Current reversals of coupled driven and damped particles evolving in a tilted potential landscape

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We explore the driven and damped dynamics of two coupled particles evolving in a symmetric and periodic substrate potential that is subjected to a static bias force. In addition, each particle is time-periodically driven with the same magnitude as, but out of phase to, its counterpart. It is shown that, for a certain parameter regime, the coupled particles can become self-organized and go against the direction of the bias force. This self-organization involves the particles becoming frequency locked with the driving force, and thus periodic motion ensues. We employ numerical arguments to show that running periodic states provide solutions of the system. Further, heuristic evidence is provided explaining how the two particles can travel against the bias force. In an effort to unearth coupling phenomena within the system, a detailed analysis of how the coupling strength affects the nonlinear dynamics is carried out. We show that within a range of coupling strengths the existence of periodic running solutions associated with negative mobility. To examine the robustness of our results we compare the deterministic system with the corresponding Langevin system. It is shown that, below a critical temperature, the qualitative behavior of the system remains the same.

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I. INTRODUCTION

Nonlinear transport processes continue to be of vital importance to the understanding of many physical systems. In particular, the transport of particles in symmetric and periodic potential landscapes has attracted considerable interest [1–6]. This ubiquitous potential lends itself to a vast number of applications, including Josephson junctions [7], charge density waves, nanoengines [8], and transport in biological systems [9]. Much of the research done in this area involves systems with an external time-periodic modulation applied, allowing for more complex dynamics. Striking effects, such as phase locking and stochastic resonance [10], are often seen in such a system. An interesting extension to problems with an externally modulated potential comes when a dc bias is introduced, serving as a constant tilt to the potential landscape [11–14]. In the single-particle case, these systems have produced some fascinating results, most notably being the existence of "absolute negative mobility" [15]. Here a particle travels against the bias with the same velocity, but with negative sign, as when the bias is removed. In this paper we consider the extension of this problem to two interacting particles. We examine the motion of two coupled damped and driven particles evolving in a tilted periodic and symmetric potential. We will highlight the anomalous transport properties of the system and aim to show that these properties are not isolated examples brought about by fine tuning of the parameters or by choice of initial condition. Our modus operandi is to explore cooperative phenomena in the system, and thus we will provide a detailed numerical investigation on how the coupling between the particles affects the overall nonlinear dynamics.

This paper is organized as follows: In the next section we introduce the system of coupled particles. In Sec. III we discuss the feature of negative mobility. We also give a brief comparison between the given system and its Langevin counterpart. Section IV explores how the coupling strength influences the nonlinear dynamics present in the system and, in particular, how it relates to the emergence of a directed flow, i.e., the current. We finish with a summary of our findings.

II. SYSTEM OF COUPLED PARTICLES

We study the dynamics of two damped, coupled particles evolving in a symmetric and periodic "washboard" potential, which has a spatial period $L = 1$. These particles are further subjected to a static bias force of magnitude $F_0 > 0$ that serves to tilt the potential landscape such that particle motion to the right is favored. The equations of motion for this system are given by

$$\ddot{q}_1 = -\sin(2\pi q_1) - \gamma\dot{q}_1 - F\sin(\Omega t + \theta_0) - \kappa(q_1 - q_2) + F_0, \tag{1}$$

$$\ddot{q}_2 = -\sin(2\pi q_2) - \gamma\dot{q}_2 + F\sin(\Omega t + \theta_0) + \kappa(q_1 - q_2) + F_0. \tag{2}$$

The two particles are driven by an external time-dependent modulation of amplitude F, frequency Ω, and phase θ_0. Notice the out-of-phase character of the periodic modulation of the two particles expressed by the different sign of the modulation amplitude F. The additional parameters γ and κ regulate the strength of the damping and coupling, respectively.

We stress that in our system the two particles, forming a dimer, are supposed to perform one-dimensional motion in parallel directions, each of them in a washboard potential. That is, for equal coordinates $q_1 = q_2 = q$ the axis of the dimer (virtual line connecting the two particles) is perpendicular to the q direction. As a realization we can suggest a simple mechanical system comprising two balls moving along parallel corrugated lines. The inclination of the latter are, apart from the static tilt, temporally modulated in an out-of-phase fashion. In addition the balls are coupled via a spring.

In general, the system will exhibit a rich and varied behavior as a function of its parameters γ, F, κ, Ω, θ_0, and F_0. However, one of our main objectives, as was previously mentioned, is to explore coupling phenomena within the system, and therefore in much of this study we will fix the remaining parameters while varying the coupling parameter κ.

III. EXISTENCE OF NEGATIVE MOBILITY

As an explanation of this phenomenon, we describe the mechanism that makes negative mobility possible. The equations of motion have been solved numerically using a fourth-order Runge-Kutta method. Figure 1 shows snapshots of a two-particle compound moving in the opposite direction of the bias force, where the particles move in the respective potential landscape given by

$$U(q_1,t) = \frac{1}{2\pi}[1 - \cos(2\pi q_1)] + F\sin(\Omega t + \theta_0)q_1 - F_0 q_1, \tag{3}$$

$$U(q_2,t) = \frac{1}{2\pi}[1 - \cos(2\pi q_2)] - F\sin(\Omega t + \theta_0)q_2 - F_0 q_2. \tag{4}$$

(Note that the potential energy as given above relates to the on-site potential not containing the particle interaction part.) These seven snapshots, taken over one period of the driving, show the relative position of each particle (henceforth called particle 1, referring to left panels in Fig. 1, and particle 2, referring to right panels in Fig. 1) versus its position in the potential landscape. In addition, arrows, where present, indicate the direction and magnitude of momentum for the respective particles, with no visible arrow indicating a vanishing momentum. It can be seen that negative mobility is the product of coupling between the particles and the effect of the time modulated potential. For example, at the beginning of the period particle 1 has a positive momentum in the direction of the bias force. However, this is countered by the height of the potential barrier and by the coupling to particle 2, which has an even stronger negative momentum. Thus motion in the direction of bias is hindered. Regarding negative mobility, we underline that the opposite time-periodic forces make it only possible that for one particle the current inclination of the washboard potential is of such form that the particle is temporarily locked in a potential well (thus hampering its dragging influence on the other particle in the unwanted direction of the tilt) while the other particle experiences a washboard potential whose current inclination favors motion against the static tilt force F_0. These phases of temporary locking and running against the tilt alternate between the particles. This cooperative effect between the particles and the finely tuned modulations of the potential combine, for the duration of the period, aiding motion against the bias force. Consequently, in one period of the driving, the dimer moves one spatial period against the bias force.

In fact, this combination even stabilizes the periodic uphill motion against perturbations induced by noise from a thermal bath. To demonstrate the robustness of negative mobility with respect to thermal fluctuations we consider the following Langevin equations:

$$\ddot{q}_1 = -\sin(2\pi q_1) - \gamma\dot{q}_1 - F\sin(\Omega t + \theta_0) - \kappa(q_1 - q_2) + F_0 + \xi_1(t), \tag{5}$$

$$\ddot{q}_2 = -\sin(2\pi q_2) - \gamma\dot{q}_2 + F\sin(\Omega t + \theta_0) + \kappa(q_1 - q_2) + F_0 + \xi_2(t), \tag{6}$$

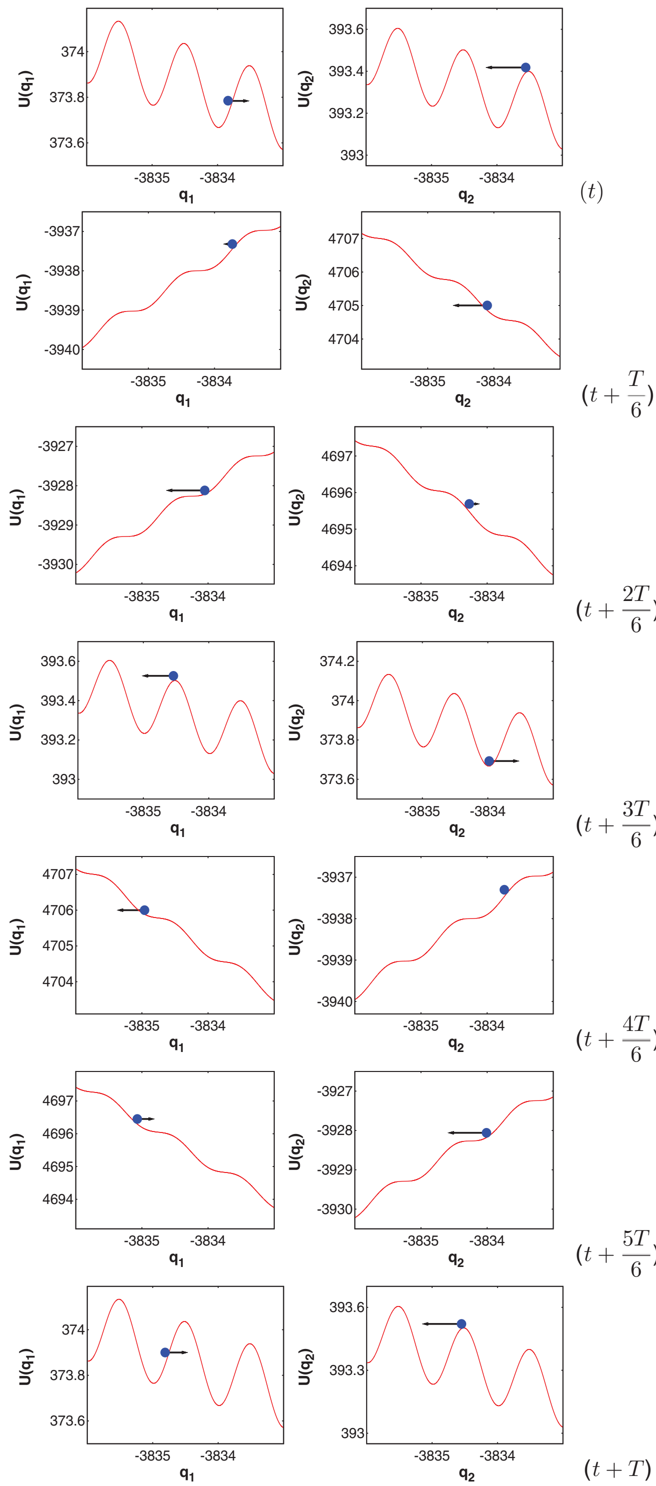

FIG. 1. (Color online) Snapshots of the dimer motion against the bias force taken over a period T of the external time-periodic modulation. Arrows indicate direction and magnitude of a particle's momentum. Left (right): particle 1 (particle 2). Here $F_0 = 0.1$, $\gamma = 0.11$, $F = 1.3$, $\Omega = 2.22$, $\theta_0 = 0$, and $\kappa = 0.372$.

where $\xi_{1,2}(t)$ denotes a Gaussian distributed thermal random force of vanishing mean and correlation $\langle\xi_m(t)\xi_n(t')\rangle = 2\gamma k_B T\delta_{m,n}\delta(t - t')$ with temperature T. Our numerical simulation results are reported in Fig. 2, displaying the temporal behavior of the mean velocity [as defined in Eq. (8)] of the

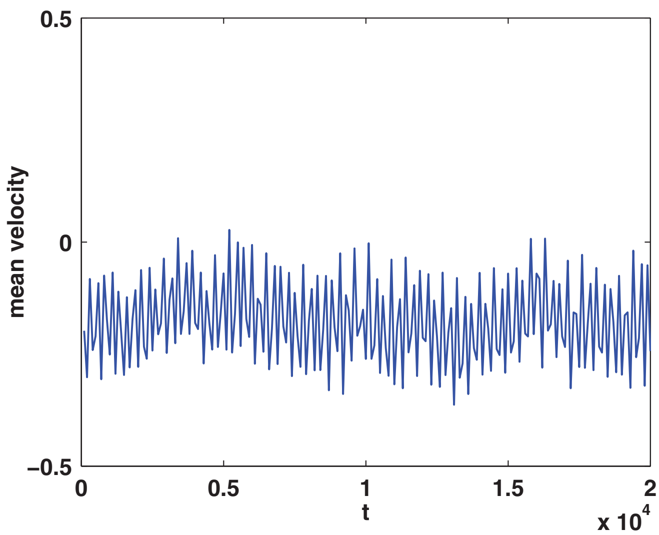

FIG. 2. (Color online) Time evolution, omitting an initial transient, of the mean velocity in the presence of a heat bath of thermal energy $k_B T = 10^{-3}\Delta E$, with $\Delta E = 1/\pi$ being the energetic barrier height of the washboard potential. The system parameters are given in Fig. 1.

two-particle system for thermal energy $k_B T = 10^{-3}\Delta E$, with $\Delta E = 1/\pi$ being the energetic barrier height of the washboard potential. The Langevin equations were numerically integrated using a second-order Heun stochastic solver. We took averages of 1000 realizations of the thermal noise. Notably, negative mean velocity results, indicating negative mobility.

IV. CURRENT

In this section we will explore in detail the effect that the coupling parameter has on the dynamics. Simulations show that, depending on the value of $\kappa > 0$, there exist both periodic and aperiodic solutions. This is clearly depicted in Fig. 3, which shows a bifurcation diagram of the particle averaged momentum defined as $p = (p_1 + p_2)/2$, where $p_{1,2} = \dot{q}_{1,2}$, taken after a suitable transient at each period of the external driving, as a function of κ. In Fig. 3 we see two windows

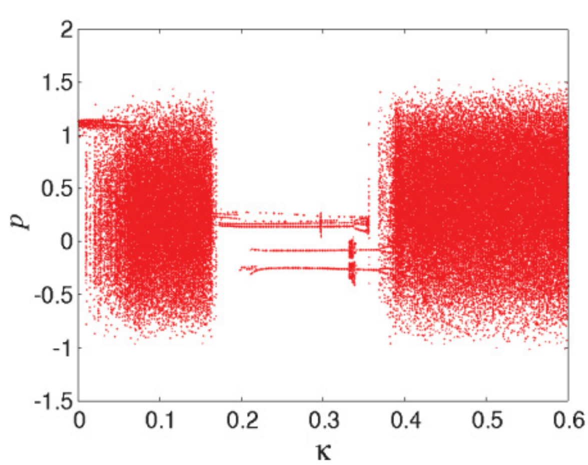

FIG. 3. (Color online) Bifurcation diagram as a function of the coupling parameter κ. The remaining parameters are given in Fig. 1.

representing periodic motion, and the other windows represent the chaotic motion in the system. The first periodic window, which is difficult to see from the scale of Fig. 3, comes in the region of very low coupling. To gain a quantitative perspective on how κ influences the dynamics we compute the current v. That is, we calculate the time averaged mean velocity for an ensemble of initial conditions, i.e.,

$$v = \frac{1}{T_s}\int_0^{T_s} dt \langle p(t) \rangle, \tag{7}$$

where T_s is the simulation time, and the ensemble average is given by

$$\langle p(t) \rangle = \frac{1}{N}\sum_{n=1}^{N}\sum_{i=1}^{2} p_{i,n}(t), \tag{8}$$

with N being the number of initial conditions. These initial conditions have been chosen such that $q_n(0)$ are uniformly distributed in the potential well centered at the origin, with $p_n(0) = 0$ for all n. For computation of the long time average, numerical integration is performed over a simulation time interval $T_s = 10^5$. The ensemble average is calculated using an ensemble of $N = 1500$ initial conditions.

The current as a function of κ is shown in Fig. 4. As we would expect, in the uncoupled case a strong positive current results, as the particles can travel freely in the direction of the bias. This is what we see in Fig. 4. However, when we couple the particles through increasing κ, we see a sharp decline in the strength of the current. Now a restoring force acts between the particles, impeding unhindered motion. This decline happens in the range $0 < \kappa \lesssim 0.08$ after which we see a slight increase in the strength of the current. The most spectacular results come in the range $0.22 \lesssim \kappa \lesssim 0.4$, which contains two windows, of notable extent, where there is a negative current and a relatively smooth transition from $v \approx 0.26$ to $v = 0$ preceding these windows of negative current. The smooth transition to a zero current is followed by a rather sharp decline into the range of negative momentum. This range of κ coincides with the window of periodic motion seen in the

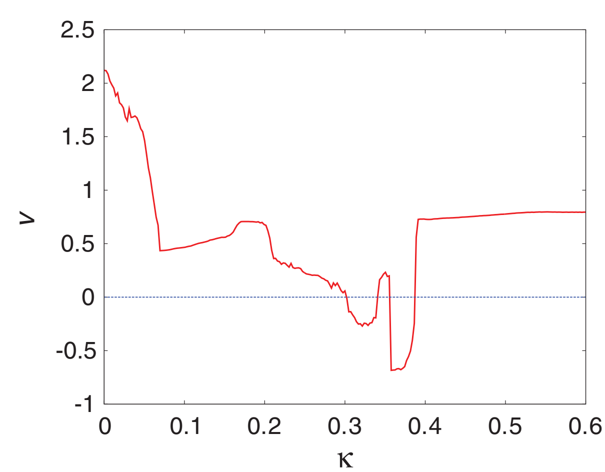

FIG. 4. (Color online) The current, as defined in Sec. IV, as a function of the coupling parameter κ. The remaining parameters are given in Fig. 1. The blue dashed line serves to highlight the regions of negative current.

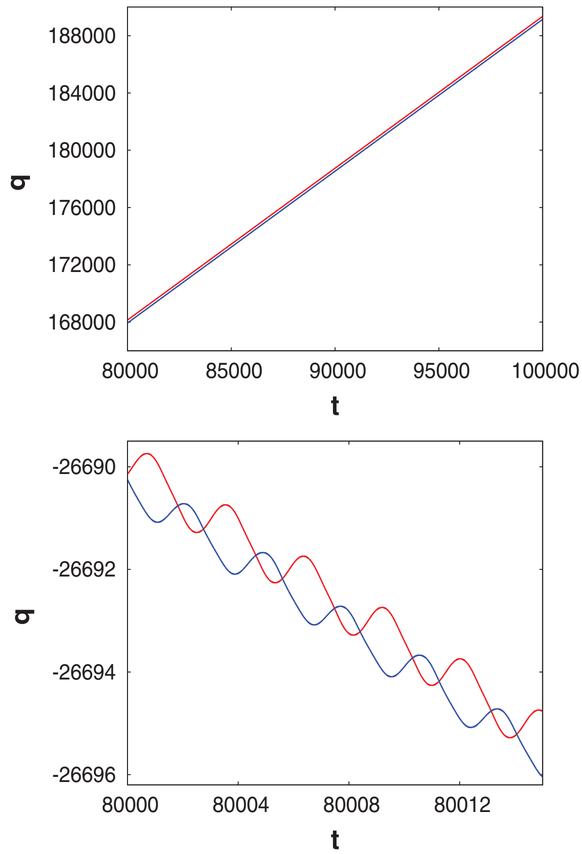

FIG. 5. (Color online) (top) Time evolution of the two uncoupled particles ($\kappa = 0$), traveling with the bias force. (bottom) Time evolution of the two coupled particles ($\kappa = 0.327$) that are traveling in the opposite direction to the bias force. The two lines represent the trajectories of the individual particles. The system parameters are given in Fig. 1. Note the different time scales.

bifurcation diagram (Fig. 3). Further, if we look closely at a climbing trajectory (Fig. 5), it can seen that this is a cooperative effect, with one particle pulling the other over one potential barrier, only for the roles to be reversed when overcoming the subsequent potential barrier. Notice that this is in compliance with our earlier consideration (Sec. III), which showed that motion against the bias can be provided by periodic running solutions, which are frequency locked to the external periodic driving force. After this window of negative current the current saturates at approximately $p = 0.75$. Previously, there was a cooperative effect helping the particles to overcome potential barriers; now it appears that for $\kappa \gtrsim 0.4$ the particles effectively act as one and are unable to overcome these barriers. Thus the dimer follows the direction of the bias, and hence a positive current is produced.

V. SUMMARY

We have explored the dynamics of two coupled, damped, and driven particles evolving in periodic and symmetric potential while being subjected to a constant bias force. Further the driving is time periodic and drives one particle in the opposite direction of its counterpart. A key finding was the existence of negative mobility, i.e., solutions in which the motion goes against the direction of bias. We demonstrated that the mechanism allowing for such motion was cooperation between the particles, where they pull each other over consecutive potential barriers. In more detail, a coordinated energy exchange between the particles allows them to collectively climb against the direction of the tilt.

Another aspect of our work dealt with directed particle transport. This involved quantifying how the coupling strength influenced the dynamics present in the system. As a first step we produced the bifurcation diagram as a function of the coupling parameter. This diagram is characterized by two windows of aperiodic motion and one fairly extended window of periodic motion. Then we computed the current as a function of the coupling parameter. It was seen that there are two windows of negative current, after which the current saturates and becomes positive. Notably, these two windows of negative current lie within the periodic window seen in the bifurcation diagram.

Finally, we have demonstrated the robustness of the motion against the bias of the tilt force with respect to thermal fluctuations.

To conclude, we remark that it may be of interest to extend the present study to include a chain of particles and to determine whether negative mobility is possible in such a system and, further, if it is possible then to determine the mechanisms that make it so.

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