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Fractional-calculus model for temperature and pressure waves in fluid-saturated porous rocks

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We study a fractional time derivative generalization of a previous Natale-Salusti model about nonlinear temperature and pressure waves, propagating in fluid-saturated porous rocks. Their analytic solutions, i.e., solitary shock waves characterized by a sharp front, are here generalized, introducing a formalism that allows memory mechanisms. In realistic wave propagation in porous media we must take into account spatial or temporal variability of permeability, diffusivity, and other coefficients due to the system "history." Such a rock fracturing or fine particulate migration could affect the rock and its pores. We therefore take into account these phenomena by introducing a fractional time derivative to simulate a memory-conserving formalism. We also discuss this generalized model in relation to the theory of dynamic permeability and tortuosity in fluid-saturated porous media. In such a realistic model we obtain exact solutions of Burgers' equation with time fractional derivatives in the inviscid case.

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I. INTRODUCTION

Modeling of the thermomechanical characteristics of fluid-saturated porous rocks implies many difficulties. The evolution of these rocks was studied by Rice and Cleary [1], who described a linear model of isothermal pressure waves. The effect of temperature was subsequently introduced by McTigue [2] in a linear model for drained (constant fluid pressure) and undrained (fully trapped fluid) porous rocks. Their equations were applied to an isotropic hemisphere, identified as the deep source of temperature and pressure, under an isotropic layer representing a fluid-saturated permeable medium. The basic features of their models are that such rocks are homogeneous, convection is neglected, and the fluid mass balance is assumed to be linear.

More recently, Bonafede [3] used the same model for an analysis of bradyseismic crisis of Campi Flegrei. In following Natale and Salusti [4], Merlani *et al*. (referred to as MNS in the following [5]) considered also nonlinear convective transports by the Darcy velocity, obtaining both nonlinear convective and diffusive waves. These are the classical Burgers solitary shock waves [6] that interestingly form a rather quick front, often much quicker than the linear solutions. Their model has found relevant geological applications in the study of bradyseismic crises at Campi Flegrei [4] and at the Izu Penisula [5]. Finally, in [7] a nonlinear model of mechanic and chemoporoelastic interactions between fluids, contaminants, and solid matrix in swelling shales has been recently presented.

Although these models deal with constant parameters in the two homogeneous hemispheres, a variability of the coefficients of the porous medium has to be considered. In fact, some coefficients, such as diffusivity and porosity, could be time dependent due to the obstruction of pores by transport of solid fine particles or due to other effects, such as rock fracturing. These effects should be considered in the modeling of realistic phenomena. To take these variations into account, Caputo [8] recently studied the behavior of fluxes in porous media using a memory formalism, in which the ordinary time derivative was replaced by a fractional derivative [9]. As a matter of fact, the Caputo fractional derivative is given by the convolution of a power law kernel and the ordinary derivative of the function. So it is a useful instrument to consider a power law frequency variability of the coefficients by a simple convolution. By using this approach one can find the macroscopic effect of microscopic variability of some coefficients.

Moreover, focusing on the frequency variability of rock coefficients, Fellah *et al*. [10] replaced the ordinary time derivative with a fractional derivative to describe transient waves in inhomogeneous porous media. These models seem to successfully take into account the delayed effect of the fluid pressure at the boundary on the flux through the medium.

In this paper we discuss a fractional time derivative generalization of the MNS model of nonlinear temperature T and pressure P waves propagating in porous rocks. The propagation of thermomechanical (T-P) nonlinear waves describes the transition in hydrothermal systems, with rock deformation and fracturing effects. Our formulation generalizes the MNS results, with the advantage of introducing a memory mechanism. Furthermore, the introduction of a nonconstraint parameter, i.e., the real order of derivation, improves the correlation between the formulation and the experimental data. We also find analytic solutions for our generalized model; i.e., we obtain exact solutions for the fractional Burgers' equation. So once again, our model describes a transient wave, but it takes into account a global delay mechanism due to the complex microscopic dynamic.

We also discuss the physical meaning of this model in relation to the theory of dynamic permeability and tortuosity in fluid-saturated porous media [11]. The aim of this approach is to highlight the reliability of using fractional modeling for these kinds of problems.

II. THE CONSTITUTIVE EQUATIONS AND THE APPROXIMATE SOLUTIONS OF MNS

The model of Bonafede [3] describes a layered system consisting of a homogeneous, isotropic horizon for $z < 0$, identified as a source of temperature or pressure change,

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TABLE I. Characteristic parameters for clay saturated with liquid water [2] and for Berea sandstone and Rhur sandstone for supercritical water [3].

Material property	Clay [2]	Berea sandstone [3]	Ruhr sandstone [3]	Units
K_f	3×10^{-16}	1.9×10^{-13}	10^{-15}	m^2
h	4×10^{-6}	10^2	10^{-1}	m^2s^{-1}
K_T	1	3	3	$Jm^{-1}s^{-1}{}^\circ C^{-1}$
k_T	2.6×10^{-7}	10^{-6}	10^{-6}	m^2s^{-1}
ϕ	0.71	0.19	0.06	
G	7.2×10^4	6×10^9	10^{10}	Pa
B	1	0.62		0.73
ν	0.5	0.20		0.15
ν_u	0.5	0.33		0.29
α_m	3×10^{-5}	3×10^{-5}	3×10^{-5}	${}^\circ C^{-1}$
α_f	3×10^{-4}	10^{-3}	10^{-3}	${}^\circ C^{-1}$
μ	1.5×10^{-3}	2×10^{-5}	2×10^{-5}	Pas
ρ_m	3×10^3	3×10^3	3×10^3	kgm^{-3}
ρ_f	10^3	10^3	10^3	kgm^{-3}
c_m	10^3	10^3	10^3	$Jkg^{-1}{}^\circ C^{-1}$
c_f	4.2×10^3	2.1×10^3	2.1×10^3	$Jkg^{-1}{}^\circ C^{-1}$

covered by another hemisphere of fluid-saturated porous-permeable rock for $z > 0$. The fluid continuity equation is

$$\frac{\partial(\rho_f \phi)}{\partial t} + \nabla(\rho_f \phi v_f) = 0, \tag{1}$$

where ρ_f is the fluid density, ϕ is the rock porosity, and v_f is the velocity of the fluid in the porous medium. The classical Darcy law gives the relation between v_f and the overpressure P of the fluid in the porous medium:

$$\phi v_f = -\frac{K_f}{\eta}\nabla p, \tag{2}$$

where K_f is the rock permeability and η is the fluid viscosity. Bonafede [3], assuming a local thermal equilibrium between the fluid and the solid, obtained the energy balance for the whole system:

$$\begin{aligned}\frac{\partial T}{\partial t} - \frac{K_T}{\phi\rho_f c_f + (1-\phi)\rho_m c_m}\nabla^2 T \\ = -\frac{\rho_f c_f}{\phi\rho_f c_f + (1-\phi)\rho_m c_m}\phi v_f \nabla T \\ -\frac{1}{\phi\rho_f c_f + (1-\phi)\rho_m c_m}\phi v_f \nabla p,\end{aligned} \tag{3}$$

where f and m refer to fluid and matrix, respectively, K_T is the average thermal conductivity, and the fluid (solid) heat capacity is $c_f(c_m)$. Equations (1)–(3) represent a one-dimensional model for waves in a homogeneous and isotropic layer, described by two coupled linear heat-like equations.

Natale and Salusti [4] gave a nonlinear solution of (1), (2), and (3). Synthetically, their equations are the thermal space-time evolution in 1 + 1 dimensions,

$$\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial z^2} = \beta\frac{\partial T}{\partial z}\frac{\partial p}{\partial z} + \chi\left(\frac{\partial p}{\partial z}\right)^2 \tag{4}$$

[where the unusual last term $\chi(\frac{\partial p}{\partial z})^2 = -\frac{1}{\phi\rho_f c_f + (1-\phi)\rho_m c_m}\phi v_f \nabla p$ schematizes rock fracturing phenomena, as discussed by MNS] and the McTigue [2] pressure space-time evolution equation,

$$\frac{\partial p}{\partial t} - h\frac{\partial^2 p}{\partial z^2} = \alpha\frac{\partial T}{\partial t}, \tag{5}$$

where h is the fluid diffusivity and α is the thermal expansion coefficient.

Equation (4) is a kind of heath evolution equation, showing how the material derivative of T is equal to diffusion and rock fracturing terms (see values and discussion for the coefficients in Appendix A and Table I).

We assume the following initial conditions at $z = 0$; i.e., these initial temperature and pressure distributions are

$$T_0(z) = \begin{cases} T_1 + T_0 & z \leqslant 0 \\ T_0 & z > 0 \end{cases}, \quad p_0(z) = \begin{cases} p_1 + p_0 & z \leqslant 0 \\ p_0 & z > 0 \end{cases}, \tag{6}$$

namely, p_1 and T_1 represent the pressure and temperature jumps at the bottom of the fluid-saturated porous layer. It can be, however, remarked that only derivatives of T and p appear in our main equations, (4) and (5). Therefore any constant T_0 and p_0 can be added to T and p, and these new functions are again solutions of (4) and (5). So the only important initial condition is the initial jumps, T_1 and p_1.

As a first interesting example we synthesize the heuristic case $h \cong 0$ of Natale and Salusti [4] in equation (5). If we can neglect h, then $p - p_0 \cong \alpha(T - T_0)$, and this gives

$$\frac{\partial T}{\partial t} = \alpha(\beta + \alpha\chi)\left(\frac{\partial T}{\partial z}\right)^2 + k\frac{\partial^2 T}{\partial z^2}. \tag{7}$$

By a z derivative we have

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial z} - 2\alpha(\beta + \alpha\chi)\frac{\partial T}{\partial z}\frac{\partial^2 T}{\partial z^2} - k\frac{\partial}{\partial z}\frac{\partial^2 T}{\partial z^2} = 0. \tag{8}$$

Introducing $c(z,t) = -2\alpha(\beta + \alpha\chi)\frac{\partial T}{\partial z}$ in (8), we have the classical Burgers' equation:

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial z} - k\frac{\partial^2 c}{\partial z^2} = 0, \tag{9}$$

whose solution is [6]

$$c(z,t) = \sqrt{k/t}\frac{(e^R - 1)e^{-\frac{z^2}{4kt}}}{\sqrt{\pi} + (e^R - 1)\int_{z/\sqrt{4kt}}^{\infty} e^{-\xi^2} d\xi}, \quad (10)$$

with $R = \Delta/k$ being a Reynolds number and $\Delta = \alpha(\beta + \alpha\chi)T_1$.

Two ranges of R are of particular interest. If nonlinear effects are negligible, i.e., $\Delta \ll k$, we can approximate (10) with

$$c(z,t) \cong \frac{1}{\sqrt{4\pi kt}} e^{-\frac{z^2}{4kt}}, \quad (11)$$

i.e., the classical solution of the heat equation with an initial δ function. But also more interesting in our context is the case where nonlinear effects are predominant, $\alpha(\beta + \alpha\chi) \gg k$, which describes the formation of a sharp front. So neglecting the diffusive term, i.e., arriving at the hyperbolic quasilinear equation of Riemann-Hopf, we have the nondiffusive solution

$$c(z,t) = \begin{cases} z/t & 0 \leqslant z = z_f \leqslant \sqrt{4\Delta t} \\ 0 & \text{otherwise} \end{cases}, \quad (12)$$

which shows the formation of a temperature front at $z = z_F(t) = \sqrt{4\Delta t}$ moving with speed $\sqrt{\Delta/t}$.

It is of great importance that the space integral of such $c(z, t)$ is time independent, namely,

$$\int_0^{\sqrt{4\Delta t}} c dz = 2\Delta. \quad (13)$$

Moreover, a similar constraint holds even if k is not null [6] and holds if R is large. Values of R for various rocks are given in Appendix B.

The corresponding solutions of the appropriate limit forms of (4) and (5) are, for $z < z_F(t) = \sqrt{4\Delta t}$,

$$T(z,t) = \frac{1}{2\alpha(\beta + \alpha\chi)} \frac{z^2}{t}, \quad (14)$$

$$p - p_0 \cong \alpha(T - T_0) \cong \alpha\left(\frac{z^2}{2\alpha(\beta + \alpha\chi)t} - T_0\right), \quad (15)$$

and otherwise they are zero.

III. THE FULL SOLUTIONS OF THE FRACTIONAL DERIVATIVE EQUATIONS

Using the Caputo fractional derivative (see Appendix C for some preliminaries regarding fractional calculus), we introduce a memory formalism in our equations. Replacing the first order time derivative with that of real order $\nu \in (0,1)$ in both Eqs. (4) and (5), we have

$$\begin{aligned} \frac{\partial^\nu T}{\partial t^\nu} - k\frac{\partial^2 T}{\partial z^2} &= \beta \frac{\partial T}{\partial z}\frac{\partial p}{\partial z} + \chi\left(\frac{\partial p}{\partial z}\right)^2, \\ \frac{\partial^\nu p}{\partial t^\nu} - h\frac{\partial^2 p}{\partial z^2} &= \alpha\frac{\partial^\nu T}{\partial t^\nu}. \end{aligned} \quad (16)$$

For initial conditions we assume (6).

We choose for the boundary a weaker condition that at $z = 0$ and $t = 0$ we have jumps of T and p.

We seek model solutions of (16) starting from an ansatz. Since, in the inviscous limit, all the previous mentioned models have solutions that are merely functions of z^2/t, we analyze a simple solution of (16) as

$$\begin{aligned} T &= m(t)\, z^2, \\ p &= r(t)\, z^2. \end{aligned} \quad (17)$$

Such a choice for T and p must clearly be checked *a posteriori*. These solutions are a sort of generalization of the similarity solution (12) of Burgers equation. So they are again valid only in the case where nonlinear effects are predominant on diffusive terms, as we will discuss. It appears obvious that (17) does not represent a realistic solution for (16) since the effects of the initial jumps arrive for any t at any distance, where such unrealistic amplitudes are as large as z^2. One can therefore assume that the constraint (13) must approximately hold also in this case, giving a space boundary at $z \sim z_F$. This is a front that somehow resembles the conservation equation (13); however, this is rigorously valid only for the classical Burgers equation.

From the definition [9] of the Caputo fractional derivative of real order $\nu \in (0,1)$,

$${}_aD_t^\nu f(t) = \frac{1}{\Gamma(a-\nu)}\int_a^t \frac{\frac{d}{dt}f(\tau)}{(t-\tau)^\nu} d\tau, \quad (18)$$

where $\Gamma(\,)$ is the Euler Γfunction, we also have (Appendix C)

$$J^\nu \partial^\nu f(t) = f(t) - f(0), \quad (19)$$

where J^ν is the Riemann-Liouville fractional integral, an antiderivative operator.

In this way, treating again the heuristic case $h \cong 0$, from the second equation of (16) we have

$$p(z,t) = \alpha T(z,t) + p_0 - \alpha T_0. \quad (20)$$

Replacing (20) in the first equation of (16) we arrive at

$$\partial_t^\nu T - (\chi\alpha^2 + \beta\alpha)(\partial_z T)^2 - k\partial_{zz} T = 0. \quad (21)$$

It is important to stress how this choice focuses the attention on the temperature, which becomes the leading quantity in this model.

If we again z differentiate (21), we have a fractional Burgers equation for $g(z,t) = -2(\chi\alpha^2 + \beta\alpha)\partial_z T$, namely,

$$\partial_t^\nu g + g\partial_z g - k\partial_{zz} g = 0. \quad (22)$$

This nonlinear fractional differential equation has been studied with different approximate and numerical methods, such as the Adomian decomposition method [12] or the homotopy perturbation method [13]. These are approximate methods, but with our model assumption (17) we can find exact, particular solutions in the case when the nonlinear effects are predominant on diffusive terms. As a matter of fact, our ansatz is valid for the hypothesis $k \cong 0$, a realistic case (see Appendix B).

We so insert $T = m(t)z^2$ in (21), neglecting the diffusive term $k\partial_{zz}T$, and we obtain

$$z^2\partial_t^\nu m(t) - 4z^2(\chi\alpha^2 + \beta\alpha)(m(t))^2 = 0. \quad (23)$$

It is simple to see that this equation has the exact solution

$$m(t) = \frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)} \frac{t^{-\nu}}{4(\chi\alpha^2+\beta\alpha)}, \quad 0 < \nu < \frac{1}{2}. \tag{24}$$

A constraint has to be assumed about the choice of the real order of derivation for physically meaningful solutions. In fact, for $1/2 < \nu < 1$ the argument of Γ in the denominator becomes negative, and this solution with the negative sign is physically unreasonable. The ordinary case $\nu = 1$ is again not exactly recovered because of the divergence of the Γ coefficients. So our solution is not rigorously a generalization of the ordinary one because of the Γ coefficients in (24). However, this is an exact solution, having the same spatial shape of the ordinary case.

From (24) we find the temperature profile in the inviscid case:

$$T(z,t) = \frac{\Gamma(1-\nu)}{\Gamma(1-\nu)} \frac{t^{-\nu} z^2}{4(\chi\alpha^2+\beta\alpha)}. \tag{25}$$

Coming back to equation (20), the pressure is given by

$$\begin{aligned} p(z,t) &= \alpha T(z,t) + p_0 - \alpha T_0 \\ &= \frac{\Gamma(1-\nu)}{\Gamma(1-\nu)} \frac{t^{-\nu} z^2}{4(\chi\alpha^2+\beta\alpha)} + p_0 - \alpha T_0. \end{aligned} \tag{26}$$

We moreover assume that the constraint (13) must hold approximately also in this case, namely, that (25) holds. This gives

$$\begin{aligned} T(z,t) &= \frac{\Gamma(1-\nu)}{\Gamma(1-\nu)} \frac{z^2 t^{-\nu}}{4(\chi\alpha^2+\beta\alpha)} \quad 0 < z < z_F, \\ z_F(t) &= 2\sqrt{\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\nu)}} t^{\frac{\nu}{2}}. \end{aligned} \tag{27}$$

It has to be remarked that assumption (17) is satisfied, and with the position (20) for this model the initial jump of p is ruled by that of T.

In Fig. 1 the effects of fractional time generalization on the propagation of the sharp front are shown. We observe a slower temporal decay when compared to the ordinary case, suggesting the presence of a time delay effect. In the next section we discuss the physical meaning of using this formalism in the theory of dynamic coefficients in fluid-saturated porous rocks.

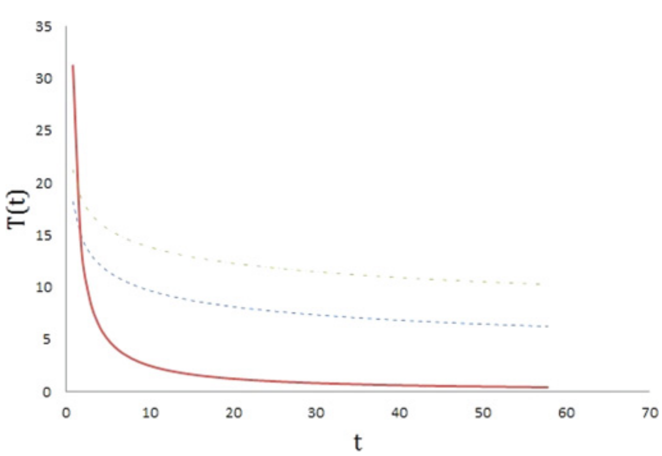

FIG. 1. (Color online) The temperature profile at $z = 1$ m for Berea sandstone and supercritical water (see Appendixes A and B) for $\nu = 1$ (solid red line), for $\nu = 1/4$ (dotted blue line), and for $\nu = 1/6$ (dot-dashed green line).

IV. APPLICATIONS OF FRACTIONAL MODELING IN THE THEORY OF DYNAMIC COEFFICIENTS IN FLUID-SATURATED MEDIA

As previously mentioned, our hypothesis is based on the idea that fractional modeling is a good instrument to describe nonlocal effects. Moreover, fractional modeling helps us to consider the stochastic variability of some coefficients in the medium of propagation in a global macroscopic way.

In this section we highlight this position in relation to the theory of dynamic coefficients in fluid-saturated rocks, referring in particular to the classical paper of Johnson *et al.* [11] on this topic. Johnson *et al.* studied the linear response of a homogeneous, isotropic porous solid to a macroscopic pressure gradient, assuming that tortuosity and permeability coefficients are frequency dependent.

The constitutive equations of their model are

$$\alpha(\omega)\rho_f \partial_t v = -\nabla P, \tag{28}$$

$$\phi v = -\frac{K_f(\omega)}{\eta} \nabla P. \tag{29}$$

The first is the linearized equation about velocity v, where ρ_f is the fluid density, $\alpha(\omega)$ is the frequency-dependent tortuosity, and ∇P is a macroscopic pressure gradient. The second equation is Darcy's law, with porosity ϕ, viscosity η, and frequency-dependent permeability $K_f(\omega)$. In their analysis the theoretical expression of frequency-dependent coefficients is given by

$$\alpha(\omega) = \alpha_\infty + \frac{i\eta\phi}{\omega k_0 \rho_f}\left(1 - \frac{4i\alpha_\infty^2 k_0^2 \rho_f \omega}{\eta \Lambda^2 \phi^2}\right)^{1/2}, \tag{30}$$

$$K_f(\omega) = \frac{k_0}{\left(1 - \frac{4i\alpha_\infty^2 k_0^2 \rho_f \omega}{\eta\Lambda^2\phi^2}\right)^{1/2} - \frac{i\alpha_\infty k_0 \rho_f \omega}{\eta\phi}}, \tag{31}$$

where the parameters α_∞, k_0, and Λ are unrelated, empirical coefficients, experimentally defined for any porous media.

These coefficients are complex power law functions in frequency (for a deeper discussion of these expressions, see [11]). This model has been experimentally validated; it opens an interesting field of research regarding wave propagation in complex media. An idea recently developed by Fellah *et al.* [10] is to include frequency variability in (28) and (29) by using a fractional derivative. In fact, as already seen, the Caputo fractional derivative includes this kind of variability by the convolution of the ordinary derivative with a power law kernel. In this sense the frequency variability of (30) and (31) is treated by using a semiderivative of order 1/2. Note that this new mathematical approach was also experimentally validated in [14].

It should also be noted that with such observations, Caputo *et al.* treated the fractional generalization of Darcy's law, also experimentally validated (see [15]).

In this framework we can give a heuristic derivation of our fractional model. Recalling the original equation (4) of MNS,

$$\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial z^2} = \beta\frac{\partial T}{\partial z}\frac{\partial p}{\partial z} + \chi\left(\frac{\partial p}{\partial z}\right)^2, \tag{32}$$

we notice that the average thermal diffusivity β is directly related to the permeability of the porous media (see Appendix A),

$$\beta = \frac{K_f\rho_f c_f}{\phi\rho_f c_f(1-\phi)\rho_m c_m}, \tag{33}$$

and the average dissipative diffusivity due to matrix-fluid friction χ is related to β by

$$\chi = \frac{\beta}{\rho_f c_f}. \tag{34}$$

If, following [11], we assume that the rock permeability is frequency-dependent, then β and χ are dynamic coefficients, proportional to K_f.

In our discussion we treated the inviscid case of (32), neglecting the diffusive term. With this hypothesis we have

$$\frac{1}{\beta(\omega)}\frac{\partial T}{\partial t} = \frac{\partial T}{\partial z}\frac{\partial p}{\partial z} + \frac{1}{\rho_f c_f}\left(\frac{\partial p}{\partial z}\right)^2. \tag{35}$$

Now we simply recall Fellah *et al.* [10] and include the power law frequency variability of β in the Caputo fractional derivative. With similar arguments we can deduce the fractional generalization of (5). It is not a rigorous derivation, but it clearly explains the physical reason to use this formalism.

In conclusion, our model can be heuristically derived from precedent analysis of dynamic coefficients. This is a use for fractionalmodeling in this field of research aimed to give more realistic treatment of wave propagation in complex media.

V. DISCUSSION

We here discuss the generalization of the original MNS model [5] of temperature and pressure waves in fluid-saturated porous rocks, introducing a memory formalism by fractional time derivative. We find a solution similar to the original model, but with a new free parameter, the fractional order ($0 < \nu < 1/2$) for the time derivative, which could be useful to reach a best fit with experimental data.

Our result is based on the introduction of two assumptions. The first one, which we can call instrumental, is the introduction of a new parameter that allows us to take both global and macroscopic delay mechanisms into account, e.g., those due to local variations of diffusivity, porosity, and permeability. Similar models of the fractional Darcy's law have also an experimental validation; see, for example, [15].

The second idea, more conceptual, is to understand in this case study the physical utility and meaning of fractional-calculus modeling. It is already clear that while using fractional time derivatives we introduce a memory in our problem. It is also interesting to investigate whether fractional calculus could be the ideal instrument to transmit randomness of the microscopic dynamic to the macroscopic scale, as discussed by Grigolini *et al.* [16]. Therefore our case study could be a physical application, representing a complex stochastic dynamic environment at the microscopic level with possible obstruction or dilatation of pores. However, a global and deterministic description of the passage of transient thermomechanical waves is required for this.

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APPENDIX A: CHARACTERISTIC PARAMETERS IN THE CONSTITUTIVE EQUATIONS

In equations (4) and (5) the average thermal diffusivity due to diffusion is $k = \frac{K_T}{\phi\rho_f c_f(1-\phi)\rho_m c_m}$, the average thermal diffusivity due to convection is $\beta = \frac{K_f\rho_f c_f}{\phi\rho_f c_f(1-\phi)\rho_m c_m}$, the average dissipative diffusivity due to matrix-fluid friction is $\chi = \frac{\beta}{\rho_f c_f}$, the fluid diffusivity is $h = \frac{K_f}{\mu}(\frac{2GB^2(1+\nu_u)^2(1-\nu)}{9(1-\nu_u)(\nu_u-\nu)})$, G is the rigidity modulus, ν (ν_u) is the drained (undrained) Poisson ratio, B is the Skempton parameter, α_m (α_f) is the volumetric thermal expansion coefficient for the solid (fluid), and the thermal expansion coefficient is $\alpha = [G\alpha_m\frac{4(1+\nu)}{(1-\nu)} + GB\phi(\alpha_f-\alpha_m)\frac{2(1+\nu)(1+\nu_u)}{3(\nu_u-\nu)}][\frac{B(1-\nu)(1-\nu_u)}{3(1-\nu)(1+\nu_u)-6(\nu_u-\nu)}]$.

The numerical estimate of such parameters for different rocks is given in Table I.

APPENDIX B

In order to find the importance of diffusion in equation (20) we now examine the related quantities χ, β, α, and k. Considering the values used by McTigue [2] and Bonafede [3], in MKS units, we have the values shown in Table II.

It results that for an initial temperature jump $T_1 = 100^\circ$ C one has $R = 3\times10^{-2}$ for clay and liquid water [2], $R = 3\times10^3$ for Berea sandstone and supercritical water [3], $R = 2\times10^3$ for Ruhr sandstone and supercritical water [3], and therefore $R \gg 0$ only for sandstones but not for clay.

APPENDIX C: PRELIMINARIES AND NOTATION FOR FRACTIONAL CALCULUS

The Caputo fractional derivative of real order $\nu \in (0,1)$ is defined by [9]

$$\begin{aligned} {}_aD_t^\nu f(t) &= \frac{1}{\Gamma(1-\nu)}\int_a^t \frac{\frac{d}{dt}f(\tau)}{(t-\tau)^\nu}d\tau \\ &= \frac{1}{\Gamma(1-\nu)}\left[\frac{df(\tau)}{dt} * K(t-\tau)\right]. \end{aligned} \tag{C1}$$

We see that it is an integro-differential operator defined by a convolution between the ordinary first derivative of the function and a power law kernel. We say that it is a nonlocal pseudodifferential operator since the value of ${}_aD_t^\nu f(t)$ depends on all the values of $f(t)$ in the interval $[a, t]$, i.e., on the entire "history" of the function. In this paper we use the symbol $\partial_t^\nu = {}_0D_t^\nu$ for the fractional partial derivative with respect to t, where the lower extreme of integration is null.

TABLE II. Characteristic parameters of equations (4) and (5) for clay, Berea, and Rhur sandstone. The natural space scale λ and time scale τ are also shown ([4]).

Parameters	Clay and liquid water	Berea and supercritical water	Ruhr sandstone and supercritical water
h	4×10^{-6}	10^{-2}	10^{-2}
α	6×10^{2}	10^{6}	2×10^{6}
k	3×10^{-7}	3×10^{-6}	3×10^{-6}
β	2×10^{-13}	7×10^{-9}	3×10^{-11}
χ	10^{-19}	10^{-14}	10^{-16}
T_1	100	100	100
λ	10^{3}	10^{3}	10^{3}
τ	3×10^{12}	4×10^{5}	10^{8}
$\chi\alpha^2 + \beta\alpha$	10^{-10}	10^{-2}	6×10^{-5}
R	3×10^{-2}	3×10^{3}	2×10^{3}

Now we introduce the fractional integral of Riemann-Liouville of order $\nu \in (0,1)$:

$$ {}_aJ_t^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\nu}} d\tau. \tag{C2} $$

The fractional Caputo derivative is related to the fractional integral of Riemann-Liouville by the following relation:

$$ D_t^\nu f(t) = J_t^{1-\nu} D_t^1 f(t) \quad \text{if } \nu \in (0,1). \tag{C3} $$

It is simple to prove the following relevant results for derivative of real order $\nu \in (0, 1)$ (for a generic order, see [8]):

$$ {}_0J_t^\nu \, {}_0D_t^\nu f(t) = f(t) - f(0), \tag{C4} $$

$$ {}_aD_t^\nu \, {}_aJ_t^\nu f(t) = f(t), \tag{C5} $$

$$ {}_aD_t^\nu t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)} t^{\beta-\nu} \quad \text{if } \beta > -1 \quad \nu > 0. \tag{C6} $$

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