# Preventing the recurrence effect in the Vlasov simulation by randomizing phase-point velocities in phase space

H. Abbasi,[1,*] M. H. Jenab,[1] and H. Hakimi Pajouh[2]
[1]*Faculty of Physics, Amirkabir University of Technology, P. O. Box 15875-4413, Tehran, Iran*
[2]*Department of Physics, Alzahra University, Tehran 19834, Iran*
(Received 28 January 2011; revised manuscript received 5 June 2011; published 6 September 2011)

The Vlasov equation is simulated by following the characteristics of phase points in phase space. It is shown that by increasing the number of phase points, without enhancing the resolution of phase-space grid, the accuracy of the simulation will be improved. In addition, the phase-point spacing introduces a smaller scale than grid spacing on which fine structures might be more conveniently handled. In order to perform simulation with a large population of phase points, an alternative to the bilinear interpolation scheme is introduced that reduces the number of operations. It is shown that by randomizing initial phase-point velocities, the recurrence effect does not happen. Finally, the standard problem of linear and nonlinear Landau damping will be examined.

DOI: 10.1103/PhysRevE.84.036702 PACS number(s): 05.10.−a, 52.65.Ff, 82.20.Fd, 52.35.−g

## I. INTRODUCTION

There are two equally important numerical approaches to the kinetic problem. The first approach, particle-in-cell (PIC) methods, self-consistently models the plasma by a finite number of macroparticles on a fixed grid. Its advantage is replacing the Vlasov equation by the ordinary differential equations of motions of macroparticles, which makes PIC codes easily extendable to multidimensional applications. However, it is well known that the numerical noise (proportional to $1/\sqrt{N}$ where $N$ is the number of macroparticles) inherent to PIC simulation becomes, in some cases, too large to allow a precise description of the distribution function (DF). Nevertheless, the PIC approach can produce accurate results when a sufficiently large number of macroparticles are involved in the simulation [1].

The second approach is direct solution of the Vlasov equation (for a collisionless plasma), that its noise is much less than a PIC simulation. The major problem of the Vlasov simulation has been the development of fine structures (filamentation) in velocity space, i.e., a problem with no seemingly simple cure. Partial treatments such as an increase in velocity resolution have sharply limited the ability to extend the above work to higher dimensions [2–4] and thus treat realistic problems. A large class of the Vlasov simulation models is based on discretizing the Vlasov equation (mainly by a splitting scheme) on a phase-space fixed grid. In the splitting method, the new $f$ was obtained as an algebraic expression in terms of the old $f$ by a suitable interpolation method [5]. There are a couple of problems with the splitting scheme: (1) It was not rigorously shown under which circumstances the coupled equations have solutions approximately consistent with the Vlasov equation; and (2) the following characteristics along the phase-space coordinates departs from the characteristics on which $f$ truly remains invariant [3]. In addition, for them the partial treatment of the filamentation, i.e., increasing the velocity resolution, is very expensive because all of the simulation operations is performed on the grid. Moreover, they suffer from the recurrence effect [6]. The recurrence effect is a kind of periodic behavior associated with the advection part of the Vlasov equation. [Please see the description after Eq. (2).]

Integration of the Vlasov equation along the collisionless phase-point trajectories has been the most promising of these methods [3]. This method is based on following the characteristics along which $f$ is constant in the collisionless case. Therefore, characteristic equations of the Vlasov equation are solved. That is one of the advantages of the scheme and makes it possible to use all the PIC simulation experiences. A complete description of the method of characteristics was presented in Ref. [3]. There, besides the grid points, are an equal number of phase points over which the DF $f_p$ is initially defined. Interpolation is performed between the phase points and the fixed background grid to obtain the DF $f_g$ on the grid. The main advantage of introducing $f_p$ is that it remains unchanged when the phase points follow their characteristics (contrary to the semi-Lagrangian method in which the DF is altered by the interpolation).

In present paper, we first improve the accuracy of the simulation by increasing the number of phase points in each grid cell, without enhancing the resolution of the phase-space grid. Better accuracy is the result of sampling DF with higher resolution. A larger population of phase points, in comparison to grid points, introduces a smaller scale than grid spacing on which fine structures might be more conveniently handled. However, increasing the number of phase points necessitates an effective interpolation scheme (IS) that reduces the number of operations while keeping the accuracy [e.g., in comparison to the bilinear interpolation scheme (BIS)]. Accordingly, a new IS is introduced. In such a way that the DF of each grid point is obtained by averaging the DF of phase points located in four cells around a grid point. Moreover, it is shown that by randomizing the initial position of the phase points along the velocity axis, the recurrence effect does not happen, and the reason is given in detail.

## II. THE NUMERICAL SCHEME AND ITS FEATURES

Our mathematical model is the one-dimensional Vlasov-Poisson system,

*abbasi@aut.ac.ir

 

$$\partial_t f + v\partial_x f - E(x,t)\partial_v f = 0,$$
$$\partial_x E = 1 - \int_{-\infty}^{+\infty} f dv, \quad (1)$$

where $f$ is the electron DF and $E$ is the electric field. In Eq. (1), and in the rest of the article, time is normalized to the inverse electron plasma frequency $\omega_{pe}^{-1}$, space is normalized to the Debye length $\lambda_D$, and velocity is normalized to the electron thermal speed $v_{Te} = \lambda_D \omega_{pe}$. Ions are taken to be motionless, and their only role is to provide a uniform, neutralizing background. Furthermore, periodic boundary conditions are assumed in $x$.

As was mentioned, the present solution of the Vlasov equation is based on following the phase-point trajectories along which phase-space DF is constant. In order to obtain the phase-point trajectory one has to solve the characteristics of the Vlasov equation,

$$dx_p/dt = v_p,$$
$$dv_p/dt = -E_p, \quad (2)$$

where subscript $p$ stands for "phase point."

Let us first begin with the free-streaming part (the advection term) of the Vlasov equation, $\partial_t f + v\partial_x f = 0$, through the following example. The solution of the advection part at a time $t$ is given as a function of the initial condition by the relation $f(x,v,t) = f(x - vt, v, 0)$. If we consider an initial Maxwellian distribution perturbed by a small perturbation, $f(x,v,0) = 1/\sqrt{2\pi}\exp(-v^2/2)[1 + \epsilon\cos(kx)]$, then the charge density $(\int_{-\infty}^{+\infty} f\,dv - 1)$ will be given by $\rho(x,t) = \exp(-k^2t^2/2)\epsilon\cos(kx)$ [6]. The analytical solution is decaying exponentially in time. For the numerical solution, the charge density is calculated at every spatial grid point by summation over all grid points in velocity space. On the Eulerian grid due to equal spacing along the velocity axis $\Delta v$, the initial condition can be reconstructed at recurrence time, $T_R = 2\pi/(k\Delta v)$ [6].

Now, let us examine the above analytical solution by the method of characteristics. For the free streaming, we just need to solve $dx/dt = v$. Since in free streaming the velocity of the phase points is constant, the characteristic equation can be exactly solved, that is, $x_p^{n+1} = x_p^n + v_p t^n$, where the superscript denotes $t = n\Delta t$. We first consider a fixed grid with regular phase-point arrangement [Fig. 1(a)]. Second, according to each $x_p$ and $v_p$, its $f_p$ is allocated. Next, $x_p$ is advanced one time step while $v_p$ remains constant. Then

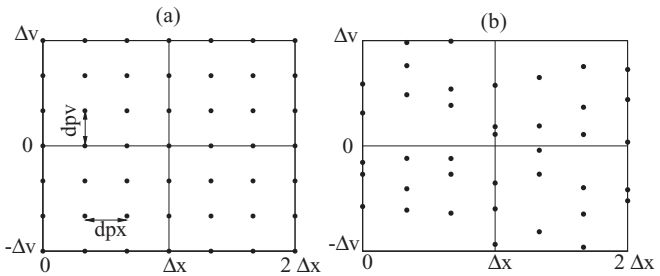


FIG. 1. A typical part of the phase-space grid. (a) Regular arrangement. (b) Random arrangement along the velocity direction.

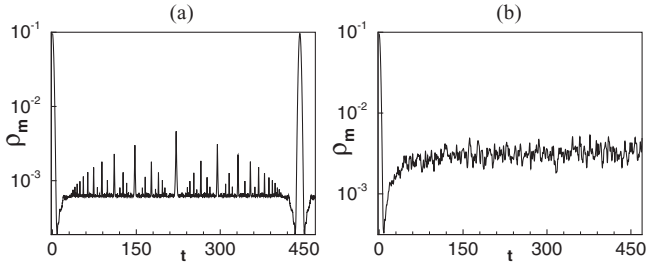


FIG. 2. The simulation result of the free-streaming part of the Vlasov equation with BIS for $\epsilon = 0.1$, $N_x \times N_v = 16 \times 32$, $k = 0.5$, and 100 phase points in each grid cell. (a) Regular arrangement. (b) Random arrangement along the velocity direction.

interpolation is performed between the phase points and the fixed grid in phase space by BIS [3,7] to obtain $f_g$. Finally, the charge density is obtained by summation (here, the Trapezoidal rule) over all grid points in velocity space.

To compare the simulation result, we chose parameters close to Ref. [6] to better benchmark the results and assess their performance, that is, $\epsilon = 0.1$, with grid points $N_x \times N_v = 16 \times 32$. The length in space is $L = 4\pi$, $k = 0.5$, and in velocity space we use $-5 \leqslant v \leqslant 5$. The recurrence time of the Eulerian codes, for this case, is $T_R = 38.95$. We put 100 phase points ($10 \times 10$) in each grid cell. The first result of the model was surprising. The recurrence took place at 442.3362 instead of 38.95 [Fig. 2(a)]. We realized that contrary to the Eulerian codes, it is not the velocity grid that specifies the recurrence time.

Thus, the main question is "How should the recurrence time be calculated?" To answer this question we have to note that it is the evolution of the phase-point arrangement that changes the interpolation weighting and therefore $f_g$ (also $\rho$). Thus, as time goes on, the recurrence will take place if there is a possibility to reconstruct the initial phase-point arrangement (arrangement at $t = 0$). According to Fig. 1(a), the velocity spacing in regular arrangement is $dpv$. That means, the smallest velocity for the moving phase point is $dpv$ (in the positive direction), and the other phase-point velocities are integer multiples of $dpv$. Since the boundary condition is periodic, if those phase points with velocity is $dpv$ move a distance $L$ within a time interval $T_R$, the other phase points will move an integer multiple of $L$ within the same time interval. As a result, all the phase points return to their initial positions and the recurrence occurs. Accordingly, in our model the recurrence time is $T_R = L/dpv$. In other words, there is a much smaller scale $dpv$, in comparison to the grid spacing, along the velocity axis that becomes an important factor in the dynamics. The influence of $dpv$ on the recurrence time is evidence regarding our claim that fine structures might be more conveniently handled. For the above example, $dpv = 0.0284$, which leads to correct recurrence time [Fig. 2(a)]. It is obvious from Fig. 2(a) that there are several other recurrences with smaller amplitude. According to our analysis, these smaller amplitude recurrences are due to subarrangements of the phase points; that is, a small group of phase points (not all) has returned to their original arrangement.

The latter analysis of the recurrence time is based on two facts. First, the boundary condition is periodic. Second, the

velocity of each phase point is an integer multiple of the others. Therefore, by randomizing the phase-point velocities [Fig. 1(b)], we can prevent the occurrence of recurrence [Fig. 2(b)]. To do that, we used a random number generator (with an uniformly distributed output) to modify the velocity of each phase point in the range $[-dpv/2, dpv/2]$ [Fig. 1(b)]. Figure 2 depicts the result after randomizing the velocities of the phase points. The recurrence that is supposed to take place at 442.3362 does not happen. That means our suggestion for the recurrence mechanism works.

As was mentioned, increasing the number of phase points necessitates an effective IS that reduces the number of operations while keeping the accuracy (e.g., in comparison to BIS). For this purpose, we develop a new IS that is almost as accurate as BIS with the difference that it does not use the weighting mechanism.

Let us call the new IS as average interpolation scheme (AIS). In AIS like BIS, all the phase points should be swept one by one to find those that are located in four cells around a specific grid point (with the important difference that in AIS, finding the host cell of the phase point is the only step; see below). Now, we denote the distance between the $i$th phase point (in those four cells) and the grid point along the $x$ axis by $\Delta x_i$ and along the $v$ axis by $\Delta v_i$. Then, by using the Taylor expansion of $f_{pi}(x_g + \Delta x_i, v_g + \Delta v_i)$ around $(x_g, v_g)$ and summing over all $J$ phase points, located in four cells, we obtain

$$f_g(x_g,v_g) = \frac{1}{J}\sum_{i=1}^{J} f_{pi}(x_g+\Delta x_i, v_g+\Delta v_i) - (\partial_x f_g)\frac{1}{J}\sum_{i=1}^{J}\Delta x_i - (\partial_v f_g)\frac{1}{J}\sum_{i=1}^{J}\Delta v_i + O(\Delta^2).$$

For an optimum total number of phase points, their initial density in phase space is almost uniform, and therefore, $\sum_{i=1}^{J}\Delta x_i \cong 0$ and $\sum_{i=1}^{J}\Delta v_i \cong 0$. Moreover, the factor of $1/J$ makes the approximation better. According to Liouville's theorem the density of system points in the vicinity of a given system point traveling through phase space is constant with time. Therefore, the uniformity of phase-point density is almost a constant of motion (to the extent of the truncation error). That means

$$f_g(x_g,v_g) = \frac{1}{J}\sum_{i=1}^{J} f_{pi}(x_{pi},v_{pi}) - O(\Delta^2).$$

Note, the phase-point positions do not explicitly interfere in AIS (as though weighting comes to play in BIS). This is an essential feature of AIS that first makes AIS easily extendable to higher dimensions and also reduces the number of operations.

Figure 3 demonstrates the simulation of free streaming that is performed using AIS. Fig. 3(a) shows the result when the initial phase-point arrangement is regular. At is expected a recurrence happens at $t = 442.3362$. However, after randomizing the initial phase-point arrangements, the recurrence does not take place.

Now, we first compare the accuracy of BIS and AIS and then show that the accuracy of the method will improve by increasing the number of phase points. In order to compare BIS and AIS, we perform two simulations with these ISs. Both of simulations are initially fed by a random phase-point arrangement with 2500 phase points in each cell. Figure 4(a) shows the result. Although BIS is slightly more accurate (maximum $1 \times 10^{-4}$), AIS interpolates much faster than BIS. It is clear that after the initial stage the accuracy of two interpolations becomes almost similar. Note that in this letter we use the simplest scheme with the parameters that are not necessarily the most appropriate ones ($\Delta x = 0.83$ and $\Delta v = 0.3125$). Recall that the accuracy of the interpolation

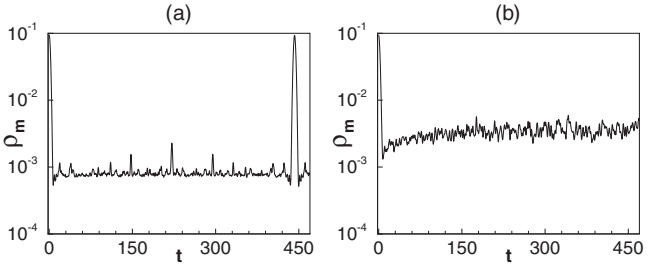


FIG. 3. The simulation result of the free-streaming part of the Vlasov equation with AIS for $\epsilon = 0.1$, $N_x \times N_v = 16 \times 32$, $k = 0.5$, and 100 phase points in each grid cell. (a) Regular arrangement. (b) Random arrangement along the velocity direction.

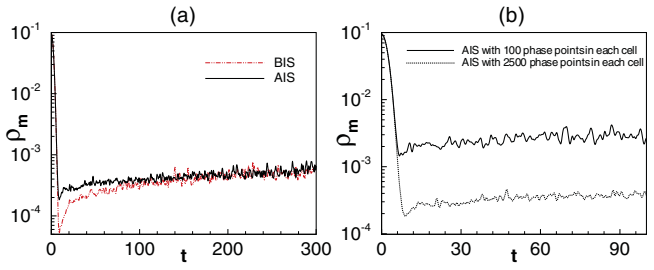


FIG. 4. (Color online) Comparison of BIS and AIS for $\epsilon = 0.1$, $N_x \times N_v = 16 \times 32$, $k = 0.5$. (a) Comparison of BIS and AIS with 2500 phase points in each cell. (b) Improvement of the accuracy of AIS by increasing the number of phase points.

Figure 5 demonstrates the simulation of free streaming that is performed using AIS. Fig. 3(a) shows the result when the initial phase-point arrangement is regular. At is expected a recurrence happens at $t = 442.3362$. However, after randomizing the initial phase-point arrangements, the recurrence does not take place.

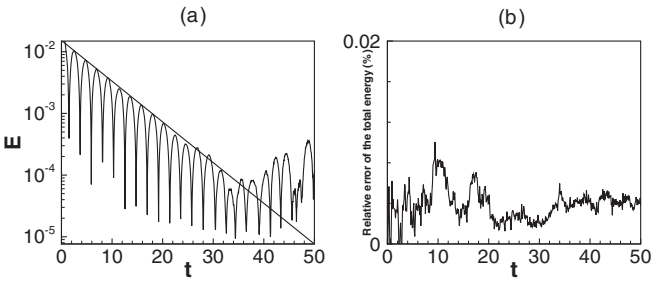


FIG. 5. The linear Landau damping for the parameters $\epsilon = 0.01$, $N_x \times N_v = 256 \times 256$, four phase points in each grid cell, $k = 0.5$, and $dt = 0.1$. (a) The exponential decay of the amplitude of the electric field with the damping rate of $\gamma = 0.153$, which is in good agreement with the result of the linear theory (solid line). (b) The relative error in the total energy in percent.

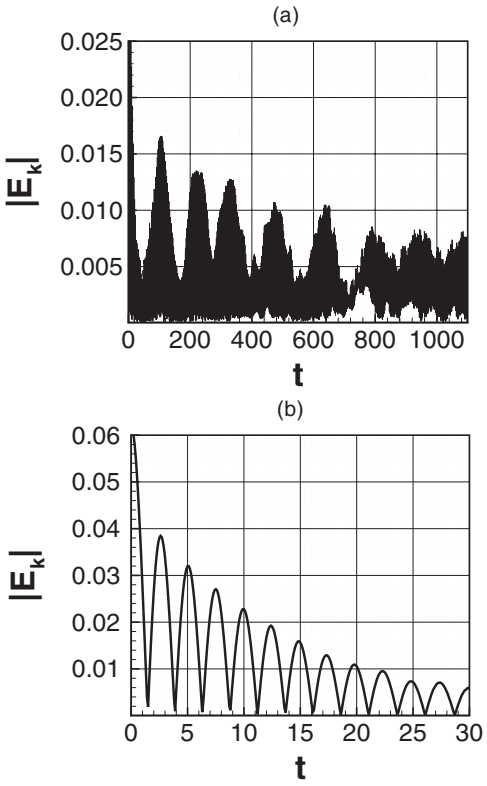


FIG. 6. The nonlinear Landau damping for the parameters $\epsilon = 0.05$, $N_x \times N_v = 1024 \times 1000$, 16 phase points in each grid cell, $k = 0.4$, and $dt = 0.1$.(a) The long-time behavior of the amplitude of the electric field that confirms BGK mode [9] construction (around $t = 900$). (b) The short-time behavior. The linear Landau damping is recovered accurately until $t \approx 25$. After the linear stage, trapping oscillations are observed.

schemes is $O(\Delta^2)$. Therefore, we have not invested in the accuracy and chose parameters close to Ref. [6] to better benchmark the results and assess their performance. Next, in order to show the improvement of accuracy of the method, we redo the simulation of free streaming by AIS for two different cases when 100 and 2500 phase points are put in each cell. Figure 4(b) demonstrates the results. It is obvious that when we put 2500 phase points in each cell, the result is one order of magnitude more accurate than the case when the DF is sampled by 100 phase points within a cell.

We now examine the classical numerical test of the linear Landau damping. In this case, we start with $f(x,v,0) = 1/\sqrt{2\pi}\exp(-v^2/2)[1+\epsilon\cos(kx)]$. The initial arrangement of the phase points is random along the $v$ axis and regular along the $x$ axis. By AIS, $f_g$ is calculated. Integrating $f_g$ over the velocity space and solving Poisson's equation leads to the electric field on the grid $E_g$. Poisson's equation is solved by the Fast Fourier Transform. Then, by a Lagrange polynomial IS [8], $E_g$ is interpolated to the position of phase points to obtain $E_p$. Having $E_p$, Eqs. (2) are solved by the Leapfrog-Trapezoidal scheme. The parameters are $\epsilon = 0.01$, with grid points $N_x \times N_v = 256 \times 256$, and in each cell we put four phase points. The periodic length is $L = 4\pi$, $k = 0.5$, $-5 \leqslant v \leqslant 5$, and $dt = 0.1$. In Fig. 5(a), the electric field is plotted against time. It shows the exponential decay of the amplitude of the electric field according to Landau's theory. The damping rate, the slope of straight line, obtained by this method is $\gamma = 0.153$, which agrees very well with values predicted by the theory [4]. Figure 5(b) exhibits the relative error in the total energy in percent, i.e., (total energy$^n$ − total energy$^0$)/total energy$^0$ $\times 100$, recalling that the superscript $n$ denotes the quantities at $t = n\Delta t$. As is seen, the scheme is capable of keeping the energy conservation.

Now, let us test the scheme for a nonlinear case, that is, the nonlinear Landau damping. To this end, we compare our scheme with the results of some accurate numerical computations for very long times, up to 1000 inverse electron plasma frequencies, that was done by Manfredi [10]. Accordingly, we choose the parameters exactly similar to Manfredi's paper. That is, $\epsilon = 0.05$, $k = 0.4$, $\Delta t = 0.1$, $-6 \leqslant v \leqslant 6$, and 16 phase points in each cell. Figure 6 shows the evolution of the fundamental mode of the electric field $E_k$. Figure 1(b) shows that the linear Landau damping is recovered accurately until $t \cong 25$, in agreement with Manfredi's results. After the linear stage, trapping oscillations are observed, while the maximum amplitude decreases at each oscillation. However, around $t \approx 900$, no further decrease is observed, and the electric field goes on oscillating around an approximately constant value [10]. It is important to note that for avoiding the recurrence effect Manfredi used 4000 and 8000 phase-space cells along the velocity axis, yielding $T_R = 5230$ in the less favorable case, which is much larger than the total time of the run ($t = 1600$). However, in our case, since by randomizing the phase-point arrangement, the recurrence effect does not take place, we could find similar results with just 1000 cells along the velocity axis.

## III. CONCLUSION

In summary, we improved the method of characteristics [3] by increasing the number of phase points in phase space. Naturally, due to phase-point dynamics the development of steep gradients can be managed without enhancing the velocity resolution. In order to reduce the number of operations, AIS was introduced, which is easily extendable to a higher dimension. By use of AIS and randomizing the phase-point arrangement, we could prevent the occurrence of the recurrence effect. The advantages of this scheme are that it is recurrence free with simpler IS (i.e., weighting is omitted). Using today's supercomputers, our method appears to be a convenient method for dealing with strongly nonlinear problems in phase space when little noise and good precision are needed.

## ACKNOWLEDGMENTS

We thank F. Kazeminezhad for his continual support and comments, B. Eliasson for his remarks on the recurrence effect, S. Kuhn for his helpful discussions, and M. Shoucri, I. Hofmann, and E. Sonnendrucker for reviewing the manuscript and their valuable comments.

[1] C. K. Birdsall and A. B. Langdon, *Plasma Physics via Computer Simulation* (Institute of Physics Publishing, Bristol/Philadelphia, 1991).

[2] N. V. Elkina and J. Buchner, J. Comput. Phys. **213**, 862 (2006).

[3] F. Kazeminezhad, S. Kuhn, and A. Tavakoli, Phys. Rev. E **67**, 026704 (2003).

[4] F. Filbet, E. Sonnendrucker, and P. Bertrand, J. Comput. Phys. **172**, 166 (2001).

[5] L. E. Johnson, J. Plasma Phys. **23**, 433 (1980).

[6] E. Pohn, M. Shoucri, and G. Kamelander, Comput. Phys. Commun. **166**, 81 (2005).

[7] O. C. Zeinkiewicz and K. Morgan, *Finite Elements and Approximation* (Wiley, New York, 1983), ch. 2.

[8] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in FORTRAN 77: The Art of Scientific Computing* (Cambridge University Press, New York, 1992), sec. 3-1.

[9] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).

[10] G. Manfredi, Phys. Rev. Lett. **79**, 2815 (1997).