Gap compactonlike solutions of coupled Kortweg–de Vries equations with linear and nonlinear dispersions

Galal A. Al-Akhaly and Bishwajyoti Dey

Department of Physics, University of Pune, Ganeshkhind, Pune 411007, India

(Received 30 April 2011; published 29 September 2011)

We show the existence of a type of excitation, which we term as "gap compactonlike," within the gap of the linear spectrum of a system of coupled Kortweg–de Vries equations with linear and nonlinear dispersions. Since the solutions lie in the gap region of the spectra, they avoid resonance with the linear oscillatory wave and, therefore, do not decay into radiations. These types of solutions are important in energy localization and transport in polymers and biopolymers, optical systems, etc.

DOI: 10.1103/PhysRevE.84.036607 PACS number(s): 03.50.−z, 05.45.Yv

I. INTRODUCTION

The class of nonlinear partial differential equations with linear dispersion, known as nonlinear evolution equations, have solitary wave solutions, which preserve their shape through nonlinear interaction. These solutions were named "solitons" by Zabusky and Kruskal in their work on Kortweg–de Vries (KdV) equations [1]. The soliton solutions are very stable, and the stability of these soliton solutions is due to the balance between the nonlinearity and dispersion. The soliton solutions are usually exponentially localized in space. They occur in many branches of sciences [2]. It has been shown that the equations possessing these solutions have an infinite sequence of conservation laws and are integrable.

It is well known that collective nonlinear excitations called gap solitons can exist in spectrum gaps forbidden for linear waves. The existence of gap soliton was shown by Chen and Mills [3] during 1987. Theoretical and numerical studies of the gap solitons have been carried out in many physical fields such as nonlinear optical systems [3–6], semiconducting systems [7,8], and super lattices [9,10], etc. The interplay of nonlinearity and lattice periodicity also results in the existence of solitary waves in the gap frequency range. A wider class of gap solitons has been found for one-dimensional diatomic lattices with nonlinearity introduced through on-site (substrate) potential [11] and in a diatomic chain of particles in harmonic potential [12]. New types of localized structures in nonlinear lattice have been shown to exist as a result of nonlinearity-induced symmetry breaking between two [13] (or more [14]) equivalent eigenmodes of the lattice. These gap solitons were later observed experimentally [15]. Harmonic gap modes can be obtained in a perfect one-dimensional chain for a variety of standard two-body potentials [16]. Bilbault, Kamga, and Remoissenet [17] have studied the nonlinear transmissivity and gap solitons characteristic in the low-amplitude or nonlinear Schrödinger limit. Similarly, Flytzanis and Malomed [18] have shown that the nonlinearity produces an effective gap in which the solitons exist when the width of the gap in the linearized system is very small. Grimshaw and Malomed showed that a new type of a two-parameter family of solitons may exist in the narrow gap in the spectrum of two linearly coupled KdV equations with opposite sign of dispersion coefficient [19]. In 1999, the gap solitonlike structures were demonstrated in Bose-Einstein condensates (BEC) in optical dipole trap [20]. Other theoretical studies of the possibility of nonlinear localization in Bose-Einstein condensates have been carried out in [21–23]. The existence of gap solitons in BEC was confirmed experimentally in 2004 for a one-dimensional optical lattice [24].

To understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman introduced and studied a family of fully nonlinear KdV equations with nonlinear dispersion. They discovered that in such a nonlinear dispersive system, a solitary wave with compact structure free of exponential tails can exist [25,26]. They termed such solutions as compactons [25]. The compacton's amplitude depends on its velocity [unlike that of the soliton, which narrows as amplitude (speed) increases], but its width is independent of its amplitude [25,27]. These compactons vanish outside a finite core region. The compacton, like the soliton [1], has a remarkable property that, after colliding with another compacton, it reemerges with the same coherent shape [25]. Unlike soliton collision in integrable systems, the point at which two compactons collide is marked by certain low amplitude compacton-anticompacton pairs Refs. [25,27]. In contrast to solitons that have an exponential tail, the compactons are a solitary wave with compact support, i.e., they are localized in a finite region of space and zero outside. The fact that the compactons are stable structures and the ability of such modes to store energy gives them a particular importance for energy localization and transport in polymers and biopolymers. The nonlinear equations with nonlinear dispersion, which support these solutions, are not completely integrable, but possess only a finite number of conservation laws [25,27]. In the presence of higher order nonlinear dispersion terms, the range of nonlinearity parameter for which the class of compacton solutions are allowed increases [28]. From their study of the linear and nonlinear stability analysis, Dey and Khare have shown that the compactons are stable structures [29]. The same result is also valid for equations with higher order nonlinear dispersion [30]. Dinda and Remoissenet [31] demonstrated the existence of stationary breather compactons in a nonlinear Klein-Gorden system. Exact discrete compactonlike breather solutions were obtained for discrete nonlinear dispersive lattice systems [32]. The shape profile study of such compacton solutions showed that the tail region decays with a faster than exponential law [33–35]. Kevrekidis and Konotop [36] considered some general classes of nonlinear lattice models, which support bright discrete compact breather solutions. They classified the models as belonging to three categories by analyzing the conditions for which such solutions are possible. Existence of such compact discrete breather solutions in a two-dimensional Fermi-Pasta-Ulam lattice system was shown by Sarkar and Dey [37], and for the same lattice, it was shown to exist in the presence of long-range interaction [38]. It has also been shown that compacton solutions also exist in a *mixed dispersion* model. The mixed dispersion model is the KdV equation where both linear and nonlinear dispersions are present [26]. This is useful as, in physical applications, often both linear and nonlinear dispersions are present.

In this paper, we demonstrate the existence of a type of nonlinear excitation in the gap region of the two linearly coupled KdV equations with mixed dispersion. These solutions have properties of compacton solutions. But, unlike compactons that have finite support, these solutions have compactonlike support in the sense that they have a finite but very small nonzero amplitude outside the core region. The small nonzero amplitude outside the finite support can be further minimized by adjusting the parameters of the system. The width of these solutions is independent of the amplitude, similar to compacton solutions [25]. We term these gap solutions as gap compactonlike solutions. We also obtain various other exact soliton solutions in the gap spectrum of the coupled KdV equations with mixed dispersion.

II. EQUATIONS OF THE SYSTEM

The general problem we consider here is a system of two linearly coupled Korteweg–de Vries equations with nonlinear dispersion. This system of coupled KdV equations we consider can be written as

$$u_t + \alpha_1(u^2)_x + u_{3x} + \beta_1(u^2)_{3x} = -\lambda v_x, \tag{1a}$$

$$v_t - \Delta v_x + \alpha_2(v^2)_x - \alpha v_{3x} + \beta_2(v^2)_{3x} = -\beta\lambda u_x, \tag{1b}$$

where $-\Delta$ is relative group velocity of linear long wave in the two subsystems, $\alpha > 0$ (corresponding to the oppositely signed dispersion in the subsystem), β_1 and β_2 are the nonlinear dispersion coefficients, β is an independent parameter (here we take $\Delta > 0$ and $\beta > 0$), λ is the small coupling constant, and α_1 and α_2 are constants. In the absence of nonlinear dispersion terms ($\beta_1 = \beta_2 = 0$) and for particular values $\alpha_1 = \alpha_2 = -1/2$, the system of coupled KdV equations (1) supports gap solitons [19]. However, we keep the value of α_1 and α_2 as arbitrary.

Looking for traveling solutions of the uncoupled equations in Eq. (1) (i.e., the coupling constant $\lambda = 0$), we get

$$cu_\eta + \alpha_1(u^2)_\eta + u_{3\eta} + \beta_1(u^2)_{3\eta} = 0, \tag{2a}$$

$$cv_\eta - \Delta v_\eta + \alpha_2(v^2)_\eta - \alpha v_{3\eta} + \beta_2(v^2)_{3\eta} = 0, \tag{2b}$$

where $\eta = x + ct$. Following the method in [28], we first check if there are any compacton solutions of the uncoupled equations (2). For this, we take the ansatz for the compacton solutions to have the form

$$u(\eta) = E_1\cos^{\delta_1}(D_1\eta)$$
$$\text{for}\quad |D_1\eta| \leqslant \pi/2,\quad u = 0\ \text{otherwise} \tag{3a}$$

and

$$v(\eta) = E_2\cos^{\delta_2}(D_2\eta)$$
$$\text{for}\quad |D_2\eta| \leqslant \pi/2,\quad v = 0\ \text{otherwise.} \tag{3b}$$

By inserting Eq. (3a) in (2a) and Eq. (3b) in (2b), we get

$$\begin{aligned}
&{D_1}^2(\delta_1-1)(\delta_1-2) + \left[c - {D_1}^2(3\delta_1-2)\right]\cos^2(D_1\eta)\\
&\quad + 4\beta_1 E_1 {D_1}^2(2\delta_1-1)(\delta_1-1)\cos^{\delta_1}(D_1\eta)\\
&\quad + \left[2\alpha_1 E_1 - 4\beta_1 E_1 {D_1}^2(2\delta_1-1)(\delta_1-1)\right.\\
&\quad \left.- 4\beta_1 E_1 {D_1}^2(3\delta_1-1)\right]\cos^{\delta_1+2}(D_1\eta) = 0,
\end{aligned} \tag{4a}$$

$$\begin{aligned}
&-\alpha {D_2}^2(\delta_2-1)(\delta_2-2) + \left[c - \Delta + \alpha {D_2}^2(3\delta_2-2)\right]\\
&\quad\times \cos^2(D_2\eta) + 4\beta_2 E_2 {D_2}^2(2\delta_2-1)(\delta_2-1)\cos^{\delta_2}(D_2\eta)\\
&\quad + \left[2\alpha_2 E_2 - 4\beta_2 E_2 {D_2}^2(2\delta_2-1)(\delta_2-1)\right.\\
&\quad \left.- 4\beta_2 E_2 {D_2}^2(3\delta_2-1)\right]\cos^{\delta_2+2}(D_2\eta) = 0.
\end{aligned} \tag{4b}$$

By equating the coefficient of the terms involving the same power of cosine in Eq. (4), we get compacton solutions for $u(\eta)$ and $v(\eta)$ for the parameter value $\delta_1 = \delta_2 = 2$ when

$${D_1}^2 = \frac{\alpha_1}{16\beta_1},\quad E_1 = \frac{4{D_1}^2 - c}{12\beta_1 {D_1}^2}, \tag{5a}$$

$${D_2}^2 = \frac{\alpha_2}{16\beta_2},\quad E_2 = \frac{\Delta - c - 4\alpha {D_2}^2}{12\beta_2 {D_2}^2}. \tag{5b}$$

The solutions satisfy the condition for the compacton solution that the width is independent of its amplitude [25]. Similarly, for $\eta = x - ct$, we get the same form of the compacton solutions of Eq. (3) by changing the sign of c in Eqs. (4) and (5) above. From Eq. (5), we see that α_1 and β_1 should have the same sign and α_2 and β_2 should also have the same sign. If we replace the nonlinear dispersion term $\beta(u^2)_{3x} = \beta(6u_x u_{2x} + 2uu_{3x})$ in Eq. (1) by $2u_x u_{2x} + uu_{3x}$, then for the appropriate choice of parameters, the compacton solutions in Eq. (3) reduce exactly to the compacton solutions of the KdV equations with mixed dispersion as obtained in [26] [Eq. (15)].

III. EXISTENCE OF THE GAP

To show that is there a gap in the system's linear spectrum opened by a weak coupling and to find its width, we study the spectrum of the system represented by Eq. (1). The spectrum is represented by the relation between the phase velocity c and the wave number k. For this, we consider the solution of the uncoupled ($\lambda = 0$) and linear case of Eq. (1) to have the form

$$u = u_0 e^{i(kx - wt)} \quad\text{and}\quad v = v_0 e^{i(kx - wt)}.$$

By substituting this solution in Eq. (1), we get

$$w = -k^3 \quad\text{and}\quad w = \alpha k^3 - \Delta k.$$

The spectra of the uncoupled subsystems is then described by $c^{(u)} = -k^2$ and $c^{(v)} = \alpha k^2 - \Delta$. Solving these two relations, we determine the cross points of phase velocity between the two spectra as

$$k = \pm k_0 = \pm\frac{\Delta^{1/2}}{(1+\alpha)^{1/2}},\quad c = c^{(0)} = \frac{-\Delta}{(1+\alpha)}, \tag{6}$$

the plot of which is shown in Fig. 1. Here, we have considered $\Delta > 0$.

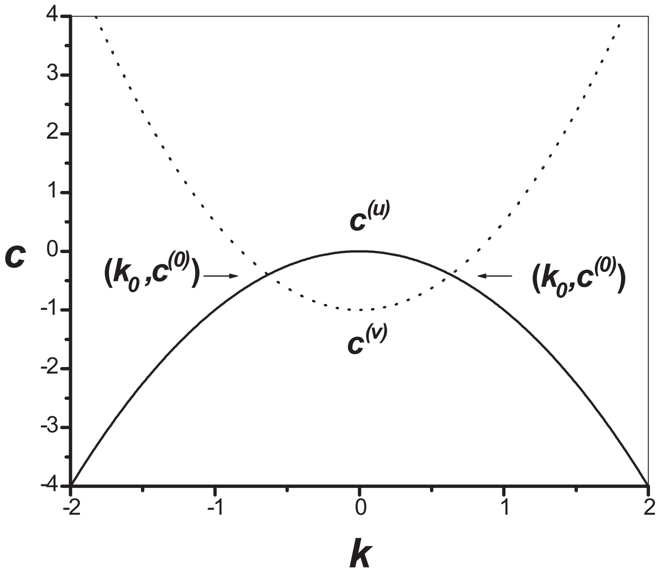

FIG. 1. Shows the linear spectrum of the system when the coupling is absent. The intersection points are at $k = \pm k_0 = 0.632\,455\,526$ and the value of c at these points is $c = c^{(0)} = -0.400\,000\,006$. The parameters are $\lambda = 0$, $\Delta = 1$, and $\alpha = 1.5$.

When coupling is switched on, it prevents the crossing of the points giving rise to the gap in the spectrum. By substituting the plane wave solution in the coupled ($\lambda \neq 0$) linear equations case of Eq. (1), we get

$$(c + k^2)u_0 - \lambda v_0 = 0, \quad -\beta\lambda u_0 + (c + \Delta - \alpha k^2)v_0 = 0.$$

These equations can be written in the form of the matrix

$$\begin{bmatrix} (c + k^2) & -\lambda \\ -\beta\lambda & c + \Delta - \alpha k^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0$$

and, from the condition of the existence of nontrivial solutions for the coupled linear equations, i.e., the determinant of the matrix should be zero, we get the new dispersion relation for the coupled equations as

$$c^2 + (\Delta - \alpha k^2 + k^2)c + k^2(\Delta - \alpha k^2) - \lambda^2\beta = 0. \quad (7)$$

The turning points of the dispersion relation ($\frac{dc}{dk} = 0$) are ${k_1}^2 = 0, {k_2}^2 = {k_0}^2 - \lambda\sqrt{\frac{\beta}{(1+\alpha)}}$, and ${k_3}^2 = {k_0}^2 - \lambda\sqrt{\frac{\beta}{\alpha(1+\alpha)}}$. The values of c at these points are $|c - c^{(0)}| = |\lambda|\sqrt{\beta}$, $|c - c^{(0)}| = 2|\lambda|\sqrt{\frac{\beta}{1+\alpha}}$, and $|c - c^{(0)}| = 2|\lambda|\sqrt{\frac{\alpha\beta}{1+\alpha}}$, respectively. The gap exists for $\alpha\beta > 0$ in the interval of the velocity

$$|c - c^{(0)}| < 2|\lambda|\frac{\sqrt{\alpha\beta}}{(1 + \alpha)}. \quad (8)$$

This is shown in Fig. 2. The gap soliton and the gap compacton solutions, if they exist in the gap region, will be stable against the decay by radiation by resonating with linear oscillatory waves.

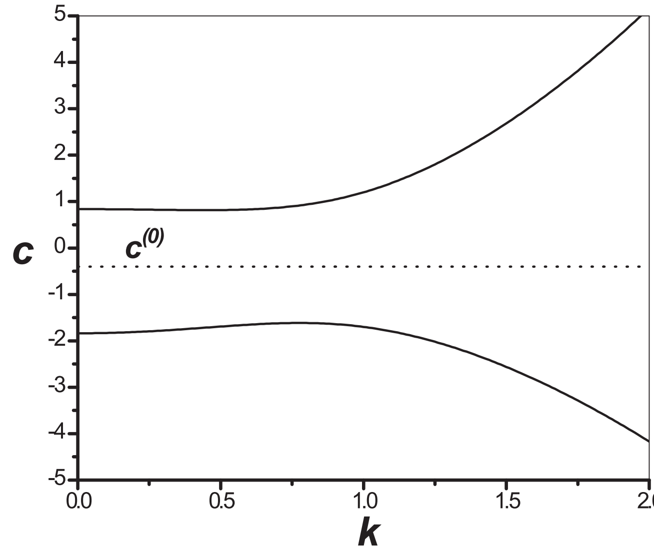

FIG. 2. Shows the opening of the gap in the linear spectrum of the system when the coupling is present. The dotted line shows the intersection velocity $c^{(0)}$. Parameters are $\lambda = 0.9$, $\Delta = 1$, $\alpha = 1.5$, and $\beta = 1.9$.

IV. DYNAMICS OF THE SYSTEM INSIDE THE SPECTRAL GAP REGION

To study the dynamics of the system inside the spectral gap region, we expand the wave field as [19]

$$u = U_1(x,t)e^{ik_0(x-c^{(0)}t)} + U_2(x,t)e^{2ik_0(x-c^{(0)}t)} + U_0(x,t) + \text{c.c.}, \quad (9a)$$

$$v = V_1(x,t)e^{ik_0(x-c^{(0)}t)} + V_2(x,t)e^{2ik_0(x-c^{(0)}t)} + V_0(x,t) + \text{c.c.} \quad (9b)$$

To obtain gap solitary waves, we consider the weak nonlinearity effect [11,39–41] and assume that the amplitudes of U and V are small and slowly varying, and the smallness produced by differentiation of slowly varying functions or squared amplitude $|U_1|^2$ and $|V_1|^2$ to be of the order of the coupling constant λ. By substituting Eq. (9) into (1) and equating the coefficient in front of the terms that have equal harmonics, we obtain for the amplitude of the second harmonics

$$U_2 = -\frac{1}{3}\left[\frac{4\Delta\beta_1 - \alpha_1(1 + \alpha)}{\Delta}\right]{U_1}^2, \quad (10a)$$

$$V_2 = \frac{1}{3}\left[\frac{4\Delta\beta_2 - \alpha_2(1 + \alpha)}{\alpha\Delta}\right]{V_1}^2. \quad (10b)$$

Here, we have taken only the terms for the amplitude of U_2 and V_2 that are less than λ. Similarly, by using Eqs. (10) and (6) and from the coefficient of the first and zeroth harmonics, considering only terms less than the square of the coupling constant, we get the equations for the amplitude of the first harmonics as

$$\frac{\partial U_1}{\partial t} - \frac{3\Delta}{(1+\alpha)}\frac{\partial U_1}{\partial x} - \frac{2}{3}ik_0(\alpha_1-\beta_1{k_0}^2)\left[\frac{4\Delta\beta_1-\alpha_1(1+\alpha)}{\Delta}\right]|U_1|^2U_1 + 4ik_0\big(\alpha_1-\beta_1{k_0}^2\big)U_1U_0 = -i\lambda k_0 V_1, \tag{11a}$$

$$\frac{\partial V_1}{\partial t} - \frac{\Delta(1-2\alpha)}{(1+\alpha)}\frac{\partial V_1}{\partial x} + \frac{2}{3}ik_0(\alpha_2-\beta_2{k_0}^2)\left[\frac{4\Delta\beta_2-\alpha_2(1+\alpha)}{\alpha\Delta}\right]|V_1|^2V_1 + 4ik_0\big(\alpha_2-\beta_2{k_0}^2\big)V_1V_0 = -i\lambda\beta k_0 U_1, \tag{11b}$$

and the equations for the amplitude of the zeroth harmonics as

$$\frac{\partial U_0}{\partial t} + \alpha_1\frac{\partial |U_1|^2}{\partial x} = 0, \tag{12a}$$

$$\frac{\partial V_0}{\partial t} - \Delta\frac{\partial V_0}{\partial x} + \alpha_2\frac{\partial |V_1|^2}{\partial x} = 0, \tag{12b}$$

where we have taken U_0 and V_0 as real functions. Defining [19]

$$U_1 = \left[\frac{3\Delta}{2k_0\big(\alpha_1-\beta_1{k_0}^2\big)[\alpha_1(1+\alpha)-4\Delta\beta_1]}\right]^{1/2} U, \tag{13a}$$

$$V_1 = \left[\frac{3\Delta\alpha}{2k_0\big(\alpha_2-\beta_2{k_0}^2\big)[\alpha_2(1+\alpha)-4\Delta\beta_2]}\right]^{1/2} V, \tag{13b}$$

$$\eta = \Delta^{-1}x + \frac{2-\alpha}{1+\alpha}t, \quad t = t \tag{14}$$

and using simple notation such as

$$M = 4k_0\big(\beta_1{k_0}^2-\alpha_1\big)U_0, \tag{15a}$$

$$N = 4k_0\big(\beta_2{k_0}^2-\alpha_2\big)V_0, \tag{15b}$$

$$\epsilon = \lambda k_0\sqrt{\alpha}\left[\frac{\big(\alpha_1-\beta_1{k_0}^2\big)[\alpha_1(1+\alpha)-4\Delta\beta_1]}{\big(\alpha_2-\beta_2{k_0}^2\big)[\alpha_2(1+\alpha)-4\Delta\beta_2]}\right]^{1/2}, \tag{15c}$$

$$\gamma = \frac{\beta}{\alpha}\left[\frac{\big(\alpha_2-\beta_2{k_0}^2\big)[\alpha_2(1+\alpha)-4\Delta\beta_2]}{\big(\alpha_1-\beta_1{k_0}^2\big)[\alpha_1(1+\alpha)-4\Delta\beta_1]}\right], \tag{15d}$$

we can write equations for first harmonics as

$$U_t - U_\eta + i|U|^2U - iMU = -i\epsilon V, \tag{16a}$$

$$V_t + V_\eta - i|V|^2V - iNV = -i\epsilon\gamma U, \tag{16b}$$

and equations for zeroth harmonics as

$$M_t + \frac{2-\alpha}{1+\alpha}M_\eta = \frac{6\alpha_1}{\alpha_1(1+\alpha)-4\Delta\beta_1}(|U|^2)_\eta, \tag{17a}$$

$$N_t + \frac{1-2\alpha}{1+\alpha}N_\eta = \frac{6\alpha_2\alpha}{\alpha_2(1+\alpha)-4\Delta\beta_2}(|V|^2)_\eta. \tag{17b}$$

To look for solitary wave solutions for Eqs. (16) and (17), we consider the general traveling solitary wave form given by

$$U(\eta-wt) = e^{-i\sigma t}A(\eta-wt), \tag{18a}$$

$$V(\eta-wt) = e^{-i\sigma t}B(\eta-wt), \tag{18b}$$

$$M = M(\eta-wt), \quad N = N(\eta-wt), \tag{19}$$

where the frequency σ is assumed to be of order λ. By substituting Eq. (18) in (17) and using Eq. (19), we obtain the zeroth harmonics amplitude M and N as

$$M = \frac{6\alpha_1(1+\alpha)}{[\alpha_1(1+\alpha)-4\Delta\beta_1][2-\alpha-w(1+\alpha)]}|A|^2, \tag{20a}$$

$$N = \frac{-6\alpha_2(1+\alpha)\alpha}{[\alpha_2(1+\alpha)-4\Delta\beta_2][2\alpha-1+w(1+\alpha)]}|B|^2. \tag{20b}$$

By substituting Eq. (18) in (16) and using Eq. (20), we get a system of coupled ordinary differential equations for the amplitude A and B as

$$-i\sigma A - (1+w)\frac{d}{d(\eta-wt)}A - iC|A|^2A = -i\epsilon B, \tag{21a}$$

$$-i\sigma B + (1-w)\frac{d}{d(\eta-wt)}B + iD|B|^2B = -i\epsilon\gamma A, \tag{21b}$$

where

$$C = \frac{6\alpha_1(1+\alpha)-[\alpha_1(1+\alpha)-4\Delta\beta_1][2-\alpha-w(1+\alpha)]}{[\alpha_1(1+\alpha)-4\Delta\beta_1][2-\alpha-w(1+\alpha)]}, \tag{22a}$$

$$D = \frac{6\alpha_2(1+\alpha)\alpha-[\alpha_2(1+\alpha)-4\Delta\beta_2][2\alpha-1+w(1+\alpha)]}{[\alpha_2(1+\alpha)-4\Delta\beta_2][2\alpha-1+w(1+\alpha)]}. \tag{22b}$$

To obtain the general solution of Eq. (21), we multiply (21a) by γA^* and add to it the complex conjugate of the resultant equation to get

$$\gamma(1+w)[A^*A' + A'^*A] = i\gamma\epsilon[A^*B - B^*A], \tag{23a}$$

where the prime denotes the derivative with respect to $\eta - wt$. Similarly, we multiply Eq. (21b) by B^* and add to it the complex conjugate of the resultant equation to get

$$(1-w)[B^*B' + B'^*B] = i\gamma\epsilon[A^*B - B^*A]. \tag{23b}$$

By equating Eqs. (23a) and (23b), we get

$$\gamma(1+w)\frac{d}{d(\eta-wt)}|A|^2 = (1-w)\frac{d}{d(\eta-wt)}|B|^2. \tag{23c}$$

By integration with respect to $(\eta - wt)$, we get

$$|B|^2 = \gamma\frac{(1+w)}{(1-w)}|A|^2. \tag{23d}$$

The solution of Eq. (23d) takes the form

$$A = \sqrt{(1-w)}Re^{i\varphi}, \quad B = \sqrt{\gamma(1+w)}Re^{i\psi}, \tag{24}$$

where R, φ, and ψ are real and $-1 < w < 1$. We note that the solution must satisfy the condition $(w^2 < 1)$ for it to be inside the gap spectrum. By multiplying Eq. (21) by A^* and adding to it the complex conjugate of the resultant equation, we get

$$\frac{dR^2}{d(\eta - wt)} = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}}R^2\sin(\varphi - \psi). \tag{25}$$

Again, by multiplying (21a) by B^* and adding to it the complex conjugate of the resultant equation, we get

$$\left[(\sigma + C|A|^2)\sqrt{\gamma(1-w^2)} - (1+w)\epsilon\gamma\cos(\varphi-\psi) + (1+w)\sqrt{\gamma(1-w^2)}\frac{d\varphi}{d(\eta-wt)}\right]2R^2\sin(\varphi-\psi) = 0. \tag{26a}$$

Also, by multiplying (21b) by A^* and adding to it the complex conjugate of the resultant equation, we get

$$-\left[(\sigma - D|B|^2)\sqrt{\gamma(1-w^2)} - (1-w)\epsilon\gamma\cos(\varphi-\psi) - (1-w)\sqrt{\gamma(1-w^2)}\frac{d\psi}{d(\eta-wt)}\right]2R^2\sin(\varphi-\psi) = 0. \tag{26b}$$

From Eqs. (26), we can write the first derivative for φ and ψ with respect to $(\eta - wt)$ as

$$\frac{d\varphi}{d(\eta-wt)} = \epsilon\sqrt{\frac{\gamma}{1-w^2}}\cos(\varphi-\psi) - \frac{[\sigma + C(1-w)R^2]}{(1+w)}, \tag{27a}$$

$$\frac{d\psi}{d(\eta-wt)} = \frac{[\sigma - D\gamma(1+w)R^2]}{(1-w)} - \epsilon\sqrt{\frac{\gamma}{1-w^2}}\cos(\varphi-\psi), \tag{27b}$$

which allows us to calculate the first derivative of the phase difference with respect to $(\eta - wt)$ as

$$\frac{d(\varphi-\psi)}{d(\eta-wt)} = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}}\cos(\varphi-\psi) - \frac{2\sigma}{(1-w^2)} - \left[\frac{C(1-w)}{(1+w)} - \frac{D\gamma(1+w)}{(1-w)}\right]R^2. \tag{28}$$

By dividing Eq. (28) by (25), we get

$$\frac{d(\phi-\psi)}{dR^2} = \frac{2\epsilon\sqrt{\frac{\gamma}{1-w^2}}\cos(\phi-\psi) - \frac{2\sigma}{1-w^2} - Q_1Q_2^{-1}R^2}{2\epsilon\sqrt{\frac{\gamma}{1-w^2}}R^2\sin(\phi-\psi)}, \tag{29a}$$

where

$$\begin{aligned} Q_1 = {} & \frac{\alpha_1(1+\alpha)}{\alpha_1(1+\alpha) - 4\Delta\beta_1}[4+\alpha+w(1+\alpha)][1-2\alpha-w(1+\alpha)](1-w)^2 \\ & + \frac{4\Delta\beta_1}{\alpha_1(1+\alpha)-4\Delta\beta_1}[2-\alpha-w(1+\alpha)][1-2\alpha-w(1+\alpha)](1-w)^2 \\ & + \frac{\gamma\alpha_2(1+\alpha)}{\alpha_2(1+\alpha)-4\Delta\beta_2}[4\alpha+1-w(1+\alpha)][2-\alpha-w(1+\alpha)](1+w)^2 \\ & + \frac{4\gamma\Delta\beta_2}{\alpha_2(1+\alpha)-4\Delta\beta_2}[2-\alpha-w(1+\alpha)][2\alpha-1+w(1+\alpha)](1+w)^2, \end{aligned} \tag{29b}$$

$$Q_2 = [2-\alpha-w(1+\alpha)][1-2\alpha-w(1+\alpha)]. \tag{29c}$$

By equating Eq. (29a) to zero (extremum of the phase difference), we get

$$\cos(\varphi-\psi) = \Omega + 2WR^2, \tag{30a}$$

which implies

$$\sin(\varphi-\psi) = [1-(\Omega+2R^2W)^2]^{\frac{1}{2}}, \tag{30b}$$

where

$$\Omega = [\gamma(1-w^2)]^{-\frac{1}{2}}\epsilon^{-1}\sigma, \tag{31a}$$

$$W = \tfrac{1}{4}[1-w^2]^{-\frac{1}{2}}\epsilon^{-1}\gamma^{-\frac{1}{2}}Q_1{Q_2}^{-1}. \tag{31b}$$

Finally, using Eqs. (30b) and (25) can be written as

$$\frac{dR^2}{d(\eta-wt)} = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}}R^2[1-(\Omega+2R^2W)^2]^{\frac{1}{2}}. \tag{32}$$

Now, to obtain the gap compacton solution of Eq. (32), we rewrite this equation in the compacton equation form [25] as

$$\left(\frac{dy}{dl}\right)^2 + p(y,p_0) = p_1, \tag{33a}$$

where

$$p(y,p_0) = 4a^2W^2y^4 + 4a^2\Omega Wy^3 - a^2(1-\Omega^2)y^2 + p_0, \tag{33b}$$

and $y = R^2$, $l = \eta - wt$, $a = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}}$, $p_0 = p_1 = 0$. Since $p_1 = 0$, Eq. (33a) implies that $p(y,p_0)$ is negative. To see for which parameter values of Ω, $p(y,p_0)$ is negative, we note that the right-hand side of Eq. (33b) has four real roots $y = 0$, a dipole root $y = \frac{1-\Omega}{2W}$, and $y = -\frac{(1+\Omega)}{2W}$. From the roots, we can see that the function $p(y,p_0)$ is negative for two cases: (i) $W > 0$ and (ii) $W < 0$. For $W > 0$, it can be checked that $p(y,p_0)$ is negative for the intervals of the parameter Ω given by

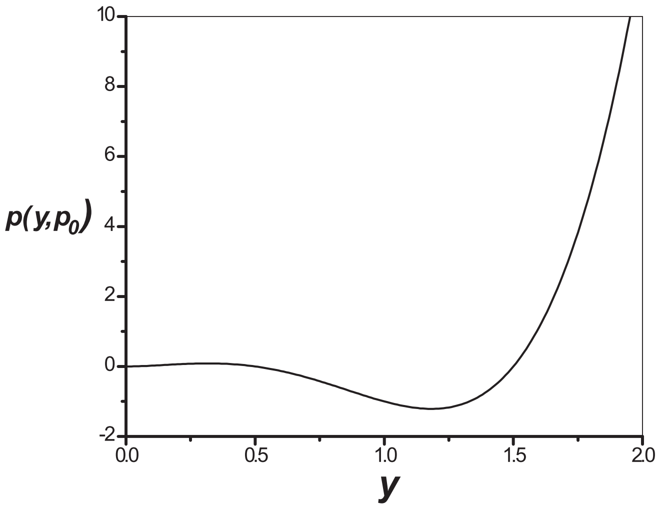

FIG. 3. Shows the variation of $p(y,p_0)$ with y for ($W > 0$) and $-\infty < \Omega < -1$. The parameters are $a = 1$, $p_0 = 0$, $W = 1$, and $\Omega = -2$.

$-\infty < \Omega < -1$, $-1 < \Omega < 1$, and $\Omega = -1$. Figure 3 shows the plot of $p(y,p_0)$ for particular values of Ω.

Similarly, for $W < 0$, $p(y,p_0)$ is negative when $1 < \Omega < \infty$, $-1 < \Omega < 1$, and $\Omega = 1$ and the corresponding $p(y,p_0)$ for particular values of Ω is shown in Fig. 4. By integrating both sides of Eq. (32), we get

$$\int \frac{d\rho}{\rho\sqrt{(1-\Omega^2) - 2\Omega\rho - \rho^2}} = 2Z + A_1, \tag{34}$$

where $\rho = 2WR^2$, A_1 is constant of integration, and

$$Z = \epsilon\sqrt{\frac{\gamma}{1-w^2}}(\eta - wt). \tag{35}$$

V. SOLITARY WAVE SOLUTIONS

As mentioned above, the solution depends on the value of the parameter Ω. Therefore, we consider the integration for three different cases of Ω.

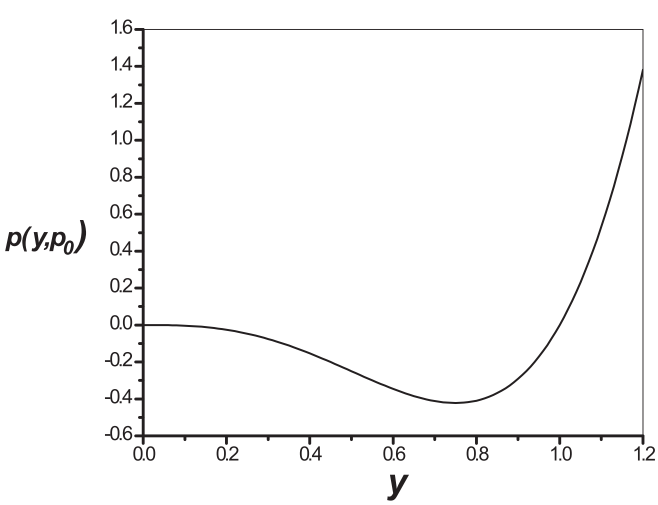

FIG. 4. Shows the variation of $p(y,p_0)$ with y for ($W < 0$) and $\Omega = 1$. The parameters are $a = 1$, $p_0 = 0$, $W = -1$, and $\Omega = 1$.

A. Gap soliton solutions

First case: $\Omega^2 < 1$. Using standard integration [42], we get

$$\rho = \frac{4(1-\Omega^2)A_2 e^{-2Z\sqrt{1-\Omega^2}}}{(A_2 e^{-2Z\sqrt{1-\Omega^2}} + 2\Omega)^2 + 4(1-\Omega^2)},$$

where $A_2 = e^{-A_1\sqrt{1-\Omega^2}}$. For a particular choice of $A_2 = 2 (A_1 < 0)$, the solution of Eq. (32) can be expressed as

$$R^2 = (2W)^{-1}\frac{(1-\Omega^2)}{2\cosh^2[Z\sqrt{1-\Omega^2}] - (1-\Omega)}. \tag{36a}$$

This represents a gap soliton solution. From the relation between $[\tan(\frac{\varphi-\psi}{2})]$ and $[\cos(\varphi-\psi)]$ and using Eq. (30a), the phase difference is given as

$$\tan\left(\frac{\varphi-\psi}{2}\right) = \sqrt{\frac{1-\Omega}{1+\Omega}}\tanh[Z\sqrt{1-\Omega^2}]. \tag{36b}$$

Equations (24), (35), and (36) therefore describe the structure of gap soliton solution of Eq. (1).

B. Weak gap soliton solutions

Second case: $\Omega^2 = 1$. Using standard integration [42], we obtain the following solutions of Eq. (32):

$$R^2 = (2W)^{-1}\left[\frac{1}{1+4Z^2}\right], \quad \tan\left(\frac{\varphi-\psi}{2}\right) = 2Z$$
$$\text{for} \quad \Omega = -1, \quad W > 0 \tag{37}$$

$$R^2 = (2W)^{-1}\left[\frac{-1}{1+4Z^2}\right], \quad \tan\left(\frac{\varphi-\psi}{2}\right) = (2Z)^{-1}$$
$$\text{for} \quad \Omega = +1, \quad W < 0. \tag{38}$$

These solutions describe weak gap soliton solutions as, unlike solitons, they do not decay exponentially.

C. Gap compactonlike solutions

Third case: $\Omega^2 > 1$. Equation (34) can be written as

$$\int \frac{d\rho}{\rho\sqrt{(\Omega^2-1) + 2\Omega\rho + \rho^2}} = 2iZ + iA_1.$$

By using standard integration [42], we get

$$\rho = \frac{4(\Omega^2-1)A_2 e^{-2iZ\sqrt{\Omega^2-1}}}{(A_2 e^{-2iZ\sqrt{\Omega^2-1}} - 2\Omega)^2 - 4(\Omega^2-1)},$$

where $A_2 = e^{-A_1 i\sqrt{\Omega^2-1}}$. For $A_2 = 2 (A_1 = ib$, b real and positive), the solution of Eq. (32) can be expressed as

$$R^2 = (2W)^{-1}\frac{(1-\Omega^2)}{(1+\Omega) - 2\cos^2[Z\sqrt{\Omega^2-1}]}. \tag{39a}$$

Similarly, from the relation between $[\tan(\frac{\varphi-\psi}{2})]$ and $[\cos(\varphi - \psi)]$ and using Eq. (30a), the phase difference can be obtained as

$$\tan\left(\frac{\varphi-\psi}{2}\right)=\sqrt{\frac{\Omega-1}{\Omega+1}}\cot[Z\sqrt{\Omega^2-1}]. \tag{39b}$$

The solution of Eq. (1) represented by Eqs. (24), (35), and (39) has two cases with respect to both W and Ω: (i) $W>0$ and $-\infty<\Omega<-1$, and (ii) $W<0$ and $1<\Omega<\infty$. For the solutions to be a compacton, it should vanish outside the core region, i.e., it should have a compact support ($|\eta|\leqslant\pi/2$). Also, for compacton solution, the width should be independent of its amplitude. But, from analysis of Eq. (33), we see that for the case (i), the solution is not localized within the compact support and, therefore, we do not consider this case further. For case (ii), the solution has almost compact support in the sense that the solution has a very small but finite nonzero amplitude outside the compact support given by

$$R^2=(2W)^{-1}\left[\frac{1-\Omega^2}{1+\Omega}\right]. \tag{40}$$

The finite but small amplitude of the solution outside the compact support can be made smaller by adjusting the parameter values. In the next section, we analyze the gap compactonlike solution for different values of the parameters.

VI. ANALYSIS OF THE GAP COMPACTONLIKE SOLUTIONS

We now analyze the structure of the gap compactonlike solution of Eq. (1) described by Eq. (39) along with Eqs. (24), (35), and (40). From Eqs. (31b) and (39), we see that the solution vanishes when Q_2 vanishes. From Eq. (29b), we see that Q_2 vanishes when $w=\frac{2-\alpha}{1+\alpha}$ or $w=\frac{1-2\alpha}{1+\alpha}$. So, the parameter α should be in the range $\frac{1}{2}<\alpha<2$, which also satisfies the condition that $w^2<1$. Also, there are no solutions when Q_1 vanishes.

A. Study of the width and the amplitude

From Eq. (39a), we see that the width of the solution depends on its amplitude. On the other hand, for the solution to be a compacton, its width should be independent of its amplitude. We can, however, adjust the parameters in Eq. (1) such that the width of Eq. (39a) becomes independent of its amplitude. From the analysis of Eq. (39a), one can show that, for the frequency of carrier wave satisfying the relation

$$\sigma^2=(1-w^2)[\epsilon^2\gamma+n^2(1-w^2)], \tag{41}$$

where n is constant, the width becomes independent of its amplitude. For this frequency, one can express Eq. (39) as

$$R^2=2(1-w^2)^{\frac{3}{2}}\left|\frac{Q_2}{Q1}\right|\left[n^2\Big/\left(\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}}+\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}+n^2(1-w^2)}-2\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}}\cos^2[n(\eta-wt)]\right)\right]$$

$$\text{for}\quad |n(\eta-wt)|\leqslant\frac{\pi}{2} \tag{42a}$$

and

$$R^2=2(1-w^2)^{\frac{3}{2}}\left|\frac{Q_2}{Q1}\right|\frac{n^2}{\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}}+\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}+n^2(1-w^2)}} \tag{42b}$$

otherwise, and

$$\tan\left(\frac{\varphi-\psi}{2}\right)=\sqrt{\frac{\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}+n^2(1-w^2)}-\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}}}{\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}+n^2(1-w^2)}+\sqrt{\lambda^2\frac{\Delta\beta}{1+\alpha}}}}\cot[n(\eta-wt)]. \tag{42c}$$

From Eq. (42a), we see that, for a fixed value of n, the width of the solution is independent of its amplitude. We fix $n=1$, which gives the minimum value of the amplitude of the solution outside the compact support Eq. (42b). Thus, Eq. (42) describes a gap compactonlike solution of a system of coupled nonlinear dispersive KdV equation (1). Figure 5 shows the plot of the gap compactonlike solution for three choices of parameter values. The three solutions in the figure have different amplitudes but the same widths, implying that the width of the solution is independent of its amplitude. Here, the width is represented by full width at half maximum (FWHM). From Eq. (42b), we see that the nonzero value of the amplitude of the gap compactonlike solution outside the compact support can be lowered for the choice of the larger value of parameters λ and β and the smaller value of α.

Finally, we can express the solutions as obtained above in terms of the original functions $u(x,t)$ and $v(x,t)$. For this, we first obtain an expression for the phases ϕ and ψ. By adding Eq. (21a) to its complex conjugate and substituting Eq. (27a) in the resultant equation, we get

$$a\sin\phi-b\cos\phi=\sin\psi, \tag{43a}$$

where $a=\Omega+2WR^2$ and $b=\sqrt{1-a^2}$ are the phase difference as a function of R [Eqs. (30a) and (30b), respectively]. Similarly, by adding Eq. (21b) to its complex conjugate and substituting Eq. (27b) in the resultant equation, we get

$$a\sin\psi+b\cos\psi=\sin\phi. \tag{43b}$$

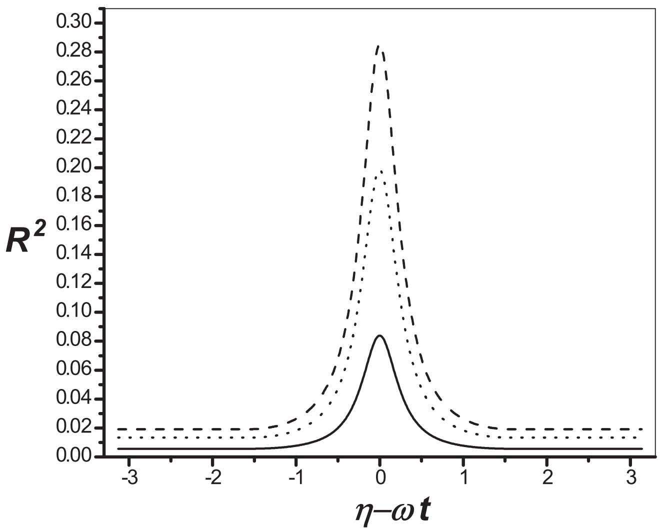

FIG. 5. Shows the compactonlike solution for three different values of parameters β_1 and β_2. From top to bottom: $\beta_1 = \beta_2 = 4$, $\beta_1 = \beta_2 = 2$, and $\beta_1 = \beta_2 = 1$, respectively. The parameters are $\alpha_1 = \alpha_2 = 1$, $\lambda = 0.9$, $\Delta = 1$, $\alpha = 1.5$, and $\beta = 1.9$.

Solving the above two equations, we obtain two independent roots $\sin\phi = a$, $\sin\psi = 1$, and $\sin\phi = -a$, $\sin\psi = -1$. Knowing R, ϕ, and ψ and using Eqs. (10), (13), (15), (18), (20), and (24), we can express the solutions in terms of the original functions Eq. (9) as

$$u(x,t) = a_1\sqrt{1-w}Re^{-i[(\sigma+k_0c^{(0)})t-k_0x-\phi]} + a_3a_1^2e^{-2i[(\sigma+k_0c^{(0)})t-k_0x-\phi]} + \frac{a_7}{a_5}(1-w)R^2 + \text{c.c.} \tag{44a}$$

and

$$v(x,t) = a_2\sqrt{\gamma(1+w)}Re^{-i[(\sigma+k_0c^{(0)})t-k_0x-\psi]} + a_4a_2^2e^{-2i[(\sigma+k_0c^{(0)})t-k_0x-\psi]} + \frac{a_8}{a_6}\gamma(1+w)R^2 + \text{c.c.}, \tag{44b}$$

where the constants a_i's are given by

$$a_1 = \left[\frac{3\Delta}{2k_0(\alpha_1 - \beta_1k_0{}^2)[\alpha_1(1+\alpha) - 4\Delta\beta_1]}\right]^{1/2},$$

$$a_2 = \left[\frac{3\Delta\alpha}{2k_0(\alpha_2 - \beta_2k_0{}^2)[\alpha_2(1+\alpha) - 4\Delta\beta_2]}\right]^{1/2},$$

$$a_3 = -\frac{1}{3}\left[\frac{4\Delta\beta_1 - \alpha_1(1+\alpha)}{\Delta}\right],$$

$$a_4 = \frac{1}{3}\left[\frac{4\Delta\beta_2 - \alpha_2(1+\alpha)}{\alpha\Delta}\right],$$

$$a_5 = 4k_0(\beta_1k_0{}^2 - \alpha_1),$$

$$a_6 = 4k_0(\beta_2k_0{}^2 - \alpha_2),$$

$$a_7 = \frac{6\alpha_1(1+\alpha)}{[\alpha_1(1+\alpha) - 4\Delta\beta_1][2 - \alpha - w(1+\alpha)]},$$

$$a_8 = \frac{-6\alpha_2(1+\alpha)\alpha}{[\alpha_2(1+\alpha) - 4\Delta\beta_2][2\alpha - 1 + w(1+\alpha)]}.$$

VII. CONCLUSION

In conclusion, we have shown the existence of a type of nonlinear localized excitation with compactlike support in the gap region of the spectrum of a system of linearly coupled KdV equations with mixed dispersion, which we term as the gap compactonlike solution. We have shown that the requirement of a compacton solution, i.e., that the width of the solution should be independent of its amplitude, can be achieved for a certain choice of the parameter values of the system. We have also shown that the nonzero amplitude of the solution outside the compact support can be reduced to a very small value for a suitable choice of the parameter values. Further, we have shown that the system also supports gap soliton solutions. It has been shown that the dynamics of various polymers and biopolymers can be modeled by KdV-type equations, which support compacton solutions [37,38]. Due to their presence in the gap region of the spectrum, the gap compactonlike solutions as obtained here are stable as they do not decay by resonating with linear phonon band and are useful for energy localization and transport in polymers and biopolymers.

[1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[2] M. Remoissenet, *Waves Called Solitons*, 2nd ed. (Springer, Heidelberg, 1995).
[3] W. Chen and D. L. Mills, Phys. Rev. Lett. **58**, 160 (1987).
[4] W. Chen and D. L. Mills, Phys. Rev. B **35**, 524 (1987).
[5] D. L. Mills and S. E. Trullinger, Phys. Rev. B **36**, 947 (1987).
[6] C. M. de Sterke and J. E. Sipe, Phys. Rev. A **38**, 5149 (1988); J. E. Sipe and H. G. Winful, Opt. Lett. **13**, 132 (1988).
[7] L. Kahn, N. S. Almeida, and D. L. Mills, Phys. Rev. B **37**, 8072 (1988).
[8] S. V. Branis, O. Martin, and J. L. Birman, Phys. Rev. Lett. **65**, 2638 (1990).
[9] D. Barday and M. Remoissenet, Phys. Rev. B **41**, 10387 (1990).
[10] D. Barday and M. Remoissenet, Phys. Rev. B **43**, 7297 (1991).
[11] Y. S. Kivshar and N. Flytzanis, Phys. Rev. A **46**, 7972 (1992).
[12] O. A. Chubykalo and Y. S. Kivshar, Phys. Rev. E **48**, 4128 (1993).
[13] Y. S. Kivshar, Phys. Rev. B **46**, 8652 (1992).
[14] Y. S. Kivshar, Phys. Rev. Lett. **70**, 3055 (1993).
[15] B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Phys. Rev. Lett. **68**, 1730 (1992).
[16] S. A. Kiselev, S. R. Bickham, and A. J. Sievers, Phys. Rev. B **48**, 13508 (1993).
[17] J. M. Bilbault, C. Tatuan Kamga, and M. Remoissenet, Phys. Rev. B **47**, 5748 (1993).
[18] N. Flytzains and B. A. Malomed, Phys. Lett. A **272**, 335 (1997).

[19] R. Grimshaw and B. A. Malomed, Phys. Rev. Lett. **72**, 949 (1994).
[20] O. Zobay, S. Potting, P. Meystre, and E. M. Wright, Phys. Rev. A **59**, 643 (1999).
[21] V. V. Konotop and M. Salerno, Phys. Rev. A **65**, 021602 (2002).
[22] P. J. Y. Louis, E. A. Ostrovskaya, C. M. Savage, and Y. S. Kivshar, Phys. Rev. A **67**, 013602 (2003).
[23] N. K. Efremidis and D. N. Christodoulides, Phys. Rev. A **67**, 063608 (2003).
[24] B. Eiermann, T. Anker, M. Albiez, M. Taglieber, P. Treutlein, K. P. Marzlin, and M. K. Oberthaler, Phys. Rev. Lett. **92**, 230401 (2004).
[25] P. Rosenau and J. M. Hyman, Phys. Rev. Lett. **70**, 564 (1993).
[26] P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994).
[27] F. Cooper, H. Shepard, and P. Sodano, Phys. Rev. E **48**, 4027 (1993).
[28] B. Dey, Phys. Rev. E **57**, 4733 (1998).
[29] B. Dey and A. Khare, Phys. Rev. E **58**, R2741 (1998).
[30] B. Dey and A. Khare, J. Phys. A: Math. Gen. **33**, 5335 (2000).
[31] P. Tchofo Dinda and M. Remoissenet, Phys. Rev. E **60**, 6218 (1999).
[32] M. Eleftheriou, B. Dey, and G. P. Tsironis, Phys. Rev. E **62**, 7540 (2000).
[33] B. Dey, M. Eleftheriou, S. Flach, and G. P. Tsironis, Phys. Rev. E **65**, 017601 (2001).
[34] P. Rosenau and S. Shochet, Chaos **15**, 015111 (2005).
[35] P. Rosenau and S. Schochet, Phys. Rev. Lett. **94**, 045503 (2005).
[36] P. G. Kevrekidis and V. V. Konotop, Phys. Rev. E **65**, 066614 (2002).
[37] R. Sarkar and B. Dey, J. Phys. A: Math. Gen. **39**, L99 (2006).
[38] R. Sarkar and B. Dey, Phys. Rev. E **76**, 016605 (2007).
[39] D. N. Christodoulides and R. I. Joseph, Phys. Rev. Lett. **62**, 1746 (1989).
[40] O. A. Chubykalo, A. S. Kovalev, and O. V. Usatenko, Phys. Rev. B **47**, 3153 (1993).
[41] A. B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).
[42] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic, New York, 1965), p. 84.