar \omega_j}{2k_B T}\right) \delta_{jk} \delta_{\alpha\beta}, \\ \langle p_{j\alpha}(0) p_{k\beta}(0) \rangle &= \frac{\hbar m_j \omega_j}{2} \coth\left(\frac{\hbar \omega_j}{2k_B T}\right) \delta_{jk} \delta_{\alpha\beta}, \\ \langle q_{j\alpha}(0) p_{k\beta}(0) \rangle &= -\langle p_{j\alpha}(0) q_{k\beta}(0) \rangle = \frac{1}{2} i \hbar \delta_{jk} \delta_{\alpha\beta}. \end{aligned} \quad (42)$$

In addition, we have the Gaussian property: the statistical average of an odd number of factors of  $q_{j\alpha}(0)$  and  $p_{j\alpha}(0)$  is zero, while that of an even number of factors is equal to the sum of products of pair averages with the order of the factors preserved.

Using the results in Eq. (42), one finds that the force operator  $\mathbf{F}(t)$ , Eq. (33), has zero mean,

$$\langle \mathbf{F}(t) \rangle = 0, \quad (43)$$

and a symmetric correlation given by

$$\begin{aligned} &\frac{1}{2} \langle F_\alpha(t) F_\beta(t') + F_\beta(t') F_\alpha(t) \rangle \\ &= \frac{\hbar \delta_{\alpha,\beta}}{2} \sum_{j=1}^N \frac{g_j^2 m_r^2 \omega_j^3}{m_j} \coth\left(\frac{\hbar \omega_j}{2k_B T}\right) \cos[\omega_j(t - t')]. \end{aligned} \quad (44)$$

In addition,  $\mathbf{F}(t)$  has the Gaussian property, which follows from the same property of the  $\mathbf{q}_j(0)$  and  $\mathbf{p}_j(0)$ .

Thus the initial distribution of the heat bath oscillators turns the force operator  $\mathbf{F}(t)$  into an operator-valued random force. On using the canonical commutation rules (3), we find that  $\mathbf{F}(t)$  has the unequal time commutator given by

$$[F_\alpha(t), F_\beta(t')] = -i \hbar \delta_{\alpha,\beta} \sum_{j=1}^N \frac{g_j^2 m_r^2 \omega_j^3}{m_j} \sin[\omega_j(t - t')]. \quad (45)$$

We are now in a position to achieve step 3 and interpret Eq. (32) with  $t \gtrsim 0$  as a QLE for the particle coordinate operator  $\mathbf{r}(t)$ , which we write again:

$$\begin{aligned} m_r \ddot{\mathbf{r}} + \int_0^t dt' \dot{\mathbf{r}}(t') \mu(t - t') + m_r \omega_0^2 \mathbf{r} + \mu_d(t) \mathbf{r}(0) \\ - \frac{e}{c} (\mathbf{v} \times \mathbf{B}) = \mathbf{F}(t), \end{aligned} \quad (46)$$

where  $\mathbf{F}(t)$  represents a random force with correlation and unequal time commutator given by Eqs. (44) and (45), respectively. The renormalized mass  $m_r$  is given by Eq. (11). The second term on the left represents a mean force characterized by the friction kernel or the memory function  $\mu(t)$ . Note the appearance of the initial value term that depends explicitly on the initial coordinate of the particle and the diagonal part of the memory function. One can absorb this term into the definition of the random force by defining  $\mathbf{G}(t) = \mathbf{F}(t) - \mu_d(t) \mathbf{r}(0)$ , and then considering the initial state (38), with particle density operator (39) and bath density operator  $\rho_B = \frac{e^{-H_B^{\text{Shifted}}/k_B T}}{Z_B}$ , where the ‘‘shifted’’ bath Hamiltonian is  $H_B^{\text{Shifted}} = \sum_{j=1}^N \left[ \frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 \left[ \mathbf{q}_j - \frac{g_j m \omega_0^2}{m_j \omega_j^2} \mathbf{r}(0) \right]^2 \right]$  [12]. This procedure guarantees that the redefined random force  $\mathbf{G}(t)$  has the same statistical properties as  $\mathbf{F}(t)$ .

We now point out some interesting features of the QLE (46), which are not present in the QLE for the case of coordinate-coordinate coupling [13]. These are (i) The coupling renormalizes the inertial mass, (ii) the harmonic potential term is also renormalized, (iii) the friction kernel has an off-diagonal part arising from the magnetic field and a diagonal part due to the harmonic potential. Similar to the coordinate-coordinate coupling, the magnetic field appears in the QLE as a quantum-generalized classical Lorentz force term, and the random force in the QLE does not depend on the magnetic field. This latter force, nevertheless, has a different form so that its symmetric correlation and unequal time commutator are modified from the corresponding expressions in the case of coordinate-coordinate coupling.

It is interesting to see that the correlation and commutator of the random force  $\mathbf{F}(t)$  may be related to the friction kernel  $\mu(t-t')$ . The Laplace transform of its diagonal part  $\mu_d(t)$ , Eq. (35), is given by

$$\begin{aligned}\tilde{\mu}_d(\omega) &\equiv \int_0^\infty dt \mu(t) e^{i\omega t}; \quad \text{Im}(\omega) > 0 \\ &= \sum_{j=1}^N \frac{g_j^2 m m_r \omega_0^2}{m_j} \int_0^\infty dt \cos(\omega_j t) e^{i\omega t} \\ &= \frac{i}{2} \sum_{j=1}^N \frac{g_j^2 m m_r \omega_0^2}{m_j} \left( \frac{1}{\omega - \omega_j} + \frac{1}{\omega + \omega_j} \right).\end{aligned}\quad (47)$$

Using the well-known result that  $1/(x+i0^+) = P(1/x) - i\pi\delta(x)$ , we have

$$\begin{aligned}\text{Re}[\tilde{\mu}_d(\omega + i0^+)] &= \frac{\pi}{2} \sum_{j=1}^N \frac{g_j^2 m m_r \omega_0^2}{m_j} \\ &\quad \times (\delta(\omega - \omega_j) + \delta(\omega + \omega_j)),\end{aligned}\quad (48)$$

so that Eq. (44) may be rewritten as

$$\begin{aligned}&\frac{1}{2} \langle F_\alpha(t) F_\beta(t') + F_\beta(t') F_\alpha(t) \rangle \\ &= \frac{\hbar \delta_{\alpha,\beta}}{\pi} \int_0^\infty d\omega \text{Re}[\tilde{\mu}_d(\omega + i0^+)] \frac{\omega^3 m_r}{\omega_0^2 m} \\ &\quad \times \coth\left(\frac{\hbar\omega}{2k_B T}\right) \cos[\omega(t-t')],\end{aligned}\quad (49)$$

and similarly, Eq. (45) as

$$\begin{aligned}[F_\alpha(t), F_\beta(t')] &= \frac{2\hbar \delta_{\alpha,\beta}}{i\pi} \int_0^\infty d\omega \text{Re}[\tilde{\mu}_d(\omega + i0^+)] \frac{\omega^3 m_r}{\omega_0^2 m} \\ &\quad \times \sin[\omega(t-t')].\end{aligned}\quad (50)$$

## V. CONCLUSIONS

In this work, we derived a quantum Langevin equation (QLE) for a charged quantum particle moving in a harmonic potential in the presence of a uniform external magnetic field and coupled linearly through the momentum variables to a collection of independent quantum harmonic oscillators constituting a heat bath. In this QLE, the magnetic field appears through a quantum-generalized classical Lorentz force term. The QLE involves a random force which does not depend on the magnetic field. These aspects are also present in the QLE for the case of particle-bath coordinate-coordinate coupling [13]. However, significant differences are also observed: (i) The random force has a modified form with symmetric correlation and unequal time commutator different from those in the case of coordinate-coordinate coupling, (ii) the inertial term and the harmonic potential term in the QLE get renormalized, and (iii) the memory function characterizing the mean force in the QLE has a field-independent diagonal part, but also an explicit field-dependent off-diagonal part.

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