Nearest neighbors, phase tubes, and generalized synchronization

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In this paper we report on the necessity of the refinement of the concept of generalized chaotic synchronization. We show that the state vectors of the interacting chaotic systems being in the generalized synchronization regime are related to each other by the functional, but not the functional relation as it was assumed until now. We propose the phase tube approach explaining the essence of generalized synchronization and allowing the detection and the study of this regime in many relevant physical circumstances. The finding discussed in this Brief Report provides great potential for different applications.

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Chaotic synchronization is a fundamental phenomenon that has both theoretical and applied significance [1], which has been widely studied recently. One of the interesting and intricate types of synchronous behavior of unidirectionally coupled chaotic oscillators is generalized synchronization (GS) [2,3]. This kind of synchronous behavior is said to mean the presence of a functional relation between the drive and response oscillator states [4,5] and has been observed in many systems both numerically [6–8] and experimentally [9–11], with many interesting features [7,12] and possible applications [13,14] of this regime being revealed.

The definition of the GS regime generally accepted hitherto is the presence of a functional relation

$$\mathbf{y}(t) = \mathbf{F}[\mathbf{x}(t)] \tag{1}$$

between the drive $\mathbf{x}(t)$ and response $\mathbf{y}(t)$ oscillator states [4,5]. Based on this definition are the different techniques that had been proposed for detecting the presence of GS between chaotic oscillators, such as the nearest-neighbor method [4,15], the auxiliary system approach [2], or the conditional Lyapunov exponent calculation [5] (with the auxiliary system approach generally being the most easy, clear, and powerful tool to study the GS regime in the model systems); for the analysis of the observed experimental time series, however, the nearest-neighbor method, as a rule, is more applicable [11].

In this Brief Report we report on the necessity of reconsidering and refining the existing concept of generalized chaotic synchronization. The main reason for this refinement is the following. Let $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{y}(t_0) = \mathbf{y}_0$ be the reference points belonging to the chaotic attractors of the drive and response oscillators being in the GS regime, respectively. For the neighbor point $\mathbf{x}(t_i) = \mathbf{x}_i$ of the drive oscillator such that $||\mathbf{x}_i - \mathbf{x}_0|| < \varepsilon$, its image $\mathbf{y}(t_i) = \mathbf{y}_i$ in the response system is also close to the reference point $\mathbf{y}_0$ (see Ref. [4] for details), i.e., $||\mathbf{y}_i - \mathbf{y}_0|| < \delta(\varepsilon)$. Having linearized Eq. (1), one obtains

$$\mathbf{y}_i - \mathbf{y}_0 = J\mathbf{F}[\mathbf{x}_0](\mathbf{x}_i - \mathbf{x}_0), \tag{2}$$

where J is the Jacobian operator. Since the form of the functional relation $\mathbf{F}[\cdot]$ cannot be found explicitly in most cases, Eq. (2) may be rewritten in the form

$$\delta\mathbf{y}_i = \mathbf{A}\delta\mathbf{x}_i, \tag{3}$$

where $\mathbf{A} = J\mathbf{F}[\mathbf{x}_0]$ is the unknown matrix and $\delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_0$ and $\delta\mathbf{y}_i = \mathbf{y}_i - \mathbf{y}_0$ are the vectors characterizing the deviation of the points under consideration $\mathbf{x}_i$ and $\mathbf{y}_i$ from the reference points $\mathbf{x}_0$ and $\mathbf{y}_0$, respectively. Without loss of generality, we shall suppose below the identical dimension m of the phase space of the drive and response systems.

Although the coefficients of the matrix $\mathbf{A}$ are unknown, the validity of Eq. (3) may be verified if there are $N > m$ nearest neighbors $\mathbf{x}_i$ of the reference point $\mathbf{x}_0$ and the corresponding vectors $\mathbf{y}_i$ of the response system. Having tested the presence of the generalized synchronization (e.g., with the help of the auxiliary system approach), we can pick out m nearest neighbors $\mathbf{x}_i$ $(i = 1,\dots,m)$ and the corresponding vectors $\mathbf{y}_i$ to determine the coefficients a_{ij} of the matrix $\mathbf{A}$ with the help of Eq. (3). To reduce the influence of the inaccuracy we select such vectors $\mathbf{x}_i$ [and $\delta\mathbf{x}_i = (\delta x_{i1},\dots,\delta x_{im})^T$, respectively] from the whole set of N vectors for which

$$|\det(\mathbf{X})| = \max, \tag{4}$$

where

$$\mathbf{X} = \begin{pmatrix} \delta x_{11} & \delta x_{12} & \cdots & \delta x_{1m} \\ \delta x_{21} & \delta x_{22} & \cdots & \delta x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \delta x_{m1} & \delta x_{m2} & \cdots & \delta x_{mm} \end{pmatrix}. \tag{5}$$

Having determined the matrix $\mathbf{A}$, we can now find the vectors $\delta\mathbf{z}_i$ $(i = m+1,\dots,N)$,

$$\delta\mathbf{z}_i = \mathbf{A}\delta\mathbf{x}_i, \tag{6}$$

and compare them with the vectors $\delta\mathbf{y}_i$ of the response system (or compare vectors $\mathbf{z}_i = \mathbf{y}_0 + \delta\mathbf{z}_i$ with $\mathbf{y}_i$) to validate the correctness of Eq. (3).

Although, at first sight, it seems that there are no fundamental causes due to which Eq. (3) may fail, in reality Eq. (3) is not correct. To illustrate this we have study numerically the synchronous behavior of two coupled chaotic Rössler oscillators

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, \quad \dot{x}_r = -\omega_r y_r - z_r + \varepsilon(x_d - x_r), \\ \dot{y}_d &= \omega_d x_d + a y_d, \quad \dot{y}_r = \omega_r x_r + a y_r, \\ \dot{z}_d &= p + z_d(x_d - c), \quad \dot{z}_r = p + z_r(x_r - c), \end{aligned} \tag{7}$$

where $\mathbf{x} = (x_d, y_d, z_d)^T$ $[\mathbf{y} = (x_r, y_r, z_r)^T]$ are the Cartesian coordinates of the drive [response] oscillator, the overdots stand for temporal derivatives, and ε is a parameter ruling the coupling strength. The other control parameters of Eq. (7) have been set to $a = 0.15$, $p = 0.2$, and $c = 10.0$, in analogy with our previous studies [3,16]. The parameter ω_r (representing the natural frequency of the response system) has been selected to be $\omega_r = 0.95$; the analogous parameter for the drive system has been fixed to $\omega_d = 0.99$. For such a choice of parameter values the boundary of the generalized synchronization regime found with the help of the auxiliary system approach is $\varepsilon_{\mathrm{GS}} \approx 0.11$.

Having chosen the reference point $\mathbf{x}_0$ of chaotic attractor of the drive oscillator randomly, one can find its nearest neighbors $\mathbf{x}_i$ $(i = 1, \dots, N)$ (and the corresponding vectors $\mathbf{y}_i$ of the response system), select [according to Eqs. (4) and (5)] the vector basis $\mathbf{x}_{1-3}$ to determine the matrix $\mathbf{A}$, and check the condition in Eq. (3) with the help of Eq. (6) and the rest of the vectors $\mathbf{x}_i$ and $\mathbf{y}_i$ $(i = 4, \dots, N)$.

In Fig. 1 the vectors $\mathbf{z}_i$ $(i = 4, \dots, 10)$ obtained with the help of Eq. (6), as well as the vectors $\mathbf{y}_i$ of the response system, are shown for the coupling strength $\varepsilon = 0.3$. The value of the coupling strength greatly exceeds the threshold $\varepsilon_{\mathrm{GS}}$ of the generalized synchronization. The GS regime demonstrates great stability; as a consequence, Eq. (3) is expected to be correct. However, contrary to expectations, the vectors $\mathbf{z}_i$ and $\mathbf{y}_i$ differ from each other sufficiently, testifying that Eq. (3) fails. As a matter of fact, the failure of Eq. (3) is also observed for other reference points of the drive Rössler oscillator as well as for other chaotic dynamical systems (e.g., Lorenz oscillators). Since Eq. (3) is just the linearization of Eq. (1), the failure of Eq. (3) is evidence of the incorrectness of Eq. (1) being the main definition of the generalized synchronization concept. At the same time, plenty of results obtained hitherto are in very good agreement with the generally accepted concept of GS. This means that the concept proposed by Rulkov *et al.* [4] works in some circumstances, but, in general, must be refined.

The core idea of this correction is the following. The state of the response system $\mathbf{y}(t)$ depends not only on the state of the drive oscillator $\mathbf{x}(t)$ at the moment of time t, but on the history of the evolution of the drive system during the time interval $(t - \tau, t]$ as well. Indeed, according to the concept of GS, synchronization means that the response oscillator $\mathbf{y}(t)$ comes to the state defined uniquely by the drive system, with the convergence time τ being connected with the largest conditional Lyapunov exponent λ_1^r, i.e., $\tau \sim 1/|\lambda_1^r|$. In other words, $\mathbf{F}[\cdot]$ in Eq. (1) must be considered a functional, but

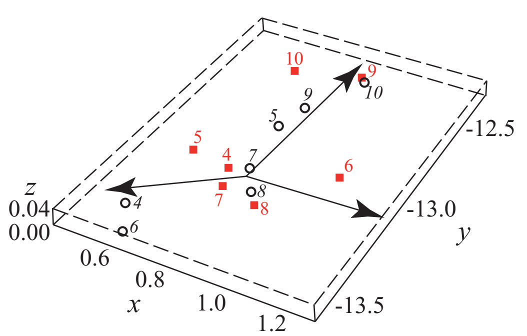

FIG. 1. (Color online) Vectors $\mathbf{y}_i$ (■) and $\mathbf{z}_i$ (○) of the response Rössler system [Eq. (7)] for $\varepsilon = 0.3$. The numbers i of the vectors are shown by the roman and italic fonts, respectively

not a functional relation. Obviously, in this case, Eq. (3) being obtained under the assumption that $\mathbf{F}[\cdot]$ is the functional relation is not satisfied, as shown above (see Fig. 1).

Considering $\mathbf{F}[\cdot]$ as the functional, one has to replace Eq. (2) by

$$\delta\mathbf{y}_i(t) = \int_{t-\tau}^{t} J\mathbf{F}[\mathbf{x}_0(s)]\delta\mathbf{x}_i(s)ds. \tag{8}$$

Having supposed that the deviation $\delta\mathbf{x}_i(s)$ from the reference trajectory $\mathbf{x}_0(s)$ $(t - \tau < s \leqslant t)$ is small, in view of the linearity, one can write

$$\delta\mathbf{x}_i(s) = \mathbf{B}(s)\delta\mathbf{x}_i(t), \quad t - \tau < s < t \tag{9}$$

[where $\mathbf{B}(s)$ is a matrix with time-dependent coefficients], which results in

$$\delta\mathbf{y}_i(t) = \int_{t-\tau}^{t} J\mathbf{F}[\mathbf{x}_0(s)]\mathbf{B}(s)\delta\mathbf{x}_i(t)ds \tag{10}$$

and, as a consequence, in

$$\delta\mathbf{y}_i(t) = \mathbf{C}(t)\delta\mathbf{x}_i(t), \tag{11}$$

where $\mathbf{C}(t)$ is the square $m \times m$ matrix defined as

$$\mathbf{C}(t) = \int_{t-\tau}^{t} J\mathbf{F}[\mathbf{x}_i(s)]\mathbf{B}(s)ds. \tag{12}$$

Thus Eq. (11) coincides formally with Eq. (3) and therefore may be validated also by the calculations of the vectors $\mathbf{z}_i$ in the same way it was done for Eq. (3). At the same time, Eq. (3) was obtained under the assumption that the vectors $\mathbf{x}_0(t)$ and $\mathbf{x}_i(t)$ are close to each other, whereas Eq. (11) was obtained under the constraint requiring the nearness of the trajectories $\mathbf{x}_0(s)$ and $\mathbf{x}_i(s)$ during the time interval $t - \tau < s \leqslant t$. Since for the chaotic systems the phase trajectories can converge in one direction of the phase space and diverge in another, the neighbor vectors $\mathbf{x}_0(t)$ and $\mathbf{x}_i(t)$ may be characterized by the very distinct phase trajectories $\mathbf{x}_0(s)$ and $\mathbf{x}_i(s)$ for $t - \tau < s \leqslant t$. The schematic representation of such a situation is given in Fig. 2. Although the vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are close to the reference point $\mathbf{x}_0(t)$, only the vector $\mathbf{x}_2(t)$ obeys Eq. (11) due to the nearness of the phase trajectories $\mathbf{x}_0(s)$ and $\mathbf{x}_2(s)$; for the vector $\mathbf{x}_1(t)$ Eq. (11) fails since the phase trajectory $\mathbf{x}_1(s)$ is not close to the reference one $\mathbf{x}_0(s)$ during the whole time interval $t - \tau < s \leqslant t$. Therefore, to verify Eq. (11) we have to consider not all vectors $\mathbf{x}_i(t)$ being nearest the reference point $\mathbf{x}_0(t)$, but only vectors that are characterized by the phase trajectories $\mathbf{x}_i(s)$ being close to the reference one $\mathbf{x}_0(s)$. Having as a basis the idea of phase-space strands [17,18] to eliminate the ineligible vectors [such as $\mathbf{x}_1(t)$

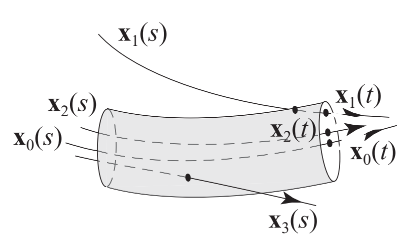

FIG. 2. Schematic representation of the nearest vectors $\mathbf{x}_i(t)$, the phase trajectories $\mathbf{x}_i(s)$, and the phase tube $\mathbb{T}_\tau(t)$.

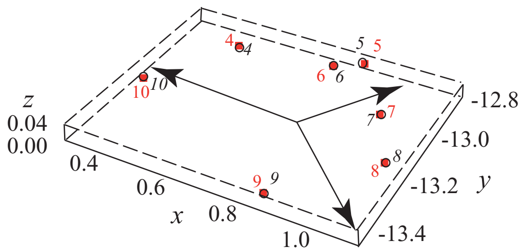

FIG. 3. (Color online) Vectors $\mathbf{y}_i$ (■) and $\mathbf{z}_i$ (○) of the response Rössler system [Eq. (7)] for $\varepsilon = 0.3$. The length of the phase tube is $\tau = 100$. The numbers i of the vectors are shown by the roman and italic fonts, respectively.

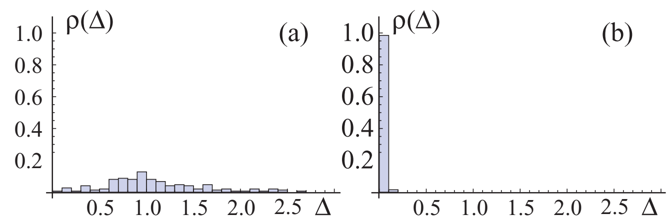

FIG. 4. Histograms of the normalized difference $\Delta = ||\delta\mathbf{y}_i(t) - \delta\mathbf{z}_i(t)||/||\delta\mathbf{y}_i(t)||$ for (a) the asynchronous dynamics ($\varepsilon = 0.06$) and (b) the generalized synchronization regime ($\varepsilon = 0.3$). The histograms have been obtained for the response Rössler system [Eq. (7)]. The length of the phase tube is $\tau = 100$.

in Fig. 2], we introduce into consideration the phase tube

$$\mathbb{T}_\tau(t) = \{\mathbf{x} : |x_{0j}(s) - x_j|\langle d_j|_{j=1}^m, \ s \in [t-\tau; t]\} \qquad (13)$$

and take into account only vectors whose phase trajectories pass through this phase tube [such as $\mathbf{x}_2(t)$ in Fig. 2].

The result of this examination for Rössler systems [Eq. (7)] with the same set of control parameter values and coupling strength as before is given in Fig. 3. The length of the phase tube is $\tau = 100$. One can see that the calculated vectors $\mathbf{z}_i(t)$ are in excellent agreement with the vectors $\mathbf{y}_i(t)$ of the response Rössler system, which confirms both the correctness of Eq. (11) and, as a consequence, the statement that $\mathbf{F}[\cdot]$ is the functional, but not the functional relation.

With an increase of the coupling strength between chaotic oscillators, the absolute value of the largest conditional Lyapunov exponent λ_1^r increases and the time interval τ [the length of the phase tube $\mathbb{T}_\tau(t)$] decreases. Finally, in the lag synchronization (LS) and complete synchronization (CS) regimes the value of τ tends to be zero. Therefore, in the LS and CS regimes Eq. (3) is satisfied for all neighbor vectors $\mathbf{x}_i(t)$ without any additional requirements concerning the phase trajectory nearness. In other words, the state vectors of any chaotic systems in the GS regime (but not in the LS or CS regime) are connected with each other by the functional, whereas in the LS and CS regimes (which are strong forms of GS) they are related to each other by the functional relation.

Though the phase tube approach has been here applied to the model systems, we expect that it can be used in many other relevant circumstances. Since the statistics for the difference between $\delta\mathbf{y}_i(t)$ and $\delta\mathbf{z}_i(t)$ vectors are radically different for synchronous and asynchronous motion (see Fig. 4), the important feature of this approach is the possibility to consider the relation between vectors [Eq. (11)] for the analysis of the registered experimental data (vector or scalar, using the Takens approach [19]) when other classical methods of GS detection are inaccurate or unapplicable. Moreover, the proposed approach may be used as a method to detect the GS regime, including the case when the chaotic oscillators are mutually coupled, since all arguments given above are also applicable for the case of bidirectional coupling.

To prove the generality of our findings we have also studied numerically two mutually coupled generators with tunnel diodes.[1] In dimensionless form the dynamics of such generators are described by the equations [20,21]

$$\begin{aligned}\dot{x}_{1,2} &= \omega_{1,2}^2[h(x_{1,2} - \varepsilon(y_{2,1} - y_{1,2})) + y_{1,2} - z_{1,2}],\\ \dot{y}_{1,2} &= -x_{1,2} + \varepsilon(y_{2,1} - y_{1,2}),\\ \mu\dot{z}_{1,2} &= x_{1,2} - f(z_{1,2}),\end{aligned} \qquad (14)$$

where $f(\xi) = -\xi + 0.002\sinh(5\xi - 7.5) + 2.9$ is the dimensionless characteristic of a nonlinear converter, $h = 0.2$, $\mu = 0.1$, $\omega_1 = 1.09$, and $\omega_2 = 1.02$ are control parameter values, and ε is the coupling parameter strength. The indices 1 and 2 correspond to the first and second coupled systems, respectively. For such values of the control parameters the threshold of the generalized synchronization regime determined by the moment of the transition of the second positive Lyapunov exponent in the field of the negative values [22,23] is $\varepsilon_{GS} \approx 0.08$.

As in the case of Rössler systems considered above, we have chosen the reference point $\mathbf{x}_0$ of the chaotic attractor of the first oscillator randomly and analyze the behavior of its nearest neighbors $\mathbf{x}_i$ ($i = 1, \ldots, N$) and the corresponding vectors $\mathbf{y}_i$ and $\mathbf{z}_i$. The vector basis $\mathbf{x}_{1-3}$ has been chosen in the same way as in the case considered above.

Figure 5 shows the vectors $\mathbf{y}_i$ and $\mathbf{z}_i$ of the second generator with a tunnel diode [Eq. (14)] for the coupling parameter strength $\varepsilon = 0.15$ greatly exceeding the threshold value of the generalized synchronization regime onset ε_{GS}. Figure 5(a) corresponds to the case in which all neighbor vectors are used, whereas in Fig. 5(b) only vectors whose phase trajectories pass through the phase tube with length $\tau = 110$ are used. It is clearly shown that in the first case the vectors $\mathbf{z}_i$ and $\mathbf{y}_i$ differ from each other sufficiently, testifying to the absence of the functional relation between the interacting system states. Conversely, for the phase tube with length $\tau = 110$ [Fig. 5(b)], the calculated vectors $\mathbf{z}_i(t)$ are in excellent agreement with the vectors $\mathbf{y}_i(t)$ of the second generator, which confirms the results obtained above for unidirectionally coupled Rössler systems. Thus, in systems with a mutual type of coupling the vector states of the interacting systems are related to each other by the functional.

In conclusion, we have reported that the concept of generalized synchronization (except for the LS and CS regimes) needs refining since the state vectors of the interacting chaotic

[1] In this case Eq. (1) should be written as $\mathbf{F}[\mathbf{x}(t),\mathbf{y}(t)] = 0$.

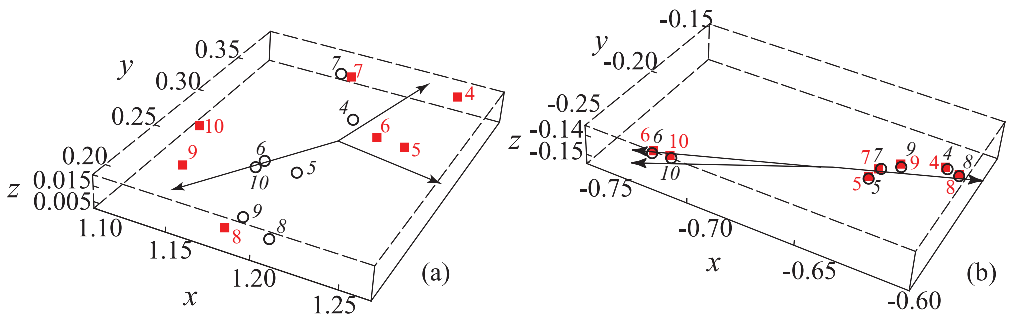

FIG. 5. (Color online) Vectors $\mathbf{y}_i$ (■) and $\mathbf{z}_i$ (○) of the second generator with a tunnel diode [Eq. (14)] for $\varepsilon = 0.15$. The numbers i of the vectors are shown by the roman and italic fonts, respectively. (a) All neighbor vectors are used. (b) Only vectors whose phase trajectories pass through the phase tube with length $\tau = 110$ are used.

systems are related to each other by the functional, but not the functional relation as it was assumed until now. Even though systems with a small number of degrees of freedom have been considered in this Brief Report, the formalism developed herein can be extended also to systems with infinite-dimensional phase space.[2] Fortunately, this modification of the generalized synchronization concept does not discard the majority of results concerning GS obtained hitherto. At the same time, this refinement has a fundamental significance in terms of understanding the core mechanisms of the phenomena considered and should offer great potential for different approaches and applications dealing with nonlinear systems.

[2]In this case the system state is defined uniquely by the function (or vector function), but not by the finite-dimensional vector as in the case of the system with a small number of degrees of freedom.

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